CENTRAL UNITS OF INTEGRAL GROUP RINGS OF NILPOTENT GROUPS

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Abstract. In this paper a finite set of generators is given for a subgroup of finite index in the group of central units of the integral group ring of a finitely generated nilpotent group.

In this paper we construct explicitly a finite set of generators for a subgroup of finite index in the centre $\mathbb{Z}(U(\mathbb{Z}G))$ of the unit group $U(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a finitely generated nilpotent group $G$. Ritter and Sehgal [4] did the same for finite groups $G$, giving generators which are a little more complicated. They also gave in [2] necessary and sufficient conditions for $\mathbb{Z}(U(\mathbb{Z}G))$ to be trivial; recall that the units $\pm G$ are called the trivial units. We first give a finite set of generators for a subgroup of finite index in $\mathbb{Z}(U(\mathbb{Z}G))$ when $G$ is a finite nilpotent group. Next we consider an arbitrary finitely generated nilpotent group and prove that a central unit of $\mathbb{Z}G$ is a product of a trivial unit and a unit of $\mathbb{Z}T$, where $T$ is the torsion subgroup of $G$. As an application we obtain that the central units of $\mathbb{Z}G$ form a finitely generated group and we are able to give an explicit set of generators for a subgroup of finite index.

1. Finite nilpotent groups

Throughout this section $G$ is a finite group. When $G$ is Abelian, it was shown in [1] that the Bass cyclic units generate a subgroup of finite index in the unit group $U(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a finitely generated nilpotent group $G$. Ritter and Sehgal [4] did the same for finite groups $G$, giving generators which are a little more complicated. They also gave in [2] necessary and sufficient conditions for $\mathbb{Z}(U(\mathbb{Z}G))$ to be trivial; recall that the units $\pm G$ are called the trivial units. We first give a finite set of generators for a subgroup of finite index in $\mathbb{Z}(U(\mathbb{Z}G))$ when $G$ is a finite nilpotent group. Next we consider an arbitrary finitely generated nilpotent group and prove that a central unit of $\mathbb{Z}G$ is a product of a trivial unit and a unit of $\mathbb{Z}T$, where $T$ is the torsion subgroup of $G$. As an application we obtain that the central units of $\mathbb{Z}G$ form a finitely generated group and we are able to give an explicit set of generators for a subgroup of finite index.

Lemma 1. The images of the Bass cyclic units of $\mathbb{Z}G$ under the natural homomorphism $j : U(\mathbb{Z}G) \rightarrow K_1(\mathbb{Z}G)$ generate a subgroup of finite index.

Let $L$ denote the kernel of this map $j$, and $B$ the subgroup of $U(\mathbb{Z}G)$ generated by the Bass cyclic units. It follows that there exists an integer $m$ such that $z^m \in LB$ for all $z \in Z(U(\mathbb{Z}G))$, and so we can write $z^m = lb_{b_2} \cdots b_k$ for some $l \in L$ and Bass cyclic units $b_i$. 

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Next, let $Z_i$ denote the $i$-th centre of $G$, and suppose from now on that $G$ is nilpotent of class $n$. For any $x \in G$ and Bass cyclic unit $b \in \mathbb{Z}(x)$, we define

$$b_{(1)} = b$$

and for $2 \leq i \leq n$

$$b_{(i)} = \prod_{g \in Z_i} b_g,$$

where $\alpha^g = g^{-1} \alpha g$ for $\alpha \in \mathbb{Z}G$. Note that by induction $b_{(i)}$ is central in $\mathbb{Z}(Z_i, x)$ and independent of the order of the conjugates in the product expression. In particular, $b_{(n)} \in Z(\mathcal{U}(\mathbb{Z}G))$.

Recall again that if $z \in Z(\mathcal{U}(\mathbb{Z}G))$, then $z^n = lb_1 b_2 \cdots b_k$ for some $l \in L$ and Bass cyclic units $b_i$. Since $K_1(\mathbb{Z}G)$ is Abelian, we can write

$$z^{m|Z_2|-|Z_n|} = (lb_1 b_2 \cdots b_k)^{m|Z_2|-|Z_n|}$$

for some $l_1 \in L$

$$= l_1 \prod_{1 \leq i \leq k} b_{i}^{m|Z_2|-|Z_n|}$$

for some $l_2 \in L$

$$= l_2 \prod_{1 \leq i \leq k} b_{i}^{1}$$

for some $l' \in L$.

Since each $b_{(n)}$ is in $Z(\mathcal{U}(\mathbb{Z}G))$, we conclude that $l' \in L \cap Z(\mathcal{U}(\mathbb{Z}G))$. But we shall show next that $L \cap Z(\mathcal{U}(\mathbb{Z}G))$ is trivial, so $l' \in \pm Z(G)$. The argument uses the same idea as in [3, Lemma 3.2].

For every primitive central idempotent $e$ in the rational group algebra $\mathbb{Q}G$, the simple ring $\mathbb{Q}Ge$ has a reduced norm which we denote by $nr_e$. Further, denote

$$m_e = \sqrt{[\mathbb{Q}Ge : Z(\mathbb{Q}Ge)]}$$

and let

$$r = \prod_{e} m_e.$$

Now let $l' \in L \cap Z(\mathcal{U}(\mathbb{Z}G))$. By definition of $K_1(\mathbb{Z}G)$ this means that a suitable matrix

$$\begin{bmatrix} l' & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

is a product of commutators. Therefore $l'e$ has reduced norm one. Since $l'e$ is also central, we obtain

$$(l'e)^{m_e} = nr(l'e)e = e.$$  

Hence

$$l'' = 1.$$

So $l'$ is a torsion central unit, and therefore is trivial [7, Corollary 1.7, page 4].

Since $Z(\mathcal{U}(\mathbb{Z}G))$ is finitely generated (see, e.g., [2]), $(Z(\mathcal{U}(\mathbb{Z}G)))^{m|Z_2|-|Z_n|}$ is of finite index. But we have just seen that the latter subgroup is contained in the subgroup generated by $\pm Z(G)$ and $\langle b_{(n)} \mid b \text{ a Bass cyclic unit} \rangle$. We have proved
Proposition 2. Let $G$ be a finite nilpotent group of class $n$. Then

$$\langle b_{(a)} \mid b \text{ a Bass cyclic} \rangle$$

is of finite index in $Z(U(\mathbb{Z}G))$.

Remark. Note that our method for constructing generators for a subgroup of finite index in $Z(U(\mathbb{Z}G))$ can be adapted for some other classes of finite groups $G$. For example if $G = D_{2n} = \langle a, b \mid x^n = 1, y^2 = 1, yx = x^{n-1}y \rangle$, the dihedral group of order $2n$, then the only nontrivial Bass cyclic units $b$ of $\mathbb{Z}D_{2n}$ belong to $\mathbb{Z}(x)$. It follows that $bb^y = b^yb$ is central. Our proof now remains valid and yields that $\langle bb^y \mid b \text{ a Bass cyclic in } \mathbb{Z}(x) \rangle$ is of finite index in $Z(U(\mathbb{Z}D_{2n}))$.

2. FINITELY GENERATED NILPOTENT GROUPS

We will now consider central units of an integral group ring of an arbitrary finitely generated nilpotent group $G$. The torsion subgroup of $G$ is denoted $T$. First we show that central units of $\mathbb{Z}G$ have the following decomposition.

Proposition 3. Let $G$ be a finitely generated nilpotent group. Every $u \in Z(U(\mathbb{Z}G))$ can be written as $u = rg$, $r \in \mathbb{Z}G$, $g \in G$.

Proof. Let $F = G/T$. Since $T$ is finite and $F$ acts on the set of primitive central idempotents of $\mathbb{Q}T$ by conjugation, by adding the idempotents in an orbit of this action we obtain

$$\mathbb{Q}T = \bigoplus (\mathbb{Q}T)e_i = \bigoplus R_i,$$

where $e_i$ are primitive central idempotents of $\mathbb{Q}G$. Then $\mathbb{Q}G$ is the crossed product

$$(\ast) \quad \mathbb{Q}G = \mathbb{Q}T \ast F = \left( \bigoplus R_i \right) \ast F = \bigoplus R_i \ast F.$$

Decompose $u$ as a sum of elements in ($\ast$):

$$u = \bigoplus_i \left( \sum_{j=1}^n u_j f_j \right), \quad 0 \neq u_j \in R_i, f_j \in G, \text{ for each } j.$$

We assume that we have put together the $u_j$’s with the same $f_j T \in G/T$, namely for $k \neq j$, $f_k T \neq f_j T$.

We claim that $n = 1$. Let us denote by $- \pi$ the projection of $\mathbb{Q}G$ onto $R_i \ast F$. Then since $u$ is central we have $\mathbb{Q}T \pi = \pi \mathbb{Q}T$, which implies $\mathbb{Q}T u_j f_j = u_j \pi f_j \mathbb{Q}T$ for all $j$. It follows that $u_j$ is not a zero divisor provided $R_i$ has only one simple (artinian) component, and so $u_j$ is a unit. The only time $u_j$ can be a nonunit is when it has some zero components in the simple components of $R_i$. However, by the construction of $R_i$, these latter components can be moved to any other place by conjugating suitably. But they must stay put due to the facts that $F$ is ordered and $\pi$ is central. It follows that $u_j$ is a unit for all $j$. Hence, working in $R_i \ast F$ and using again that $F$ is ordered, it follows by a classical argument that $\pi = \sum_j u_j f_j$ is simply equal to $u_a f_n$ as claimed.

Changing notation, we write

$$u = \bigoplus_i \alpha f, \quad \alpha \in R_i, f \in G.$$
Let \( k = |\text{Aut}(T)| \), so \( f^k \) commutes with \( T \) for \( f \in G \). Hence
\[
    u^k = \bigoplus (\alpha f)^k = \bigoplus \beta f^k, \quad \beta \in R_t
\]
(note that the number of summands in \( u^k \) is the same as the number of summands in \( u \), because each \( \alpha \) is a unit in \( R_t \), and thus
\[
    u^k = (u^k)^t = \bigoplus (\beta f^k)^t = \bigoplus \beta f^k, \quad t \in T.
\]
The last step follows from the fact that conjugation will preserve the order on the \( fT \)'s in the ordered group \( F \). Since \((f^k)^t = tf^k\), we can choose \( k \) large enough so that all the \( f^k \) commute with each other and with \( T \). Thus we may assume that
\[
    u^k = \bigoplus \beta f^k.
\]
Again, we put together all \( \beta \) with the same \( f^kT \). In other words, we assume that \( u^k = \bigoplus \beta f^k \) with all \( f^kT \) different. Note that these new values of \( \beta \) all lie in \( ZT \).
Furthermore, we now obtain for each \( t \in T \),
\[
    u^k = (u^k)^t = \bigoplus (\beta f^k)^t = \bigoplus \beta f^k,
\]
and thus \( \beta^t = \beta \). So the ring \( R \) generated by all the \( \beta \) is commutative. Again, if necessary, replacing \( k \) by a high enough power, we may assume that the group \( A \) generated by all the \( f^k \) in the summation of \( u^k \) is a torsion-free Abelian group, and thus a free Abelian group. Consequently
\[
    u^k \in RA,
\]
the commutative group ring of \( A \) over \( R \). Let \( N = \text{Rad}(R) \) be the set of nilpotent elements of \( R \). Now \( ZT \) has only trivial idempotents \([6, \text{Theorem 2.20}, \text{page } 25]\). Hence since \( R \subseteq ZT \) and since idempotents of \( R/N \) can be lifted to \( R \), it follows that \( R/N \) also has only trivial idempotents. Therefore \([6, \text{Lemma 3.3}, \text{page } 55]\)
together with an inductive argument tells us that \((R/N)A\) has only trivial units. It follows that
\[
    u^k = \beta f^k + \text{nilpotent elements}.
\]
But as each \( \beta \) is a sum of units in various \( R_t \), it follows that the last term must be zero. Hence \( u^k = \beta f^k \), and thus all \( f \)'s in the original decomposition of \( u = \bigoplus \alpha f \) were in the same coset of \( T \). Thus \( u = rf \) as required. \( \square \)

We give two important consequences of the last result. We say that \( Z(\mathcal{U}(ZG)) \) is trivial if it contains only trivial units.

**Corollary 4.** Let \( G \) be a finitely generated nilpotent group. If \( Z(\mathcal{U}(ZT)) \) is trivial, then \( Z(\mathcal{U}(ZG)) \) is trivial.

**Proof.** Let \( u \in Z(\mathcal{U}(ZG)) \) be nontrivial. Then the support of \( u \) contains two different elements, say \( x \) and \( y \). Since finitely generated nilpotent groups are residually finite, there exists a finite factor \( G/N = \overline{G} \) so that \( \overline{x} \neq \overline{y} \) in \( \overline{G} \) (see \([5, \text{page } 149]\)). Hence \( \overline{\tau} \) has in its support at least two different elements, and thus \( \overline{\tau} \) is of infinite order \([7, \text{Corollary 1.7}, \text{page } 4]\)). By Proposition 3 we write \( u = rg, \ r \in ZT, \ g \in G \). Since \( u \) is central, \( r \) commutes with \( g \). It then follows easily that \( \overline{\tau} \), and hence also \( r \), is of infinite order. Moreover, there exists a positive integer \( n \) such that \( (g^n, T) = 1 \). Consequently it follows from \( u^n = r^n g^n \) that \( r^n \) commutes with \( T \). Thus \( r^n \) is a nontrivial unit of \( Z(\mathcal{U}(ZT)) \). \( \square \)
Corollary 5. Let $G$ be a finitely generated nilpotent group. Then $Z(U(ZG))$ is
finitely generated. Furthermore, $(Z(U(ZG)) \cap Z(U(ZT)))Z(G)$ is of finite index in
$Z(U(ZG))$.

Proof. Let $S = Z(U(ZG)) \cap Z(U(ZT))$. First we show that $Z(U(ZG)) / SZ(G)$
is a torsion group of bounded exponent. Indeed, let $u \in Z(U(ZG))$. Because of
Proposition 3 write $u = rg$, with $r \in U(ZT)$ and $g \in G$. Considering the natural
epimorphism $ZG \to Z(G/T)$ and using the fact that $Z(U(Z(G/T)))$ is trivial
because $G/T$ is ordered, it follows that $gT \in Z(G/T)$. Hence $(g^kT) = 1$ and
$g^k \in Z(G)$ for $k = [\text{Aut}(T)]$ and $l = k[T]$. Now since $u$ is central, $r$ and $g$
commute. Therefore
$$u' = r^lg^k$$
Consequently $u' \in SZ(G)$, and the claim follows.

As a subgroup of the finitely generated group $Z(U(ZT))$, the group $S$ is itself
finitely generated. Hence so is $SZ(G)$. Since the torsion subgroup of $Z(U(ZG))$
is finite (see for example [6, page 46]), the above claim now easily yields that
$Z(U(ZG))$ is indeed finitely generated.

We will now construct finitely many generators for the central units of any finitely
generated nilpotent group.

Let $n$ be the nilpotency class of $T$ and $h$ the Hirsch number of $G/T$. Let $k =
[\text{Aut}(T)]$. Further let $x_1, \ldots, x_h$ be elements of $G$ such that for each
$1 \leq i \leq h$ the group $G_i = (T, x_1, \ldots, x_i)$ is normal in $G$ and $G_i / G_{i-1} \cong Z$, where $G_0 = T$. For
any generator $b(u)$ described in Proposition 2 define
$$b^{(0)}(u) = b(u)$$
and for $1 \leq i \leq h$
$$b^{(i)}(u) = \prod_{0 \leq j < k} \left( b^{(i-1)}(b(u)) \right)^{x_j}. $$
Since each $b(u)$ is in $Z(U(ZT))$, the order of the conjugates in the product expression
for $b^{(i)}(u)$ is unimportant. It follows by induction that $b^{(i)}(u)$ is in $Z(U(ZG_i))$. In
particular, $b^{(h)}(u) \in Z(U(ZG))$.

Theorem 6. Let $G$ be a finitely generated nilpotent group. Suppose $n$ is the nilpo-
tency class of $T$ and $h$ is the Hirsch number of $G/T$. Then
$$\langle b^{(h)}(u) | b \text{ a Bass cyclic of } ZT \rangle Z(G)$$
is of finite index in $Z(U(ZG))$.

Proof. Because of Corollary 5 the group $SZ(G)$ with $S = Z(U(ZG)) \cap Z(U(ZT))$
is of finite index in $Z(U(ZG))$. Let $\alpha_1, \ldots, \alpha_p$ be a finite set of generators for $S$.
By Proposition 2 there exists a positive integer $m$ such that all $\alpha_1^m, \ldots, \alpha_p^m$ are in
$\langle b(u) | b \text{ a Bass cyclic in } ZT \rangle$. For simplicity, write $\alpha = \alpha_1^m$. Then
$$\alpha = \prod b(u),$$
where the product runs over a finite number of Bass cyclic units of $ZT$. Since $\alpha$ is
in $Z(U(ZG))$, and using the notation introduced above, we obtain
$$\alpha^k = \alpha \alpha^{x_1} \cdots \alpha^{x_{h-1}}.
As each $b(n)$ is central in $\mathbb{Z}T$, this implies
$$\alpha_k^i = \prod_{b(n)}^i.$$
Continuing this process one obtains that
$$\alpha_k h = \prod_{b(n)}^i.$$
Since the group generated by $\alpha_{mk} h, \ldots, \alpha_{mp} h$ is of finite index in $S$, the result follows.

Note that Corollary 4 can now also be obtained as an easy consequence of Theorem 6.

We now give an example showing that the converse of Corollary 4 does not hold.

**Example.** Let $G = \langle a, x \mid a^8 = a^3 = 1 \rangle$. Clearly $G$ is a nilpotent group with $T = \langle a \rangle$, a cyclic group of order 8. From Higman’s result (see [6]) it follows that $Z(U(\mathbb{Z}G))$, modulo the trivial units, is a free Abelian group of rank 1. We now show that $Z(U(\mathbb{Z}G))$ contains only trivial units. For this suppose $u$ is a nontrivial central unit in $\mathbb{Z}G$. By Proposition 3, we can write $u = rx^i$ for some integer $i$ and $r \in U(\mathbb{Z}T)$. We know from the above that $r$ is of infinite order, and since $r$ commutes with $x$, it must be in $Z(U(\mathbb{Z}G))$.

Because the only Bass cyclic unit, up to inverses, in $\mathbb{Z}T$ is
$$b = (1 + a + a^2)^4 - 10\hat{a}, \quad \hat{a} = 1 + a + \cdots + a^7,$$
Proposition 2 yields that
$$r^k = b^l,$$
for some nonzero integers $k, l$. Observe, however, that $b^5 = b^{-1}$. Since $b^l = r^k$ is central in $\mathbb{Z}G$, we obtain $b^l = b^{-1}$, contradicting the fact that $b$ is of infinite order.

**References**


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