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CENTRAL UNITS OF INTEGRAL GROUP RINGS OF NILPOTENT GROUPS

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Abstract. In this paper a finite set of generators is given for a subgroup of finite index in the group of central units of the integral group ring of a finitely generated nilpotent group.

In this paper we construct explicitly a finite set of generators for a subgroup of finite index in the centre $Z(\mathbf{U}(\mathbf{Z}G))$ of the unit group $\mathcal{U}(\mathbf{Z}G)$ of the integral group ring $\mathbf{Z}G$ of a finitely generated nilpotent group G. Ritter and Sehgal [4] did the same for finite groups G, giving generators which are a little more complicated. They also gave in [2] necessary and sufficient conditions for $Z(\mathcal{U}(\mathbf{Z}G))$ to be trivial; recall that the units $\pm G$ are called the trivial units. We first give a finite set of generators for a subgroup of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$ when G is a finite nilpotent group. Next we consider an arbitrary finitely generated nilpotent group and prove that a central unit of $\mathbf{Z}G$ is a product of a trivial unit and a unit of $\mathbf{Z}T$, where T is the torsion subgroup of G. As an application we obtain that the central units of $\mathbf{Z}G$ form a finitely generated group and we are able to give an explicit set of generators for a subgroup of finite index.

1. Finite nilpotent groups

Throughout this section G is a finite group. When G is Abelian, it was shown in [1] that the Bass cyclic units generate a subgroup of finite index in the unit group. Using a stronger version of this result, also proved by Bass in [1], we will construct a finite set of generators from the Bass cyclic units when G is finite nilpotent.

Our notation will follow that in [6]. The following lemma is proved in [1].

Lemma 1. *The images of the Bass cyclic units of* $\mathbf{Z}G$ *under the natural homomorphism* $j : \mathcal{U}(\mathbf{Z}G) \to K_1(\mathbf{Z}G)$ *generate a subgroup of finite index.*

Let L denote the kernel of this map j, and B the subgroup of $\mathcal{U}(\mathbf{Z}G)$ generated by the Bass cyclic units. It follows that there exists an integer m such that $z^m \in LB$ for all $z \in Z(\mathcal{U}(\mathbf{Z}G))$, and so we can write $z^m = lb_1b_2\cdots b_k$ for some $l \in L$ and Bass cyclic units b_i.

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Next, let Z_i denote the i-th centre of G, and suppose from now on that G is nilpotent of class n. For any $x \in G$ and Bass cyclic unit $b \in \mathbf{Z}\langle x\rangle$, we define

$$b_{(1)} = b$$

and for $2 \le i \le n$

$$b_{(i)} = \prod_{g \in Z_i} b^g_{(i-1)},$$

where $\alpha^g = g^{-1}\alpha g$ for $\alpha \in \mathbf{Z}G$. Note that by induction $b_{(i)}$ is central in $\mathbf{Z}\langle Z_i, x\rangle$ and independent of the order of the conjugates in the product expression. In particular, $b_{(n)} \in Z(\mathcal{U}(\mathbf{Z}G))$.

Recall again that if $z \in Z(\mathcal{U}(\mathbf{Z}G))$, then $z^m = lb_1b_2\cdots b_k$ for some $l \in L$ and Bass cyclic units b_i. Since $K_1(\mathbf{Z}G)$ is Abelian, we can write

$$\begin{aligned} z^{m|Z_2||Z_3|\cdots|Z_n|} &= (lb_1b_2\cdots b_k)^{|Z_2||Z_3|\cdots|Z_n|} \\ &= l_1 \textstyle\prod_{1\le i\le k} b_i^{|Z_2||Z_3|\cdots|Z_n|} && \text{for some } l_1 \in L \\ &= l_2 \textstyle\prod_{1\le i\le k} b_{i(2)}^{|Z_3|\cdots|Z_n|} && \text{for some } l_2 \in L \\ &= l' \textstyle\prod_{1\le i\le k} b_{i(n)} && \text{for some } l' \in L. \end{aligned}$$

Since each $b_{i(n)}$ is in $Z(\mathcal{U}(\mathbf{Z}G))$, we conclude that $l' \in L \cap Z(\mathcal{U}(\mathbf{Z}G))$. But we shall show next that $L \cap Z(\mathcal{U}(\mathbf{Z}G))$ is trivial, so $l' \in \pm Z(G)$. The argument uses the same idea as in [3, Lemma 3.2].

For every primitive central idempotent e in the rational group algebra $\mathbf{Q}G$, the simple ring $\mathbf{Q}Ge$ has a reduced norm which we denote by nr_e. Further, denote

$$m_e = \sqrt{[\mathbf{Q}Ge : Z(\mathbf{Q}Ge)]}$$

and let

$$r = \prod_e m_e.$$

Now let $l' \in L \cap Z(\mathcal{U}(\mathbf{Z}G))$. By definition of $K_1(\mathbf{Z}G)$ this means that a suitable matrix

$$\begin{bmatrix} l' & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdots & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

is a product of commutators. Therefore $l'e$ has reduced norm one. Since $l'e$ is also central, we obtain

$$(l'e)^{m_e} = nr(l'e)e = e.$$

Hence

$$l'^r = 1.$$

So l' is a torsion central unit, and therefore is trivial [7, Corollary 1.7, page 4].

Since $Z(\mathcal{U}(\mathbf{Z}G))$ is finitely generated (see, e.g., [2]), $(Z(\mathcal{U}(\mathbf{Z}G)))^{m|Z_2||Z_3|\cdots|Z_n|}$ is of finite index. But we have just seen that the latter subgroup is contained in the subgroup generated by $\pm Z(G)$ and $\{b_{(n)} \mid b \text{ a Bass cyclic unit}\}$. We have proved

Proposition 2. *Let G be a finite nilpotent group of class n. Then*

$$\langle b_{(n)} \mid b \text{ a Bass cyclic}\rangle$$

is of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$.

Remark. Note that our method for constructing generators for a subgroup of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$ can be adapted for some other classes of finite groups G. For example if $G = D_{2n} = \langle a, b \mid x^n = 1,\ y^2 = 1,\ yx = x^{n-1}y\rangle$, the dihedral group of order $2n$, then the only nontrivial Bass cyclic units b of $\mathbf{Z}D_{2n}$ belong to $\mathbf{Z}\langle x\rangle$. It follows that $bb^y = b^y b$ is central. Our proof now remains valid and yields that $\langle bb^y \mid b \text{ a Bass cyclic in } \mathbf{Z}\langle x\rangle\rangle$ is of finite index in $Z(\mathcal{U}(\mathbf{Z}D_{2n}))$.

2. Finitely generated nilpotent groups

We will now consider central units of an integral group ring of an arbitrary finitely generated nilpotent group G. The torsion subgroup of G is denoted T. First we show that central units of $\mathbf{Z}G$ have the following decomposition.

Proposition 3. *Let G be a finitely generated nilpotent group. Every $u \in Z(\mathcal{U}(\mathbf{Z}G))$ can be written as $u = rg$, $r \in \mathbf{Z}T$, $g \in G$.*

Proof. Let $F = G/T$. Since T is finite and F acts on the set of primitive central idempotents of $\mathbf{Q}T$ by conjugation, by adding the idempotents in an orbit of this action we obtain

$$\mathbf{Q}T = \bigoplus(\mathbf{Q}T)e_i = \bigoplus R_i,$$

where e_i are primitive central idempotents of $\mathbf{Q}G$. Then $\mathbf{Q}G$ is the crossed product

$$(*) \qquad \mathbf{Q}G = \mathbf{Q}T * F = \Big(\bigoplus R_i\Big) * F = \bigoplus R_i * F.$$

Decompose u as a sum of elements in $(*)$:

$$u = \bigoplus_i \left(\sum_{j=1}^{n} u_j f_j\right), \quad 0 \neq u_j \in R_i,\ f_j \in G, \quad \text{for each } j.$$

We assume that we have put together the u_j's with the same $f_jT \in G/T$, namely for $k \neq j$, $f_kT \neq f_jT$.

We claim that $n = 1$. Let us denote by $-$ the projection of $\mathbf{Q}G$ onto $R_i * F$. Then since u is central we have $\overline{\mathbf{Q}T}\,\overline{u} = \overline{u}\,\overline{\mathbf{Q}T}$, which implies $\overline{\mathbf{Q}T}u_jf_j = u_jf_j\overline{\mathbf{Q}T}$ for all j. It follows that u_j is not a zero divisor provided R_i has only one simple (artinian) component, and so u_j is a unit. The only time u_j can be a nonunit is when it has some zero components in the simple components of R_i. However, by the construction of R_i, these latter components can be moved to any other place by conjugating suitably. But they must stay put due to the facts that F is ordered and $\overline{u}$ is central. It follows that u_j is a unit for all j. Hence, working in $R_i * F$ and using again that F is ordered, it follows by a classical argument that $\overline{u} = \sum_j u_jf_j$ is simply equal to u_nf_n as claimed.

Changing notation, we write

$$u = \bigoplus_i \alpha f, \quad \alpha \in R_i,\ f \in G.$$

Let $k = |Aut(T)|$, so f^k commutes with T for $f \in G$. Hence

$$u^k = \bigoplus (\alpha f)^k = \bigoplus \beta f^k, \quad \beta \in R_i$$

(note that the number of summands in u^k is the same as the number of summands in u, because each α is a unit in R_i), and thus

$$u^k = \left(u^k\right)^{f_1^k} = \bigoplus \left(\beta f^k\right)^{f_1^k} = \bigoplus \beta t f^k, \quad t \in T.$$

The last step follows from the fact that conjugation will preserve the order on the fT's in the ordered group F. Since $\left(f^k\right)^{f_1^k} = tf^k$, we can choose k large enough so that all the f^k commute with each other and with T. Thus we may assume that

$$u^k = \bigoplus \beta f^k.$$

Again, we put together all β with the same f^kT. In other words, we assume that $u^k = \bigoplus \beta f^k$ with all f^kT different. Note that these new values of β all lie in $\mathbf{Z}T$. Furthermore, we now obtain for each $t \in T$,

$$u^k = \left(u^k\right)^t = \bigoplus (\beta)^t f^k,$$

and thus $\beta^t = \beta$. So the ring R generated by all the β is commutative. Again, if necessary, replacing k by a high enough power, we may assume that the group A generated by all the f^k in the summation of u^k is a torsion-free Abelian group, and thus a free Abelian group. Consequently

$$u^k \in RA,$$

the commutative group ring of A over R. Let $N = Rad(R)$ be the set of nilpotent elements of R. Now $\mathbf{Z}T$ has only trivial idempotents [6, Theorem 2.20, page 25]. Hence since $R \subseteq \mathbf{Z}T$ and since idempotents of R/N can be lifted to R, it follows that R/N also has only trivial idempotents. Therefore [6, Lemma 3.3, page 55] together with an inductive argument tells us that $(R/N)A$ has only trivial units. It follows that

$$u^k = \beta f^k + \text{ nilpotent elements.}$$

But as each β is a sum of units in various R_i, it follows that the last term must be zero. Hence $u^k = \beta f^k$, and thus all f's in the original decomposition of $u = \bigoplus_i \alpha f$ were in the same coset of T. Thus $u = rf$ as required. □

We give two important consequences of the last result. We say that $Z(\mathcal{U}(\mathbf{Z}G))$ is trivial if it contains only trivial units.

Corollary 4. *Let G be a finitely generated nilpotent group. If $Z(\mathcal{U}(\mathbf{Z}T))$ is trivial, then $Z(\mathcal{U}(\mathbf{Z}G))$ is trivial.*

Proof. Let $u \in Z(\mathcal{U}(\mathbf{Z}G))$ be nontrivial. Then the support of u contains two different elements, say x and y. Since finitely generated nilpotent groups are residually finite, there exists a finite factor $G/N = \overline{G}$ so that $\overline{x} \neq \overline{y}$ in $\overline{G}$ (see [5, page 149]). Hence $\overline{u}$ has in its support at least two different elements, and thus $\overline{u}$ is of infinite order ([7, Corollary 1.7, page 4]). By Proposition 3 we write $u = rg$, $r \in \mathbf{Z}T$, $g \in G$. Since u is central, r commutes with g. It then follows easily that $\overline{r}$, and hence also r, is of infinite order. Moreover, there exists a positive integer n such that $(g^n, T) = 1$. Consequently it follows from $u^n = r^n g^n$ that r^n commutes with T. Thus r^n is a nontrivial unit of $Z(\mathcal{U}(\mathbf{Z}T))$. □

Corollary 5. *Let G be a finitely generated nilpotent group. Then $Z(\mathcal{U}(\mathbf{Z}G))$ is finitely generated. Furthermore, $(Z(\mathcal{U}(\mathbf{Z}G)) \cap Z(\mathcal{U}(\mathbf{Z}T)))\, Z(G)$ is of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$.*

Proof. Let $S = Z(\mathcal{U}(\mathbf{Z}G)) \cap Z(\mathcal{U}(\mathbf{Z}T))$. First we show that $Z(\mathcal{U}(\mathbf{Z}G))/SZ(G)$ is a torsion group of bounded exponent. Indeed, let $u \in Z(\mathcal{U}(\mathbf{Z}G))$. Because of Proposition 3 write $u = rg$, with $r \in \mathcal{U}(\mathbf{Z}T)$ and $g \in G$. Considering the natural epimorphism $\mathbf{Z}G \to \mathbf{Z}(G/T)$ and using the fact that $Z(\mathcal{U}(\mathbf{Z}(G/T)))$ is trivial because G/T is ordered, it follows that $gT \in Z(G/T)$. Hence $(g^k, T) = 1$ and $g^l \in Z(G)$ for $k = |Aut(T)|$ and $l = k|T|$. Now since u is central, r and g commute. Therefore

$$u^l = r^l g^l \text{ and } r^l \in S.$$

Consequently $u^l \in SZ(G)$, and the claim follows.

As a subgroup of the finitely generated group $Z(\mathcal{U}(\mathbf{Z}T))$, the group S is itself finitely generated. Hence so is $SZ(G)$. Since the torsion subgroup of $Z(\mathcal{U}(\mathbf{Z}G))$ is finite (see for example [6, page 46]), the above claim now easily yields that $Z(\mathcal{U}(\mathbf{Z}G))$ is indeed finitely generated. □

We will now construct finitely many generators for the central units of any finitely generated nilpotent group.

Let n be the nilpotency class of T and h the Hirsch number of G/T. Let $k = |Aut(T)|$. Further let $x_1, \cdots, x_h$ be elements of G such that for each $1 \leq i \leq h$ the group $G_i = \langle T, x_1, \cdots, x_i \rangle$ is normal in G and $G_i/G_{i-1} \cong \mathbf{Z}$, where $G_0 = T$. For any generator $b_{(n)}$ described in Proposition 2 define

$$b_{(n)}^{(0)} = b_{(n)}$$

and for $1 \leq i \leq h$

$$b_{(n)}^{(i)} = \prod_{0 \leq j < k} \left(b_{(n)}^{(i-1)}\right)^{x_i^j}.$$

Since each $b_{(n)}$ is in $Z(\mathcal{U}(\mathbf{Z}T))$, the order of the conjugates in the product expression for $b_{(n)}^{(i)}$ is unimportant. It follows by induction that $b_{(n)}^{(i)}$ is in $Z(\mathcal{U}(\mathbf{Z}G_i))$. In particular, $b_{(n)}^{(h)} \in Z(\mathcal{U}(\mathbf{Z}G))$.

Theorem 6. *Let G be a finitely generated nilpotent group. Suppose n is the nilpotency class of T and h is the Hirsch number of G/T. Then*

$$\langle b_{(n)}^{(h)} \mid b \text{ a Bass cyclic of } \mathbf{Z}T \rangle\, Z(G)$$

is of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$.

Proof. Because of Corollary 5 the group $SZ(G)$ with $S = Z(\mathcal{U}(\mathbf{Z}G)) \cap Z(\mathcal{U}(\mathbf{Z}T))$ is of finite index in $Z(\mathcal{U}(\mathbf{Z}G))$. Let $\alpha_1, \cdots, \alpha_p$ be a finite set of generators for S. By Proposition 2 there exists a positive integer m such that all $\alpha_1^m, \cdots, \alpha_p^m$ are in $\langle b_{(n)} \mid b \textit{ a Bass cyclic in } \mathbf{Z}T \rangle$. For simplicity, write $\alpha = \alpha_1^m$. Then

$$\alpha = \prod b_{(n)},$$

where the product runs over a finite number of Bass cyclic units of $\mathbf{Z}T$. Since α is in $Z(\mathcal{U}(\mathbf{Z}G))$, and using the notation introduced above, we obtain

$$\alpha^k = \alpha \alpha^{x_1} \cdots \alpha^{x_1^{k-1}}.$$

As each $b_{(n)}$ is central in $\mathbf{Z}T$, this implies

$$\alpha^k = \prod b_{(n)}^{(1)}.$$

Continuing this process one obtains that

$$\alpha^{k^h} = \prod b_{(n)}^{(h)}.$$

Since the group generated by $\alpha_1^{mk^h}, \cdots, \alpha_p^{mk^h}$ is of finite index in S, the result follows. $\square$

Note that Corollary 4 can now also be obtained as an easy consequence of Theorem 6.

We now give an example showing that the converse of Corollary 4 does not hold.

Example. Let $G = \langle a, x \mid a^x = a^3,\ a^8 = 1\rangle$. Clearly G is a nilpotent group with $T = \langle a\rangle$, a cyclic group of order 8. From Higman's result (see [6]) it follows that $Z(\mathcal{U}(\mathbf{Z}T))$, modulo the trivial units, is a free Abelian group of rank 1. We now show that $Z(\mathcal{U}(\mathbf{Z}G))$ contains only trivial units. For this suppose u is a nontrivial central unit in $\mathbf{Z}G$. By Proposition 3, we can write $u = rx^i$ for some integer i and $r \in \mathcal{U}(\mathbf{Z}T)$). We know from the above that r is of infinite order, and since r commutes with x, it must be in $Z(\mathcal{U}(\mathbf{Z}G))$.

Because the only Bass cyclic unit, up to inverses, in $\mathbf{Z}T$ is

$$b = (1 + a + a^2)^4 - 10\hat{a}, \quad \hat{a} = 1 + a + \cdots + a^7,$$

Proposition 2 yields that

$$r^k = b^l,$$

for some nonzero integers k, l. Observe, however, that $b^x = b^{-1}$. Since $b^l = r^k$ is central in $\mathbf{Z}G$, we obtain $b^l = b^{-l}$, contradicting the fact that b is of infinite order.

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