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TWO DEFINITIONS OF EXPONENTIAL DICHOTOMY FOR SKEW-PRODUCT SEMIFLOW IN BANACH SPACES

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ABSTRACT. In this paper we introduce a concept of exponential dichotomy for linear skew-product semiflows (LSPS) in infinite dimensional Banach spaces, which is an extension of the classical concept of exponential dichotomy for time dependent linear differential equations in Banach spaces. We prove that the concept of exponential dichotomy used by Sacker-Sell and Magalhães in recent years is stronger than this one, but they are equivalent under suitable conditions. Using this concept we where able to find a formula for all the bounded negative continuations. After that, we characterize the stable and unstable subbundles in terms of the boundedness of the corresponding projector along (forward/backward) the LSPS and in terms of the exponential decay of the semiflow. The linear theory presented here provides a foundation for studying the nonlinear theory. Also, this concept can be used to study the existence of exponential dichotomy and the roughness property for LSPS.

1. INTRODUCTION

The concept of exponential dichotomy of linear differential equations was introduced by Perron [14], which is concerned with the problem of conditional stability of a system $\dot{x} = A(t)x$ and its connection with the existence of bounded solutions of the equation $\dot{x} = A(t)x + f(x,t)$, where the state space is a Banach space X and $t \to A(t) : \mathbb{R} \to L(X)$ is bounded, continuous in the strong operator topology. An important contribution to these problems is the work done by Massera-Schäffer [12], Daleckii-Krein [5], Levinson [8], Coppel [4], Sacker-Sell [15] and Palmer [13].

The need for a new approach arose from the fact that for a time dependent linear differential equation with unbounded operator $A(t)$, the solutions, generally speaking, either cannot be extended in the direction of the negative times, or can be extended, but not uniquely. For example, for parabolic partial differential equations many authors have studied these problems, including Henry [7], Xiao-Biao Lin [10] and J. Hale [6]. For the case of functional differential equations we can see the work done by X.B. Lin [9].

All the problems above can be treated in the unified setting of a linear skew-product semiflow (LSPS). In [16] Sacker-Sell use a concept of exponential dichotomy for skew-product semiflow with the restriction that the unstable subspace has finite

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dimension, and they give a sufficient condition for the existence of exponential dichotomy for skew-product semiflow. This concept is also used by Magalhães in [11]. In this work we introduce a concept of exponential dichotomy for skew-product semiflow weaker than the concept used by Sacker-Sell and Magalhães; here we allow the unstable subspace to have infinite dimension. We prove that the concept of exponential dichotomy used by Sacker-Sell and Magalhães implies this one, and that they are equivalent, if we suppose that the unstable subspace has finite dimension (or infinite dimension) in both definitions. Using this concept, we will find a formula for all the bounded negative continuations. After that, we will characterize the stable and unstable subbundles in terms of the boundedness of the corresponding projector along (forward/backward) the LSPS and in terms of the exponential decay of the semiflow. The linear theory presented here provides a foundation for studying the nonlinear theory. Also this concept can be used to study the existence of exponential dichotomy and the roughness property for LSPS.

2. Preliminaries

In this section we shall present some definitions, notations and results about skew-product semiflow in infinite dimensional Banach spaces.

2.1. Linear skew–product semiflow.

We begin with the notion of skew-product semiflow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$, where X is a fixed a Banach space (the state space) and Θ is a compact Hausdorff space.

Definition 2.1. Suppose that $\sigma(\theta, t) = \theta \cdot t$ is a flow on Θ , i.e., the mapping $(\theta, t) \to \theta \cdot t$ is continuous, $\theta \cdot 0 = \theta$ and $\theta \cdot (s + t) = (\theta \cdot s) \cdot t$, for all $s, t \in \mathbb{R}$. A **linear skew-product semiflow** $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is a mapping $\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t)$ for $t \geq 0$, with the following properties:
(1) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$.
(2) $\lim_{t\to 0^+} \Phi(\theta, t)x = x$, uniformly in θ . This means that for every $x \in X$ and every $\epsilon > 0$ there is a $\delta = \delta(x, \epsilon) > 0$ such that $\|\Phi(\theta, t)x - x\| \leq \epsilon$, for all $\theta \in \Theta$ and $0 \leq t \leq \delta$.
(3) $\Phi(\theta, t)$ is a bounded linear operator from X into X that satisfies the cocycle identity:

$$\Phi(\theta, t + s) = \Phi(\theta \cdot t, s)\Phi(\theta, t), \qquad \theta \in \Theta, \quad 0 \leq s, t. \tag{2.1}$$

(4) For all $t \geq 0$ the mapping from $\mathcal{E}$ into X given by

$$(x, \theta) \to \Phi(\theta, t)x$$

is continuous.

The properties (2) and (3) imply that for each $(x, \theta) \in \mathcal{E}$ the solution operator $t \to \Phi(\theta, t)x$ is right continuous for $t \geq 0$. In fact :

$$\|\Phi(\theta, t + h)x - \Phi(\theta, t)x\| = \|[\Phi(\theta \cdot t, h) - I]\Phi(\theta, t)x\|,$$

which goes to 0 as h goes to 0^+ .

Since $\mathcal{E} = X \times \Theta$ is a trivial Banach bundle, then for any subset $\mathcal{F} \subset \mathcal{E}$ we define the fiber

$$\mathcal{F}(\theta) := \{x \in X : (x, \theta) \in \mathcal{F}\}, \quad \theta \in \Theta. \tag{2.2}$$

So $\mathcal{E}(\theta) = X, \quad \theta \in \Theta$.

2.2. **Projectors and subbundles.** A mapping $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is said to be a projector if $\mathbf{P}$ is continuous and has the form $\mathbf{P}(x, \theta) = (P(\theta)x, \theta)$, where $P(\theta)$ is a bounded linear projection on the fiber $\mathcal{E}(\theta)$.

For any projector $\mathbf{P}$ we define the range and null space by

$$\mathcal{R} = \mathcal{R}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = x\},$$

$$\mathcal{N} = \mathcal{N}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = 0\}$$

The continuity of $\mathbf{P}$ implies that the fibers $\mathcal{R}(\theta)$ and $\mathcal{N}(\theta)$ vary continuously in θ . This also means that $P(\theta)$ is strongly continuous as a function of θ. The following result can be found in Sacker-Sell [16].

Lemma 2.1. *Let $\mathbf{P}$ be a projector on $\mathcal{E}$. Then $\mathcal{R}$ and $\mathcal{N}$ are closed subsets in $\mathcal{E}$ and we have*

$$\mathcal{R}(\theta) \cap \mathcal{N}(\theta) = \{0\}, \quad \mathcal{R}(\theta) + \mathcal{N}(\theta) = \mathcal{E}(\theta) \quad \textit{for all} \quad \theta \in \Theta.$$

Definition 2.2. A subset $\mathcal{V}$ is said to be a subbundle of $\mathcal{E}$, if there is a projector $\mathbf{P}$ on $\mathcal{E}$ with the property that $\mathcal{R}(\mathbf{P}) = \mathcal{V}$; in this case $\mathcal{W} = \mathcal{N}(\mathbf{P})$ is a complementary subbundle , i.e., $\mathcal{E} = \mathcal{V} + \mathcal{W}$ as a Whitney sum of subbundles.

2.3. **The stable, unstable and the initial bounded sets.**

Definition 2.3. A point $(x, \theta) \in \mathcal{E}$ is said to have a negative continuation with respect to π if there exists a continuous function $\phi = \phi(x, \theta)$, $\phi : (-\infty, 0] \to \mathcal{E}$, satisfying the following properties:
(1) $\phi(t) = (\phi^x(t), \theta \cdot t)$ where $\phi^x : (-\infty, 0] \to X$,
(2) $\phi(0) = (x, \theta)$,
(3) $\pi(\phi(s), t) = \phi(s + t)$ for each $s \le 0$ and $0 \le t \le -s$,
(4) $\pi(\phi(s), t) = \pi(x, \theta, t + s)$, for each $0 \le -s \le t$.

In this case the function ϕ is said to be a **negative continuation of the point** (x, θ).

Now we shall define the following sets:
$\mathcal{M} := \{(x, \theta) \in \mathcal{E} : (x, \theta)$ has a negative continuation $\phi\}$,
$\mathcal{X}_u := \{(x, \theta) \in \mathcal{M} :$ there is a negative continuation ϕ of (x, θ) satisfying $\|\phi^x(t)\| \to 0$ as $t \to -\infty\}$,
$\mathcal{B}^+ := \{(x, \theta) \in \mathcal{E} : \sup_{t \ge 0} \|\Phi(\theta, t)x\| < \infty\}$,
$\mathcal{B}_u^- := \{(x, \theta) \in \mathcal{M} : (x, \theta)$ has a unique bounded negative continuation $\phi\}$,
$\mathcal{B}^- := \{(x, \theta) \in \mathcal{M} :$ there is a bounded negative continuation ϕ of $(x, \theta)\}$,
$\mathcal{X}_s := \{(x, \theta) \in \mathcal{E} : \|\Phi(\theta, t)x\| \to 0$ as $t \to \infty\}$,
$\mathcal{B} := \mathcal{B}^+ \cap \mathcal{B}^-$.
The set $\mathcal{X}_u$ is called the **unstable set**, $\mathcal{X}_s$ is the **stable set** and $\mathcal{B}$ is the **initial bounded set**.

Definition 2.4. For $\theta \in \Theta$ we shall call the fibers $\mathcal{X}_s(\theta)$ and $\mathcal{X}_u(\theta)$ the stable and unstable linear space of $\pi = (\Phi, \sigma)$ respectively.

Proposition 2.1. *Let ϕ and ψ be negative continuations of (x, θ) and (y, θ) respectively. Then*
(a) $h(t) = (h^{x \pm y}(t), \theta \cdot t) = (\phi^x(t) \pm \psi^y(t), \theta \cdot t)$, $t \le 0$, is a negative continuation of $(x \pm y, \theta)$.
(b) For all $\lambda \in \mathbb{R}$, $h_\lambda(t) = (\lambda \phi^x(t), \theta \cdot t)$, $t \le 0$, is a negative continuation of $(\lambda x, \theta)$.

Proof. It follows directly from the Definition 2.2. □

Proposition 2.2. *If* $\mathcal{B}_u^- \neq \emptyset$ *then* $\mathcal{B}_u^- = \mathcal{B}^-$.

Proof. Clearly $\mathcal{B}_u^- \subset \mathcal{B}^-$. It easy to see that $0 \in \mathcal{B}_u^-(\theta)$ for all $\theta \in \Theta$. Now, suppose that $\phi(t) = (\phi^x(t), \theta \cdot t)$ and $\psi(t) = (\psi^x(t), \theta \cdot t)$ are two bounded negative continuations of the point $(x, \theta) \in \mathcal{B}^-$. Then $h(t) = (\phi^x(t) - \psi^x(t), \theta \cdot t),\ \ t \leq 0$, is a bounded negative continuation of $(0, \theta)$. Therefore, $\phi^x(t) = \psi^x(t),\ \ t \leq 0 \iff \phi = \psi$. This means that each point of $\mathcal{B}^-$ has only one bounded negative continuation. Hence, $\mathcal{B}^- \subset \mathcal{B}_u^-$. □

3. Exponential dichotomy for linear skew-product semiflow

Now we shall introduce two concepts of exponential dichotomy for skew-product semiflow in infinite dimensional Banach spaces. The first one is used by Sacker and Sell in [16] and by Magalhães in [11]. The second one is an extension of the concept of exponential dichotomy for evolution operator given in Henry [7].

Definition 3.1. A projector **P** on $\mathcal{E}$ is say to be **invariant** if it satisfies the following property:

$$P(\theta \cdot t)\Phi(\theta, t) = \Phi(\theta, t)P(\theta), \quad t \geq 0, \ \ \theta \in \Theta, \tag{3.1}$$

i.e.,

$$\mathbf{P} \circ \pi(\cdot, t) = \pi(\cdot, t) \circ \mathbf{P}, \ \ t \geq 0.$$

Proposition 3.1. *(a) For all* $\theta \in \Theta$, $\mathcal{B}_u^-(\theta)$ *is a linear subspace of* X.

(b) For all invariant projectors $\boldsymbol{P}$ *and* $(x, \theta) \in \mathcal{B}_u^-$ *with the corresponding negative bounded continuation* $\phi(t) = (\phi^x(t), \theta \cdot t)$*, if for* $t \leq 0$ *we define* $\Phi(\theta, t)x := \phi^x(t)$*, then we have that:* $\Phi(\theta, t)$ *is linear mapping from* $\mathcal{B}_u^-(\theta)$ *to* $\mathcal{B}_u^-(\theta \cdot t)$ *and*

$$\Phi(\theta, t + s)x = \Phi(\theta \cdot t, s)\Phi(\theta, t)x, \quad s, t \in \mathbb{R}, \tag{3.2}$$

$$P(\theta \cdot t)\Phi(\theta, t)x = \Phi(\theta, t)P(\theta)x, \quad t \in \mathbb{R}. \tag{3.3}$$

Definition 3.2 (Sacker-Sell)**.** We shall say that a linear skew-product semiflow π on $\mathcal{E}$ has an **exponential dichotomy over** Θ , if $dimRange(I - P(\theta)) < \infty$ and $Range(I - P(\theta)) \subset \mathcal{B}_u^-(\theta)$ for each $\theta \in \Theta$, and there are constants $k \geq 1, \ \ \beta > 0$ such that the following inequalities hold :

$$\|\Phi(\theta, t)P(\theta)\| \leq ke^{-\beta t}, \quad t \geq 0, \ \ \theta \in \Theta,$$

$$\|\Phi(\theta, t)(I - P(\theta)\| \leq ke^{\beta t}, \quad t \leq 0, \ \ \theta \in \Theta.$$

Remark 3.1. It is easy to see that if $\Phi(\theta, t)$ is one-to-one for all $t > 0$, then every negative continuation is unique. Uniqueness of negative continuations is a common feature in the study of partial differential equations, see, for example, Hale [6].

The following definition of exponential dichotomy for a skew-product semiflow is weaker than Definition 3.2. Basically, the unstable subspace is not required to be finite dimensional. But, they are equivalent if the unstable subspace is finite in both definitions (or if the unstable subspace is infinite in both definitions). Both definitions do allow for the possibility that the linear operator $\Phi(\theta, t)$ need not be one-to-one for some $t > 0$, i.e., $\Phi(\theta, t)$ may has a nontrivial null space. Because of this, it maybe possible for a point $(x, \theta) \in \mathcal{E}$ to have more than one negative continuation.

Definition 3.3. We shall say that a linear skew-product semiflow π on $\mathcal{E}$ has an **exponential dichotomy over** Θ , if there are constants $k \geq 1$, $\beta > 0$ and invariant projector $\mathbf{P}$ such that for all $\theta \in \Theta$ we have the following:
(1) $\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot t))$, $t \geq 0$, is an isomorphism with inverse:

$$\Phi(\theta \cdot t, -t) : \mathcal{N}(P(\theta \cdot t)) \to \mathcal{N}(P(\theta)), \quad t \geq 0.$$

(2) $\|\Phi(\theta, t)P(\theta)\| \leq ke^{-\beta t}$, $t \geq 0$.
(3) $\|\Phi(\theta, t)(I - P(\theta)\| \leq ke^{\beta t}$, $t \leq 0$.
From $\mathcal{N}(P(\theta)) = \mathcal{R}(I - P(\theta))$ and the Open Mapping Theorem we have that $\Phi(\theta, t)(I - P(\theta))$ is well defined and is a linear bounded operator for $t \leq 0$.

Proposition 3.2. *Definition 3.2 (Sacker-Sell) implies Definition 3.3.*

Proof. We only have to prove that

$$\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot t)), \quad t \geq 0,$$

is an isomorphism. In fact, since

$$Range(I - P(\theta)) = \mathcal{N}(P(\theta)) \subset \mathcal{B}_u^-(\theta), \quad \theta \in \Theta,$$

then for all $x \in \mathcal{N}(P(\theta))$ the point (x, θ) has a unique bounded negative continuation $\phi(t) = (\phi^x(t), \theta \cdot t)$. Then for $t \leq 0$ we shall define $\Phi(\theta, t)x := \phi^x(t)$. Moreover, from Definition 3.1 we get

$$\Phi(\theta, t + s)x = \Phi(\theta \cdot t, s)\Phi(\theta, t)x, \quad s, t \in \mathbb{R}.$$

Hence

$$x = \Phi(\theta \cdot t, -t)\Phi(\theta, t)x, \quad t \in \mathbb{R}. \tag{3.4}$$

So, if $\Phi(\theta, t)x = 0$, then $x = 0$. On the other hand, from Definition 3.1 we have that

$$P(\theta \cdot t)\Phi(\theta, t)x = \Phi(\theta, t)P(\theta)x \quad t \in \mathbb{R}.$$

Therefore, $\Phi(\theta, t)x \in \mathcal{N}(P(\theta \cdot t)$. Finally, if $y \in \mathcal{N}(P(\theta \cdot t))$, then $y \in \mathcal{B}_u^-(\theta \cdot t)$. So, if we put $x = \Phi(\theta \cdot t, -t)y$, then we get $y = \Phi(\theta, t)x$. □

Lemma 3.1. *If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ which admits an exponential dichotomy over Θ according to Definition 3.3 with an invariant projector $\mathbf{P}$, then for all $\theta \in \Theta$ we have that:*

$$\mathcal{B}(\theta) = \{0\}, \quad \mathcal{X}_s(\theta) = \mathcal{R}(P(\theta)) = \mathcal{B}^+(\theta) \quad \textit{and} \quad \mathcal{X}_u(\theta) = \mathcal{N}(P(\theta)) = \mathcal{B}^-(\theta).$$

Moreover,

$$X = \mathcal{R}(P(\theta)) + \mathcal{N}(P(\theta) = \mathcal{X}_s(\theta) + \mathcal{X}_u(\theta)$$

Proof. Consider $x \in \mathcal{B}(\theta)$ and $\phi(t) = (\phi^x(t), \theta \cdot t)$ the corresponding bounded negative continuation of the point (x, θ). Set $y = P(\theta)x$ and $z = (I - P(\theta))x$. Then $x = y + z$. From the Definition 2.3 of negative continuation we get that

$$\Phi(\theta, t + s)x = \Phi(\theta \cdot t, s)\phi^x(t), \qquad 0 \leq -t \leq s.$$

So, if we put $s = -t$, then $x = \Phi(\theta \cdot t, -t)\phi^x(t)$, for $t \leq 0$. Therefore, for $t \leq 0$ we have the following:

$$y = P(\theta)x = P(\theta \cdot t \cdot (-t))\Phi(\theta \cdot t, -t)\phi^x(t) = \Phi(\theta \cdot t, -t)P(\theta \cdot t)\phi^x(t).$$

Then $\|y\| \leq ke^{\beta t}\|\phi^x(t)\|$, $t \leq 0$. Since, $\phi^x(t)$ is bounded, then $y = 0$.

From the Definition 3.3 of exponential dichotomy, we know that

$$\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot t)), \quad t \geq 0,$$

is an isomorphism with inverse:

$$\Phi(\theta \cdot t, -t) : \mathcal{N}(P(\theta \cdot t)) \to \mathcal{N}(P(\theta)), \quad t \geq 0.$$

Since $z = (I - P(\theta))x \in \mathcal{N}(P(\theta)) = \mathcal{R}(I - P(\theta))$, we get that

$$\begin{aligned} z &= \Phi(\theta \cdot t, -t)\Phi(\theta, t)z \\ &= \Phi(\theta \cdot t, -t)\Phi(\theta, t)(I - P(\theta))x \\ &= \Phi(\theta \cdot t, -t)(I - P(\theta \cdot t))\Phi(\theta, t)x, \quad t \geq 0. \end{aligned}$$

Hence, $\|z\| \leq ke^{-\beta t}\|\Phi(\theta, t)x\|, \quad t \geq 0$. Since $\Phi(\theta, t)x$ is bounded, then $z = 0$. Therefore, $x = 0$. So $\mathcal{B}(\theta) = \{0\}$.

Clearly, $\mathcal{N}(P(\theta)) \subset \mathcal{X}_u(\theta) \subset \mathcal{B}^+(\theta)$ and $\mathcal{R}(P(\theta)) \subset \mathcal{X}_s(\theta) \subset \mathcal{B}^-(\theta)$.

The proof follows from $X = \mathcal{R}(P(\theta)) + \mathcal{N}(P(\theta))$. □

Remark 3.2. From Proposition 3.1 and Lemma 3.1 we get that in Definition 3.2 the condition $\mathcal{R}(I - P(\theta)) \subseteq \mathcal{B}_u^-(\theta), \quad \theta \in \Theta$, is equivalent to $\mathcal{R}(I - P(\theta)) = \mathcal{B}_u^-(\theta), \quad \theta \in \Theta$. From now on, we will work with Definition 3.3.

Proposition 3.3. *If the skew-product semiflow $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ according to Definition 3.3, then for all $\theta \in \Theta$ and $t, s \in \mathbb{R}$ we have that*

$$\Phi(\theta, t + s)(I - P(\theta)) = \Phi(\theta \cdot t, s)\Phi(\theta, t)(I - P(\theta)).$$

Proof. (i) If $t, s \geq 0$, then it follows from the cocycle property (2.1).

(ii) If $t < 0$ and $s < 0$, then

$$\begin{aligned} &\Phi(\theta \cdot t, s)\Phi(\theta, t)(I - P(\theta)) \\ &= (\Phi(\theta \cdot (t + s), -s)|_{\mathcal{N}(P(\theta \cdot t))})^{-1}(\Phi(\theta \cdot t, -t)|_{\mathcal{N}(P(\theta))})^{-1}(I - P(\theta)) \\ &= [\Phi(\theta \cdot t, -t)\Phi(\theta \cdot (t + s), -s)|_{\mathcal{N}(P(\theta))}]^{-1}(I - P(\theta)). \end{aligned}$$

Now, using the cocycle property (2.1), we get that

$$\begin{aligned} \Phi(\theta \cdot t, s)\Phi(\theta, t)(I - P(\theta)) &= (\Phi(\theta \cdot (t + s), -(t + s))|_{\mathcal{N}(P(\theta))})^{-1}(I - P(\theta)) \\ &= \Phi(\theta, t + s)(I - P(\theta)). \end{aligned}$$

(iii) If $t > 0, \quad s < 0$ and $t + s < 0$, then

$$\begin{aligned} &\Phi(\theta \cdot t, s)\Phi(\theta, t)(I - P(\theta)) \\ &= (\Phi(\theta \cdot (t + s), -s)|_{\mathcal{N}(P(\theta \cdot t))})^{-1}(\Phi(\theta \cdot t, -t)|_{\mathcal{N}(P(\theta))})^{-1}(I - P(\theta)) \\ &= [\Phi(\theta \cdot t, -t)\Phi(\theta \cdot (t + s), -s)|_{\mathcal{N}(P(\theta))}]^{-1}(I - P(\theta)) \\ &= [\Phi(\theta \cdot t, -t)\Phi(\theta \cdot (t + s), -(t + s) + t)|_{\mathcal{N}(P(\theta))}]^{-1}(I - P(\theta)). \end{aligned}$$

Since $-(t + s) > 0$ and $t > 0$, we can apply the cocycle property (2.1) to get

$$\begin{aligned} &\Phi(\theta \cdot t, s)\Phi(\theta, t)(I - P(\theta)) \\ &= [\Phi(\theta \cdot t, -t)\Phi(\theta, t)\Phi(\theta \cdot (t + s), -(t + s))|_{\mathcal{N}(P(\theta))}]^{-1}(I - P(\theta)) \\ &= (\Phi(\theta \cdot (t + s), -(t + s))|_{\mathcal{N}(P(\theta))})^{-1}(I - P(\theta)) \\ &= \Phi(\theta, t + s)(I - P(\theta)). \end{aligned}$$

The case (iv) $t > 0, \quad s < 0$ and $t + s > 0$ is similar. □

Proposition 3.4. *If the skew-product semiflow $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ according to Definition 3.3 with an invariant projector $\mathbf{P}$ on $\mathcal{E}$, then for all $\theta \in \Theta$ and $t, s \in \mathbb{R}$ we have that*

$$\Phi(\theta, t)(I - P(\theta)) = (I - P(\theta \cdot t))\Phi(\theta, t),$$

on $\mathcal{N}(P(\theta))$.

Proof. If $t \geq 0$ there is nothing to prove. Suppose $t < 0$. Then, from (3.1) we get that

$$\Phi(\theta \cdot t, -t)(I - P(\theta \cdot t)) = (I - P(\theta))\Phi(\theta \cdot t, -t).$$

Therefore,

$$(\Phi(\theta, t)|_{\mathcal{N}(P(\theta))})^{-1}(I - P(\theta \cdot t)) = (I - P(\theta))(\Phi(\theta, t)|_{\mathcal{N}(P(\theta))})^{-1}.$$

Then,

$$(I - P(\theta \cdot t)) = \Phi(\theta, t)(I - P(\theta))(\Phi(\theta, t)|_{\mathcal{N}(P(\theta))})^{-1}.$$

So,

$$(I - P(\theta \cdot t))\Phi(\theta, t) = \Phi(\theta, t)(I - P(\theta)).$$

□

Proposition 3.5. *If the skew-product semiflow $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ according to Definition 3.3, then for all $x \in X$ fixed, the mapping*

$$t \to \phi^x(t) := \Phi(\theta, t)(I - P(\theta))x$$

is continuous in $\mathbb{R}$. Moreover, the mapping $\phi(t) := (\phi^x(t), \theta \cdot t)$ is a negative continuation of the point $((I - P(\theta))x, \theta)$.

Proof. First, we shall prove the continuity at $t = 0$, which is enough to prove that

$$\lim_{t \to 0^-} \Phi(\theta, t)(I - P(\theta))x = (I - P(\theta))x.$$

In fact, taking $\epsilon > 0$ and using Proposition 3.3, we get that

$$\begin{aligned}
\lim_{t \to 0^-} \phi^x(t) &= \lim_{t \to 0^-} \Phi(\theta, t)(I - P(\theta))x \\
&= \lim_{t \to 0^-} \Phi(\theta, -\epsilon + t + \epsilon)(I - P(\theta))x \\
&= \lim_{t \to 0^-} \Phi(\theta \cdot (-\epsilon), t + \epsilon)\Phi(\theta, -\epsilon)(I - P(\theta))x, \quad t + \epsilon > 0.
\end{aligned}$$

From Definition 2.1 we get that for all $z \in X$ the mapping $s \to \Phi(\theta, s)z$ is continuous for $s \geq 0$ uniformly on Θ. Therefore,

$$\begin{aligned}
\lim_{t \to 0^-} \phi^x(t) &= \Phi(\theta \cdot (-\epsilon), \epsilon)\Phi(\theta, -\epsilon)(I - P(\theta))x \\
&= (I - P(\theta))x.
\end{aligned}$$

Hence

$$\lim_{t \to 0} \Phi(\theta, t)(I - P(\theta))x = (I - P(\theta))x.$$

Now, consider $t < 0$ and $h \in \mathbb{R}$ small enough. Then from Propositions 3.3 and 3.4 we get:

$$\begin{aligned}
\lim_{h \to 0} \Phi(\theta, t + h)(I - P(\theta))x &= \lim_{h \to 0} \Phi(\theta \cdot t, h)\Phi(\theta, t)(I - P(\theta))x \\
&= \lim_{h \to 0} \Phi(\theta \cdot t, h)(I - P(\theta \cdot t))\Phi(\theta, t)x.
\end{aligned}$$

If we put $z = \Phi(\theta, t)x$, then

$$\lim_{h\to 0} \Phi(\theta, t+h)(I - P(\theta))x = \lim_{h\to 0} \Phi(\theta\cdot t, h)(I - P(\theta\cdot t))z$$
$$= (I - P(\theta\cdot t))z = \Phi(\theta, t)(I - P(\theta))x.$$

□

Corollary 3.1. *If the skew-product semiflow $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ according to Definition 3.3 with projector $\mathbf{P}$, then each $(x, \theta) \in \mathcal{N}(\mathbf{P})$ has a bounded negative continuous*

$$\phi(t) = (\phi^x(t), \theta\cdot t) := (\Phi(\theta, t)(I - P(\theta))x, \theta\cdot t), \quad t \le 0. \tag{3.5}$$

Moreover, $\|\phi^x(t)\| \le k e^{t\beta}\|x\|, \quad t \le 0.$

Corollary 3.2. *If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ which admits an exponential dichotomy over Θ according to Definition 3.3 with an invariant projector $\mathbf{P}$ and $\mathcal{B}_u^- \neq \emptyset$, then for all $\theta \in \Theta$ we have that:*

$$\mathcal{X}_u(\theta) = \mathcal{N}(P(\theta)) = \mathcal{B}^-(\theta) = \mathcal{B}_u^-(\theta).$$

Moreover, all the bounded negative continuations all given by formula (3.5).

Proof. From Proposition 2.2 and Lemma 3.1 we get that $\mathcal{N}(P(\theta)) = \mathcal{B}^-(\theta) = \mathcal{B}_u^-(\theta), \quad \theta \in \Theta$. Then from Corollary 3.1 we get that all bounded negative continuations are given by the formula (3.5). □

Theorem 3.1. *If in both Definitions 3.3 and 3.2 of exponential dichotomy we assume that $dim\mathcal{R}(I - P(\theta)) < \infty, \quad \theta \in \Theta$ (or $dim\mathcal{R}(I - P(\theta)) = \infty, \quad \theta \in \Theta$), then Definitions 3.3 and 3.2 are equivalent.*

3.1. **Characterization of the stable and unstable manifolds.** We begin this section with the following definition:

Definition 3.4. Given a point $(x, \theta) \in \mathcal{E} = X \times \Theta$, we shall say that $\Phi(\theta, t)x$ is **well defined on** $\mathbb{R}$ if it is a continuous function on $t \in \mathbb{R}$ and satisfies

(a) $\Phi(\theta, t+s)x = \Phi(\theta\cdot t, s)\Phi(\theta, t)x, \quad t, s \in \mathbb{R},$

(b) $P(\theta\cdot t)\Phi(\theta, t)x = \Phi(\theta, t)P(\theta)x, \quad t \in \mathbb{R}.$

Also, we define the set

$$\mathcal{M}_w := \{(x, \theta) \in \mathcal{E} : \Phi(\theta, t)x \text{ is well defined}\}.$$

Remark 3.3. Clearly $\mathcal{B}_u^- \subset \mathcal{M}_w$. Also, if $\pi = (\Phi, \sigma)$ has an exponential dichotomy according to Definition 3.3, then $\mathcal{N}(\mathbf{P}) \subset \mathcal{M}_w$.

Lemma 3.2. *If $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ, then*

(3.6)
$$\mathcal{X}_s(\theta) = \mathcal{R}(P(\theta)) = \{x \in X : \sup_{t\ge 0} \|(I - P(\theta\cdot t))\Phi(\theta, t)x\| < \infty\} =: \mathcal{Z}_u(\theta),$$

(3.7)
$$\mathcal{X}_u(\theta) = \mathcal{N}(P(\theta)) = \{x \in \mathcal{M}_w(\theta) : \sup_{t\le 0} \|P(\theta\cdot t)\Phi(\theta, t)x\| < \infty\} =: \mathcal{Z}_s(\theta)$$

for all $\theta \in \Theta$.

Proof. Suppose $x \in \mathcal{R}(P(\theta)) = \mathcal{X}_s(\theta)$. Then $P(\theta)x = x$. So we get

$$\|(I - P(\theta \cdot t))\Phi(\theta, t)x\| = \|\Phi(\theta, t)(I - P(\theta))x\| = 0.$$

Therefore, $x \in \mathcal{Z}_s(\theta)$. So, $\mathcal{X}_s(\theta) \subset \mathcal{Z}_s(\theta)$.

Suppose $x \in \mathcal{Z}_s(\theta)$ Then there exists a constant $C > 0$ such that

$$\|(I - P(\theta \cdot t))\Phi(\theta, t)x\| \leq C < \infty, \quad \text{for } \ t \geq 0.$$

Then

$$\begin{aligned}\Phi(\theta, t)x &= \Phi(\theta, t)(I - P(\theta))x + \Phi(\theta, t)P(\theta)x \\ &= (I - P(\theta \cdot t))\Phi(\theta, t)x + \Phi(\theta, t)P(\theta)x.\end{aligned}$$

So

$$\|\Phi(\theta, t)x\| \leq C + ke^{-\beta t}\|x\|, \quad t \geq 0.$$

Hence, $x \in \mathcal{B}^+ = \mathcal{R}(P(\theta)) = \mathcal{X}_s(\theta)$. So, $\mathcal{Z}_s(\theta) = \mathcal{X}_s(\theta)$.

Now, suppose that $x \in \mathcal{N}(P(\theta)) = \mathcal{X}_u(\theta)$. Then $P(\theta)x = 0$. Hence, using Proposition 2.1, we get the following:

$$P(\theta \cdot t)\Phi(\theta, t)x = \Phi(\theta, t)P(\theta)x = 0, \quad t \leq 0.$$

Therefore, $\mathcal{X}_u(\theta) \subset \mathcal{Z}_u(\theta)$.

Suppose that $x \in \mathcal{Z}_u(\theta)$. Then there exists $C > 0$ such that

$$\|P(\theta \cdot t)\Phi(\theta, t)x\| < C, \quad t \leq 0.$$

On the other hand, from Definition 3.4 we get

$$\begin{aligned}\Phi(\theta, t)x &= (I - P(\theta \cdot t))\Phi(\theta, t)x + P(\theta \cdot)\Phi(\theta, t)x \\ &= \Phi(\theta, t)(I - P(\theta))x + P(\theta \cdot t)\Phi(\theta, t)x.\end{aligned}$$

Then

$$\|\Phi(\theta, t)x\| \leq C + ke^{\beta t}\|x\|, \quad t \leq 0.$$

So, $\Phi(\theta, t)x$ is bounded for $t \leq 0$. Therefore $x \in \mathcal{B}^-(\theta) = \mathcal{N}(P(\theta)) = \mathcal{X}_u(\theta)$. Hence, $\mathcal{X}_u(\theta) = \mathcal{Z}_u(\theta)$. □

Lemma 3.3. *If* $\pi = (\Phi, \sigma)$ *has an exponential dichotomy over* Θ*, then for all* $\eta \in (0, \beta)$ *we have*

$$\mathcal{X}_s(\theta) = \{x \in X : \sup_{t \geq 0} e^{-\eta t}\|\Phi(\theta, t)x\| < \infty\} \tag{3.8}$$

$$\mathcal{X}_u(\theta) = \{x \in \mathcal{M}_w(\theta) : \sup_{t \leq 0} e^{\eta t}\|\Phi(\theta, t)x\| < \infty\} \tag{3.9}$$

for all $\theta \in \Theta$.

Proof. Denote the right side of (3.8) by $\mathcal{Z}_s(\theta)$. Then clearly $\mathcal{X}_s(\theta) \subset \mathcal{Z}_s(\theta)$.

Assume $x \in \mathcal{Z}s(\theta)$. Then

$$\|\Phi(\theta, t)x\| \leq Ce^{\eta t}, \quad t \geq 0, \quad \text{and} \quad x = P(\theta)x + (I - P(\theta))x.$$

It is enough to prove that $(I - P(\theta))x = 0$. In fact, for $t \leq 0$ we get

$$\begin{aligned}\|(I - P(\theta))x\| &= \|\Phi(\theta \cdot (-t), t)\Phi(\theta, -t)(I - P(\theta))x\| \\ &= \|\Phi(\theta \cdot (-t), t)(I - P(\theta \cdot (-t)))\Phi(\theta, -t)x\| \\ &\leq \|\Phi(\theta \cdot (-t), t)(I - P(\theta \cdot (-t)))\|\|\Phi(\theta, -t)x\| \\ &\leq ke^{\beta t}Ce^{-\eta t} = kCe^{(\beta - \eta)t} \to 0, \quad \text{as} \quad t \to -\infty.\end{aligned}$$

Hence $\mathcal{Z}_s(\theta) \subset \mathcal{X}_s(\theta)$.

Denote the right side of (3.9) by $\mathcal{Z}_u(\theta)$. Then clearly $\mathcal{X}_u(\theta) \subset \mathcal{Z}_u(\theta)$.
Suppose $x \in \mathcal{Z}_u(\theta)$. Then

$$x \in \mathcal{M}_w(\theta), \quad \|\Phi(\theta, t)x\| \le Ce^{-\eta t}, \quad t \le 0, \quad \text{and} \quad x = P(\theta)x + (I - P(\theta))x.$$

It is enough to prove that $P(\theta)x = 0$. In fact, for $-t \le 0$, from Definition 3.4 we get that

$$\begin{aligned} \|P(\theta)x\| &= \|\Phi(\theta \cdot (-t), t)\Phi(\theta, -t)P(\theta)x\| \\ &= \|\Phi(\theta \cdot (-t), t)P(\theta \cdot (-t))\Phi(\theta, -t)x\| \\ &\le \|\Phi(\theta \cdot (-t), t)P(\theta \cdot (-t))\|\|\Phi(\theta, -t)x\| \\ &\le ke^{-\beta t}Ce^{\eta t} = kCe^{(\eta-\beta)t} \to 0, \quad \text{as} \quad t \to +\infty. \end{aligned}$$

Hence $\mathcal{Z}_u(\theta) \subset \mathcal{X}_u(\theta)$. □

In conclusion we have the following theorem:

Theorem 3.2. *If $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ, then we have the following*:
(a) $\mathcal{X}_s$, $\mathcal{X}_u$ are invariant subbundles of $\mathcal{E}$ under the flow π and

$$\mathcal{E} = \mathcal{X}_s + \mathcal{X}_u \quad \text{and} \quad \mathcal{X}_s = \mathcal{B}^+, \quad \mathcal{X}_u = \mathcal{B}^-.$$

(the Whitney sum of two subbundles).
(b) We get the following characterization of $\mathcal{X}_s$ and $\mathcal{X}_u$:

$$\mathcal{X}_s = \{(x, \theta) \in \mathcal{E} : \sup_{t \ge 0} \|(I - P(\theta \cdot t))\Phi(\theta, t)x\| < \infty\}, \tag{3.10}$$

$$\mathcal{X}_u = \{(x, \theta) \in \mathcal{M}_w : \sup_{t \le 0} \|P(\theta \cdot t)\Phi(\theta, t)x\| < \infty\}. \tag{3.11}$$

(c) For $\eta \in (0, \beta)$ we get

$$\mathcal{X}_s = \{(x, \theta) \in \mathcal{E} : \sup_{t \ge 0} e^{-\eta t}\|\Phi(\theta, t)x\| < \infty\}, \tag{3.12}$$

$$\mathcal{X}_u = \{(x, \theta) \in \mathcal{M}_w : \sup_{t \le 0} e^{\eta t}\|\Phi(\theta, t)x\| < \infty\}. \tag{3.13}$$

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