PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

# HNN BASES AND HIGH-DIMENSIONAL KNOTS

DANIEL S. SILVER

(Communicated by James West )

ABSTRACT. There exists a 3-knot group having HNN bases of two types: bases that are arbitrarily large finitely presented and bases that are arbitrarily large finitely generated but not finitely presented. Any $n$-knot with such a group has a Seifert manifold that can be converted to a minimal one by a finite sequence of ambient 0- and 1-surgeries, but cannot be converted by 1-surgeries alone.

## 1. Introduction

Let $G$ be a group and $h : G \twoheadrightarrow \mathbf{Z}$ an epimorphism. We can describe $G$ as an HNN extension $\langle x, B |\ xsx^{-1} = \phi(s)\ \forall s \in S\rangle$, where $h(x) = 1$, $B \le \ker h$, $S \le B$, and $\phi$ is an isomorphism $\phi : S \xrightarrow{\sim} T \le B$. The group $B$ is called an *HNN base* (or *base*) for $G$ relative to $h$, while $S$ and $T$ are called *associated subgroups*. When $G$ is finitely presented, we can find such an extension in which $B, S$, and $T$ are finitely generated [**3**].

Section 2 of this paper is concerned with the case that $G$ is an $n$-knot group. A *base* for an $n$-knot group will mean an HNN base relative to the abelianization homomorphism. We recall that an *$n$-knot*, for $n \ge 1$, is a smoothly embedded $n$-sphere $\mathcal{K} \subset \mathcal{S}^{n+2}$. The group $\pi_1(\mathcal{S}^{n+2} - \mathcal{K})$ is called an $n$-knot group or the *group* of $\mathcal{K}$. For $n \ge 1$, any $n$-knot group is also an $(n+1)$-knot group (see [**7**]). A *Seifert manifold* for $\mathcal{K}$ is a compact, connected, oriented $(n+1)$-manifold $\mathcal{V} \subset \mathcal{S}^{n+2}$ with boundary equal to $\mathcal{K}$. By [**8**] any $n$-knot possesses a Seifert manifold. A Seifert manifold $\mathcal{V}$ is *minimal* if the inclusion map $\iota$: *int* $\mathcal{V} \hookrightarrow \mathcal{S}^{n+2} - \mathcal{K}$ induces a monomorphism of fundamental groups.

If an $n$-knot $\mathcal{K}$ has a minimal Seifert manifold, then the group of $\mathcal{K}$ has a finitely presented base. Such a base can be realized as the fundamental group of the compact manifold obtained by "splitting" $\mathcal{S}^{n+2}$ along the Seifert manifold [**8**]. This fact was used in [**13**] in order to produce examples of $n$-knots, for $n > 2$, with no minimal Seifert manifolds. Whether or not there exists a 2-knot with no minimal Seifert manifold is an open question. (See [**9**] for an example of a knotted torus $\mathcal{T}$ in $\mathcal{S}^4$ such that $\pi_1(\mathcal{S}^4 - \mathcal{T})$ has no HNN decomposition with finitely presented base.) In higher dimensions the situation is clearer. We proved in [**14**] that any $n$-knot $\mathcal{K}$, for $n > 2$, has a minimal Seifert manifold if and only if its group has a finitely presented base. We showed moreover that in this case any Seifert manifold for $\mathcal{K}$ can be converted to a minimal one by a finite sequence of ambient 0- and 1-surgeries. (Ambient $i$-surgery is also called $(i+1)$-handle exchange [**8**].)

Received by the editors May 17, 1994.
1991 *Mathematics Subject Classification.* Primary 57Q45; Secondary 20E06, 20F05.

In view of the above-mentioned results it is natural to ask whether a finitely presented group can have an HNN base (relative to some $h$) that is finitely presented and another that is finitely generated but not finitely presented. Can an $n$-knot group have this property? We begin with a simple construction that answers the first question affirmatively.

Let $H$ be any finitely presented group with a subgroup $H_0$ that is finitely generated but not finitely presented. For definiteness one can use the example of J. Stallings [**16**] in which

$$H = \langle t, a_1, a_2, b \mid ta_1t^{-1} = a_1,\ ta_2t^{-1} = a_2,\ tbt^{-1} = a_1ba_1^{-1},\ a_1ba_1^{-1} = a_2ba_2^{-1}\rangle$$

while $H_0$ is the subgroup $gp(a_1, a_2, b)$ generated by $a_1,\ a_2$, and $b$. Given any such pair $(H, H_0)$, we define

$$G = \langle x, H \mid x^2hx^{-2} = h\ \forall h \in H_0\rangle.$$

Since $H_0$ is finitely generated, $G$ has a finite presentation. Let $\hat{H}_0$ be an isomorphic copy of $H_0$ with isomorphism $\hat{h} \mapsto h$. Then

$$G \cong \langle x, H * \hat{H}_0 \mid x^2hx^{-2} = h\ \forall h \in H_0,\ \hat{h} = x^{-1}hx\ \forall \hat{h} \in \hat{H}_0\rangle$$

$$\cong \langle x, H * \hat{H}_0 \mid xhx^{-1} = \hat{h}\ \forall h \in H_0,\ x\hat{h}x^{-1} = h\ \forall \hat{h} \in \hat{H}_0\rangle.$$

This last presentation exhibits $G$ as an HNN extension with base $B = H * \hat{H}_0$ relative to the homomorphism $h : G \twoheadrightarrow \mathbf{Z}$ that maps $x$ to 1 and $H * \hat{H}_0$ to 0. The associated subgroups $S$ and $T$ are the same, namely $H_0 * \hat{H}_0$, while $\phi : S \to T$ is the isomorphism that interchanges factors. Since $H$ and $H_0$ are finitely generated, so is the base $B$. However, since $H_0$ is not finitely presented, neither is $B$.

In order to show that $G$ has a finitely presented HNN base, we introduce some notation. For any subgroup $A \leq B$ and integer $\nu$, we let $A_\nu$ denote the subgroup $x^\nu Ax^{-\nu} \leq \ker h$. With this notation, $\ker h$ can be described as the infinite amalgamated free product of the groups $B_\nu$, $\nu \in \mathbf{Z}$, in which $S_{\nu+1}$ is identified with $T_\nu$ by the mapping $x^{\nu+1}sx^{-\nu-1} \mapsto x^\nu\phi(s)x^{-\nu}\ \forall s \in S$. For integers $i \leq j$ let $B_{i,j}$ denote the subgroup of $\ker h$ generated by $\bigcup_{i\leq\nu\leq j} B_\nu$. Whenever $i < j$ we can describe $G$ as an HNN extension with base $B_{i,j}$, associated subgroups $B_{i,j-1}, B_{i+1,j}$ and isomorphism $B_{i,j-1} \to B_{i+1,j}$ described by the map $\sigma : x^\nu gx^{-\nu} \mapsto x^{\nu+1}gx^{-\nu-1}$. In particular, the base $B_{0,1}$ is the amalgamated free product of two copies of $H * \hat{H}_0$ in which $H_0 * \hat{H}_0$ in the first factor is identified with $H_0 * \hat{H}_0$ in the second by an isomorphism that interchanges the factors. Clearly, this base is isomorphic to $H * H$, a finitely presented group.

A byproduct of our construction is a finitely presented group $H * H$ with a nontrivial amalgamated free product decomposition in which the factors, both isomorphic to $H * \hat{H}_0$, and the amalgamated subgroups, isomorphic to $H_0 * \hat{H}_0$, are finitely generated but not finitely presented. The first such example was given by G. Baumslag and P. Shalen in [**2**] using wreath products of groups. Our example is relatively elementary, and it suggests the following.

**Proposition 1.1.** *Let $H$ be a finitely presented group. Then $H$ has a subgroup that is finitely generated but not finitely presented if and only if $H * H$ has an amalgamated free product decomposition in which some factor or amalgamated subgroup is finitely generated but not finitely presented.*

*Proof.* We have seen the forward implication. In order to prove the converse, assume that $H * H$ has an amalgamated free product decompostion $A *_C B$ in which at least one of $A, B$ or $C$ is finitely generated but not finitely presented. By the Kurosh Subgroup Theorem that subgroup is a free product of a free group and certain subgroups of conjugates of $A$ and/or $B$. Each of the factors is finitely generated, but at least one must fail to be finitely presented. Hence $H$ contains a subgroup that is finitely generated but not finitely presented. □

In the construction above we showed that the base $B_{0,1}$ is finitely presented. More generally, one sees that $B_{i,j}$ is finitely presented whenever $j - i > 0$. This follows also from the next result.

**Proposition 1.2.** *Assume that $B$ is an HNN base for a finitely presented group $G$. If $B$ is finitely presented, then for all integers $i < j$, the base $B_{i,j}$ is also finitely presented.*

*Proof.* Assume that $S$ and $T$ are the associated subgroups corresponding to $B$. For any integers $i < j$, the base $B_{i,j}$ is the free product of $j - i + 1$ copies of $B$ amalgamated along copies of $S$, and by [**1**] such a group is finitely presented if and only if $S$ is finitely generated. Associated subgroups $S$, $T$ of an HNN decomposition for $G$ need not be finitely generated. However, they must be finitely generated if the base $B$ is finitely presented. To see this, consider the presentation

$$\langle x,\ B \mid xsx^{-1} = \phi(s)\ \forall s \in S\rangle.$$

Since $G$ is finitely presented, only finitely many relators are needed [**11**], say those of a finite presentation of $B$ together with $xs_1x^{-1} = \phi(s_1), \ldots, xs_nx^{-1} = \phi(s_n)$. Let $\tilde{S}$ be the subgroup of $S$ generated by $s_1, \ldots, s_n$, and let $\tilde{T} = \phi(\tilde{S})$. Replacing $S$ and $T$ with their respective subgroups $\tilde{S}$ and $\tilde{T}$, known to be finitely generated, we can conclude that the base $B_{i,j}$ is finitely presented. Of course, we can also conclude that the original associated subgroups $S$ and $T$ were finitely generated, since $B_{i,j}$ cannot be both finitely presented and nonfinitely presented. □

## 2. High-dimensional knot groups

Proposition 1.2 implies that if a finitely presented group $G$ has an HNN base that is finitely presented, then it has finitely presented bases that are "arbitrarily large" in the following sense.

**Definition 2.1.** A group $G$ has arbitrarily large finitely presented (resp. finitely generated but not finitely presented) HNN bases relative to $h : G \twoheadrightarrow \mathbf{Z}$ if $G$ has finitely presented (resp. finitely generated but not finitely presented) HNN bases $B(k),\ k = 1, 2 \ldots$, such that $\bigcup_k B(k) = \ker h$.

**Theorem 2.2.** *There exists a 3-knot group $G$ with arbitrarily large finitely presented HNN bases and arbitrarily large finitely generated but not finitely presented*

*HNN bases. Any $n$-knot with such a group has a Seifert manifold that can be converted to a minimal one by a finite sequence of ambient* 0*- and* 1*-surgeries, but cannot be converted by ambient* 1*-surgeries alone.*

*Remark.* The last statement of Theorem 2.2 says that certain Seifert manifolds can be made minimal by ambient surgery only if their fundamental groups are first enlarged by attaching "hollow 1-handles" to the Seifert manifolds.

**Lemma 2.3.** *Let $G = \langle x, B |\ xsx^{-1} = \phi(s)\ \forall s \in S\rangle$ be a group with HNN base $B$ relative to $h : G \twoheadrightarrow \mathbf{Z}$ and associated subgroups $S$ and $T$, and let $A$ be a subgroup of $B$. Assume that $A$ and $T$ are free factors of the subgroup that they generate. Then $x^{-1}Ax$ and $B$ are free factors of $gp(x^{-1}Ax, B) \leq \ker h$, and $gp(x^{-1}Ax, B)$ is a base for $G$ relative to $h$.*

*Proof.* Let $\hat{A}$ be an isomorphic copy of $A$ with isomorphism $\hat{a} \mapsto a$, and consider the presentation

$$\langle x, \hat{A} * B |\ x\hat{a}x^{-1} = a\ \forall \hat{a} \in \hat{A},\ xgx^{-1} = \phi(g)\ \forall g \in S\rangle. \tag{1.1}$$

The relations $x\hat{a}x^{-1} = a$ can be rewritten as $\hat{a} = x^{-1}ax$, thereby identifying $\hat{A}$ with $x^{-1}Ax \leq G$; consequently, (1.1) is a presentation for $G$. Since $gp(A, T) \cong A * T$, the mappings $\hat{a} \mapsto a\ \forall a \in \hat{A}$ and $g \mapsto \phi(g)\ \forall g \in S$ determine an isomorphism between $\hat{A} * S \leq \hat{A} * B$ and $gp(A, T)$. It follows that (1.1) presents $G$ as an HNN extension with base $\hat{A} * B$. Since $\hat{A}$ is identical to $x^{-1}Ax$ in $G$, the lemma is proved. □

*Proof of Theorem 2.2.* J. Hillman showed in [**6**] that the following presentation describes the group of a 2-knot $\mathcal{K}$:

$$\langle x, a, b |\ a^2 = (ab)^3 = b^5,\ xax^{-1} = a,\ xbx^{-1} = b^{-1}a^{-1}b^2ab\rangle.$$

The commutator subgroup of this group is the binary icosahedral group $I$, a perfect group of order 120. Observe that the element $a \in I$ commutes with the meridianal generator $x$. Define $U$ to be the group of the product 2-knot $\mathcal{K}\sharp\mathcal{K}$ with presentation

$$\langle x_U, a_1, b_1, a_2, b_2 |\ a_i^2 = (a_ib_i)^3 = b_i^5,\ x_U a_i x_U^{-1} = a_i,\ x_U b_i x_U^{-1} = b_i^{-1}a_i^{-1}b_i^2a_ib_i, i = 1, 2\rangle.$$

The commutator subgroup $U'$ of $U$ is the free product $I * I$, a finitely generated perfect group. The element $l = a_1a_2$ has infinite order and commutes with the meridianal generator $x_U$. The subgroup $gp(l, x_U)$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

Let $V$ be the group of any 3-knot for which no minimal Seifert manifold exists. The commutator subgroup of $V$ must contain a finitely generated subgroup $W$ that is not finitely presented. In fact, arguments of [**14**, p. 106] show that $W$ can be chosen to be $\iota_*\pi_1(int\ \mathcal{V})$, where $\mathcal{V}$ is any Seifert manifold for the 3-knot. The main idea is that if $\iota_*\pi_1(int\ \mathcal{V})$ were finitely presented, then the kernel of $\iota_*$ would be the normal closure of only finitely many elements of $\pi_1(int\ \mathcal{V})$, elements that can be represented by embedded loops in $\mathcal{V}$ and then killed by ambient 1-surgery.

Let $\tilde{U}$ be an isomorphic copy of $U$ with isomorphism $\tilde{u} \mapsto u$. Define

$$B \cong \langle V * U * \tilde{U} |\ x_V = x_U = \tilde{l}, l = \tilde{x}_{\tilde{U}}\rangle.$$

The group $B$ is the amalgamated free product of the 3-knot groups $\langle V * U | x_{_V} = x_{_U} \rangle$ and $\tilde{U}$ in which the free abelian subgroup generated by $x_{_U}$ and $l$ in the first factor is identified with the free abelian subgroup generated by $\tilde{l}$ and $\tilde{x}_{_{\tilde{U}}}$ in the second factor. We record the observations for later use that the subgroup of $B$ generated by $V'$ and $U'$ is the commutator subgroup of $\langle V * U | x_{_V} = x_{_U} \rangle$, and that this group is simply the free product of $V'$ and $U'$.

Clearly $H_1 B$ is trivial. Using the Mayer-Vietoris sequence [**4**] (or see [**16**]) and the fact that every $n$-knot group has vanishing second homology [**7**], one checks that $B$ is *superperfect*; i.e., $H_2 B$ is also trivial. Finally define

$$G \cong \langle x, B | \; xgx^{-1} = x_{_U} g x_{_U}^{-1} \; \forall g \in U' \rangle. \tag{2.1}$$

Since $B$ is finitely presented and $U'$ is finitely generated, $G$ is finitely presented. Presentation (2.1) displays $G$ as an HNN extension with base $B$, associated subgroups $S$, $T$ equal to $U'$, and isomorphism $\phi : S \tilde{\to} T$ given by the meridianal automorphism $g \mapsto x_{_U} g x_{_U}^{-1}$ in $U$. Since $B$ is perfect, $H_1 G$ is an infinite cyclic group generated by $x$. It is a straightforward exercise to check that $x$ normally generates $G$. One begins by killing $x$, thereby introducing the relations $g = x_{_U} g x_{_U}^{-1} \forall \; g \in U'$ which kill $U'$ (since $U$ is normally generated by $x_{_U}$) thereby killing $l$, etc. Also, the exact sequence for HNN extensions [**4**] (or see [**5**]) combined with the facts that $H_2 B = 0$ and $H_1 U' = 0$ enable one to check that $H_2 G = 0$. Hence by Kervaire's theorem [**7**] the group $G$ is a 3-knot group.

Using the notation established in Section 1, we can describe $G$ as an HNN extension with base $B_{i,j}$, associated subgroups $B_{i,j-1}, B_{i+1,j}$ and isomorphism $B_{i,j-1} \to B_{i+1,j}$ determined by $\sigma : x^{\nu} g x^{-\nu} \mapsto x^{\nu+1} g x^{-\nu-1}$. In this way we obtain arbitrarily large finitely presented bases $B(k) = B_{-k,k}, k > 0$, for $G$.

We now produce arbitrarily large bases $B^*(k)$ for $G$ that are finitely generated by not finitely presented. Recall that $V'$ contains a subgroup $W$ that is finitely generated but not finitely presented. For any integers $i < j$, the base $B_{i-1,j}$ is the amalgamated free product of $B_{i-1}$ and $B_{i,j}$ in which $U'_{i-1}$ is identified with $U'_i$. Earlier we observed that $gp(V', U')$ is the free product $V' * U'$. Hence $gp(W, U')$ is $W * U'$. It follows that the subgroup of $B_{i-1,j}$ generated by $W_{i-1}$, $U'_{i-1}$, and $B_{i,j}$ is the amalgamated free product of $W_{i-1} * U'_{i-1}$ and $B_{i,j}$ in which $U'_{i-1}$ is identified with $U'_i \leq B_{i,j}$, and this group is clearly $W_{i-1} * B_{i,j}$. Hence $W_{i-1}$ and $B_{i,j}$ are free factors of the subgroup of $B_{i-1,j}$ that they generate. By Lemma 2.3 the subgroup $W_{i-1} * B_{i,j}$ is a base $B^*_{i,j}$ for $G$. Of course, $B^*_{i,j}$ is finitely generated since each of its factors is, but it is not finitely presented since $W_{i-1}$ is not. For $k > 0$, we define $B^*(k)$ to be $W_{-k-1} * B_{-k,k}$. Then $\bigcup_k B^*(k)$ is equal to $G'$, and the first statement of Theorem 2.2 is proved.

Assume that $\mathcal{K}$ is any 3-knot with group $G$, and let $\mathcal{V}$ be a Seifert manifold for $\mathcal{K}$. The image $\iota_* \pi_1(int\ \mathcal{V})$ in $\pi_1(\mathcal{S}^3 - \mathcal{K})$ is a finitely generated subgroup of $G'$. Choosing $k > 0$ sufficiently large, we can assume that the image is contained in the base $B^*(k)$, and since $B^*(k)$ is finitely generated, we can perform ambient 0-surgeries on $\mathcal{V}$ until the image coincides with $B^*(k)$ (see [**14**] pp. 106 – 107). The new Seifert manifold $\mathcal{V}'$ that we obtain cannot be converted to a minimal one by any finite sequence of ambient 1- surgeries alone because $\iota_* \pi_1(int\ \mathcal{V}')$ is not finitely related. □

## 3. A conjecture about knot-like groups

Following Rapaport [**12**] we will say that a group $G$ is *knot-like* if $G/G' \cong \mathbf{Z}$ and $G$ has a finite presentation in which the number of generators exceeds the number of relators by one. Any 1-knot group is a knot-like group as are the groups of many $n$-knots for $n > 1$. By a *base* for a knot-like group we will mean an HNN base with respect to the abelianization homomorphism. By [**3**] every knot-like group has a finitely generated base.

**Conjecture 3.1.** *Every finitely generated base for a knot-like group is finitely presented.*

It is well known and not difficult to prove that if the commutator subgroup $G'$ of a knot-like group is finitely generated, then any finitely generated base for $G$ is equal to $G'$. A conjecture of Rapaport [**12**] asserts that in this case any finitely generated base for $G$ is in fact a free group. Partial results about this conjecture were obtained in [**12**] and [**15**] (see also [**10**]).

## References

1. G. Baumslag, *A remark on generalized free products*, Proc. Amer. Math. Soc. **13** (1962), 53–54. MR **26:**202
2. G. Baumslag, P.B. Shalen, *Amalgamated products and finitely presented groups*, Comment. Math. Helv. **65** (1990), 243–254. MR **91j:**20071
3. R. Bieri, R. Strebel, *Almost finitely presented soluble groups*, Comment. Math. Helv. **53** (1978), 258–278. MR **58:**16890
4. I.M. Chiswell, *Exact sequences associated with a graph of groups*, J. Pure Appl. Alg. **8** (1976), 63–74. MR **53:**3147
5. C.McA. Gordon, *Homology of groups of surfaces in the 4-sphere*, Math. Proc. Camb. Phil. Soc. **89** (1981), 113–117. MR **83d:**57016
6. J. Hillman, *High dimensional knot groups which are not two-knot groups*, Bull. Austral. Math. Soc. **16** (1977), 449–462. MR **58:**31098
7. M. Kervaire, *On higher dimensional knots*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) (S.S. Cairns, ed.), Princeton University Press, Princeton, 1965, pp. 105–109. MR **31:**2732
8. J. Levine, *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16. MR **31:**4045
9. T. Maeda, *Knotted surfaces in the 4-sphere with no minimal Seifert manifolds*, preprint.
10. D.I. Moldavanskii, *Certain subgroups of groups with one defining relation* (in Russian), Sibirsk. Math. Z. **8** (1967), 1370–1384. MR **36:**3862
11. B.H. Neumann, *Some remarks on infinite groups*, J. London Math. Soc. **12** (1937), 4–11.
12. E.S. Rapaport, *Knot-like groups*, Annals of Math. Studies, vol. 84, Princeton Univ. Press, Princeton, 1975, pp. 119–133.
13. D.S. Silver, *Examples of 3-knots with no minimal Seifert manifolds*, Math. Proc. Camb. Phil. Soc. **110** (1991), 417–420. MR **92f:**57030
14. D.S. Silver, *On the existence of minimal Seifert manifolds*, Math. Proc. Camb. Phil. Soc. **114** (1993), 103–109. MR **94c:**47039
15. D.S. Silver, *On knot-like groups and ribbon concordance*, J. Pure Appl. Alg. **82** (1992 99–105). MR **94a:**57021
16. J. Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Amer. Journal of Math. **95** (1963), 541–543. MR **28:**2139

Department of Mathematics and Statistics, University of South Alabama, Mobile, Alabama 36688

*E-mail address*: silver@mathstat.usouthal.edu