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A COMPLETELY REGULAR SPACE WHICH IS THE T_1-COMPLEMENT OF ITSELF

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ABSTRACT. Two topologies τ and σ on a fixed set are T_1-complements if $\tau \cap \sigma$ is the cofinite topology and $\tau \cup \sigma$ is a sub-base for the discrete topology. In 1967, Steiner and Steiner showed that of any two T_1-complements on a countable set, at least one is not Hausdorff. In 1969, Anderson and Stewart asked whether a Hausdorff topology on an uncountable set can have a Hausdorff T_1-complement. We construct two homeomorphic completely regular T_1-complementary topologies.

Theorem 1. *There are two Hausdorff topologies σ_0 and σ_1 on a set such that $\sigma_0 \vee \sigma_1$ is the discrete topology, such that $\sigma_0 \wedge \sigma_1$ is the cofinite topology, and such that σ_0 and σ_1 are homeomorphic topologies.*

If σ_0 and σ_1 are topologies which satisfy

- $\sigma_0 \vee \sigma_1$ is the discrete topology,
- $\sigma_0 \wedge \sigma_1$ is the cofinite topology,

then we say σ_0 and σ_1 are T_1-complements.

In 1967, Steiner and Steiner [2] showed that of any pair of T_1-complements on a countable set, at least one is not Hausdorff. In 1969, Anderson and Stewart [1] showed that of any pair of T_1-complements, at least one is not first countable Hausdorff. Anderson and Stewart also asked: Can a Hausdorff topology on an (uncountable) set have a Hausdorff T_1-complement ? We answer this question affirmatively.

First we need a little finite combinatorics:

Example 1. There is a directed graph on a finite set containing the distinguished element p whose edges can be colored with three colors: green, red, and yellow, and two shades: light and dark, so that

1. The subgraph of light edges decomposes into components each of which is one of
    - the graph formed by identifying the initial vertex of a light green edge with the initial vertex of another light green edge,
    - the graph formed by identifying the terminal vertices of a light green edge and a light red edge,

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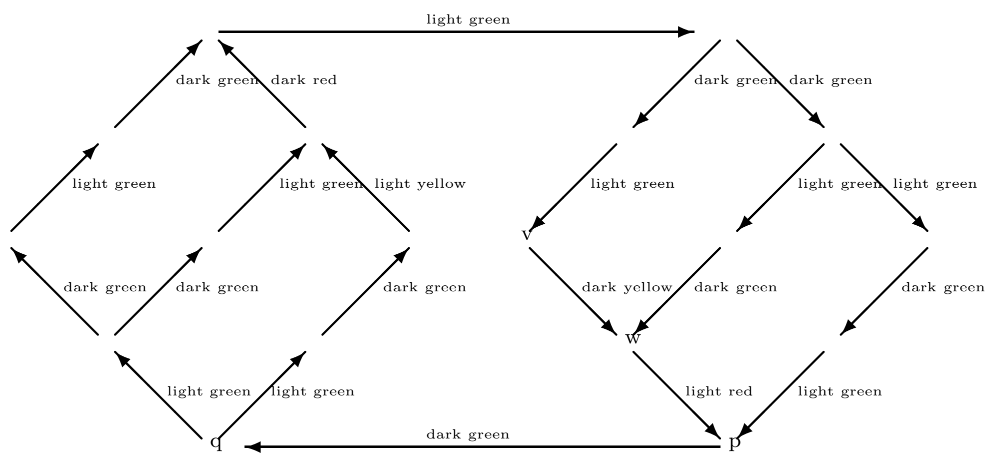
- the graph formed by identifying the terminal vertices of a light green edge and a light yellow edge,
- a graph consisting of a single light green edge.

2. Every vertex is either not initial in any light edge or not initial in any dark edge.
3. There is a directed graph isomorphism which sends any light edge to a dark edge of the same color and any dark edge to a light edge of the same color.
4. There is a dark yellow edge whose terminal vertex w is the initial vertex of a light red edge whose terminal vertex is p.
5. There is a path from any vertex to any other vertex and also a path from p to any other vertex entirely consisting of green edges.

Proof.



Construction 1. *Let (V, E) be a directed graph in which each edge is colored one of three colors: green, red and yellow, whose components are all of one of four kinds:*

- *the graph formed by identifying the initial vertex of a green edge with the initial vertex of another green edge,*
- *the graph formed by identifying the terminal vertices of a green edge and a red edge,*
- *the graph formed by identifying the terminal vertices of a green edge and a yellow edge,*
- *a graph consisting of a single green edge.*

Let $\{x_\alpha : \alpha \in \kappa\}$ be a list of distinct free ultrafilters on κ which extend the filter of closed unbounded subsets of κ and contain $\{\alpha \in \kappa : cf(\alpha) = \omega_1\}$. Let $\{y_\alpha : \alpha \in \kappa\}$ be a list of distinct free ultrafilters on κ each containing a countable set such that

$$(\forall B \in [\kappa]^\omega)(\exists C \in [\kappa]^\kappa) : (\forall \alpha \in C) B \in y_\alpha.$$

Let the set $\kappa \times V$ have the topology in which, for each component H in the graph, $\kappa \times H$ is clopen and has one of the following four kinds of topology:

- *Suppose the component is the graph formed by identifying the initial vertex of a green edge with the initial vertex of another green edge. Let Q be the space obtained by taking $\beta\kappa \times \{0,1\}$, identifying the subspaces $(\beta\kappa - \kappa) \times \{0\}, (\beta\kappa - \kappa) \times \{1\}$ by the natural two-to-one quotient map, and taking the subspace which is the image of $(\{y_\alpha : \alpha \in \kappa\} \times 2) \cup (\kappa \times 2)$. Now if a is the common initial vertex of the two green edges, identify $\kappa \times \{a\}$ with $\{y_\alpha : \alpha \in \kappa\}$ and, if $\{b_i : i \in \{0,1\}\}$ are the two terminal vertices, identify $\kappa \times \{b_i\}$ with $\kappa \times \{i\}$. Thus we construct a topology so that $\kappa \times \{a, b_0, b_1\}$ is homeomorphic to a subspace of a quotient space of $\beta\kappa$.*
- *Suppose the component is the graph formed by identifying the terminal vertices of a green edge and a red edge. Let R be the space which is the subspace of the Stone-Čech compactification of $\{\alpha \in \kappa : cf(\alpha) \neq \omega\}$ which consists of the successor ordinals of κ, $\{y_\alpha : \alpha \in \kappa\}$ viewed as ultrafilters on the successor ordinals, and $\{\alpha \in \kappa : cf(\alpha) = \omega_1\}$. Now if a is the common terminal vertex of the two edges, then identify $\kappa \times \{a\}$ with the successor ordinals in R. If b is the initial vertex of the green edge, then identify $\kappa \times \{b\}$ with $\{y_\alpha : \alpha \in \kappa\}$. If c is the initial vertex of the red edge, then identify $\kappa \times \{c\}$ with $\{\alpha \in \kappa : cf(\alpha) = \omega_1\}$ in an order-preserving manner. Thus we construct a topology so that $\kappa \times \{a, b, c\}$ is homeomorphic to a subspace of $\beta(\{\alpha \in \kappa : cf(\alpha) \neq \omega\})$*
- *Suppose the component is the graph formed by identifying the terminal vertices of a green edge and a yellow edge. Let S be the space which is the subspace of the Stone-Čech compactification of κ which consists of $\kappa \cup \{y_\alpha : \alpha \in \kappa\} \cup \{x_\alpha : \alpha \in \kappa\}$. Now if a is the common terminal vertex of the edges, then identify $\kappa \times \{a\}$ with $\kappa \subset \beta\kappa$. If b is the initial vertex of the green edge, then identify $\kappa \times \{b\}$ with $\{y_\alpha : \alpha \in \kappa\}$. If c is the initial vertex of the yellow edge, then identify $\kappa \times \{c\}$ with $\{x_\alpha : \alpha \in \kappa\}$. Thus we construct a topology so that $\kappa \times \{a, b, c\}$ is homeomorphic to a subspace of $\beta\kappa$.*
- *Suppose the component is the graph formed by a single green edge. Let T be the space which is the subspace of the Stone-Čech compactification of κ which consists of $\kappa \cup \{y_\alpha : \alpha \in \kappa\}$. Now if a is the terminal vertex of the edge, then identify $\kappa \times \{a\}$ with $\kappa \subset \beta\kappa$. If b is the initial vertex of the edge, then identify $\kappa \times \{b\}$ with $\{y_\alpha : \alpha \in \kappa\}$. Thus we construct a topology so that $\kappa \times \{a, b\}$ is homeomorphic to a subspace of $\beta\kappa$.*

This completes the construction of a topology on $\kappa \times V$.

Note that the construction requires κ to satisfy

1. $\kappa^\omega = \kappa$,
2. $\kappa \leq 2^{2^\omega}$,
3. $cf(\kappa) > \omega_1$.

In a ZFC construction we can let $\kappa = (2^\omega)^+$.

Proof of the Theorem. Apply Construction 1 to Example 1 using the light edges to get σ_0 and using the dark edges to get σ_1.

First we show that $\sigma_0 \vee \sigma_1$ is the discrete topology. This is a direct consequence of property 2 in Example 1 since this implies the stronger property that each point is an isolated point in one of the topologies.

Next we show that $\sigma_0 \wedge \sigma_1$ is the cofinite topology. Suppose U is a nonempty open set in $\sigma_0 \wedge \sigma_1$. If (x, y) is any edge, then any open set in $\sigma_0 \wedge \sigma_1$ which contains any point of $\kappa \times \{x\}$ must contain some point of $\kappa \times \{y\}$. Since there is a path from any vertex to any other vertex, this implies that U must intersect each $\kappa \times \{v\}$.

We now look at the edges mentioned in property 4 of Example 1 and use the fact that U must intersect $\kappa \times \{v\}$ where v is the initial vertex of the dark yellow edge mentioned there. By the subspace topology put on the third type of component, U must then contain a stationary subset of $\kappa \times \{w\}$ where w is the terminal vertex of the dark yellow edge. Now w is the initial vertex of a light red edge and so, by the subspace topology put on the second type of component, U must then contain all but a subset of size less than κ of $\kappa \times \{p\}$ where p is the terminal vertex of this light red edge. Now p is the initial vertex of a dark green edge. By the subspace topology put on the fourth type of component, U must then contain all but finitely-many points of $\kappa \times \{q\}$ where q is the terminal vertex of this dark green edge. Now we can use property 5 of Example 1 repeatedly to show that U is cofinite.

Next we note that property 3 of Example 1 and the definition of the topologies σ_0 and σ_1 imply that the two topologies are homeomorphic by a homeomorphism induced by the product of the identity and the directed graph isomorphism.

Finally we note that σ_0 (and so σ_1) are Tychonoff (and even zero-dimensional). This follows directly from the fact that each of these topologies decomposes as a free union of the four topologies constructed in Construction 1. Of these four topologies, two are subspaces of $\beta\kappa$, one is a subspace of the Stone-Čech compactification of a subspace of an ordinal space and the remaining one is a subspace of a Hausdorff quotient space of two compact Hausdorff spaces.

Conjecture 1. *It is consistent that no Hausdorff topology which is its own T_1 complement can lie on a set of cardinality $\aleph_1$.*

After this example was presented in Prague in 1986, Bohdan Aniszczyk constructed two compact Hausdorff topologies which are T_1-complementary.

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