BASIC DIFFERENTIAL FORMS
FOR ACTIONS OF LIE GROUPS

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ABSTRACT. A section of a Riemannian $G$-manifold $M$ is a closed submanifold $\Sigma$ which meets each orbit orthogonally. It is shown that the algebra of $G$-invariant differential forms on $M$ which are horizontal in the sense that they kill every vector which is tangent to some orbit, is isomorphic to the algebra of those differential forms on $\Sigma$ which are invariant with respect to the generalized Weyl group of $\Sigma$, under some condition.

1. Introduction

A section of a Riemannian $G$-manifold $M$ is a closed submanifold $\Sigma$ which meets each orbit orthogonally. This notion was introduced by Szenthe [26], [27], and in a slightly different form by Palais and Terng in [19], [20]. The case of linear representations was considered by Bott and Samelson [4] and Conlon [9], and then by Dadok [10] who called representations admitting sections polar representations and completely classified all polar representations of connected compact Lie groups. Conlon [8] considered Riemannian manifolds admitting flat sections. We follow here the notion of Palais and Terng.

If $M$ is a Riemannian $G$-manifold which admits a section $\Sigma$, then the trace on $\Sigma$ of the $G$-action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$. Palais and Terng [19] showed that then the algebras of invariant smooth functions coincide, $C^\infty(M, \mathbb{R})^G \cong C^\infty(\Sigma, \mathbb{R})^{W(\Sigma)}$.

In this paper we will extend this result to the algebras of differential forms. Our aim is to show that pullback along the embedding $\Sigma \to M$ induces an isomorphism $\Omega^p_{\text{hor}}(M)^G \cong \Omega^p(\Sigma)^{W(\Sigma)}$ for each $p$, where a differential form $\omega$ on $M$ is called horizontal if it kills each vector tangent to some orbit. For each point $x$ in $M$, the slice representation of the isotropy group $G_x$ on the normal space $T_x(G.x)^\perp$ to the tangent space to the orbit through $x$ is a polar representation. The first step is to show that the result holds for polar representations. This is done in Theorem 3.7 for polar representations whose generalized Weyl group is really a Coxeter group, i.e., is generated by reflections. Every polar representation of a connected compact Lie group has this property. The method used there is inspired by Solomon [25]. Then the general result is proved under the assumption that each slice representation

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has a Coxeter group as a generalized Weyl group. The last section gives some perspective to the result.

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2. Basic differential forms

2.1. Basic differential forms. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and multiplication $\mu : G \times G \to G$, and for $g \in G$ let $\mu_g, \mu^g : G \to G$ denote the left and right translations.

Let $\ell : G \times M \to M$ be a left action of the Lie group $G$ on a smooth manifold $M$. We consider the partial mappings $\ell_g : M \to M$ for $g \in G$ and $\ell^x : G \to M$ for $x \in M$ and the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ given by $\zeta_X(x) = T_x(\ell^x)X$.

Since $\ell$ is a left action, the negative $-\zeta$ is a Lie algebra homomorphism.

A differential form $\varphi \in \Omega^p(M)$ is called $G$-invariant if $(\ell_g)^* \varphi = \varphi$ for all $g \in G$ and horizontal if $\varphi$ kills each vector tangent to a $G$-orbit: $i_{\zeta_X} \varphi = 0$ for all $X \in \mathfrak{g}$.

We denote by $\Omega^p_{\text{hor}}(M)^G$ the space of all horizontal $G$-invariant $p$-forms on $M$. They are also called basic forms.

2.2. Lemma. Under the exterior differential $\Omega^p_{\text{hor}}(M)^G$ is a subcomplex of $\Omega(M)$.

Proof. If $\varphi \in \Omega^p_{\text{hor}}(M)^G$, then the exterior derivative $d \varphi$ is clearly $G$-invariant. For $X \in \mathfrak{g}$ we have

$$i_{\zeta_X} d \varphi = i_{\zeta_X} d \varphi + d i_{\zeta_X} \varphi = \mathcal{L}_{\zeta_X} \varphi = 0,$$

so $d \varphi$ is also horizontal.

2.3. Sections. Let $M$ be a connected complete Riemannian manifold, and let $G$ be a Lie group which acts isometrically on $M$ from the left. A connected closed smooth submanifold $\Sigma$ of $M$ is called a section for the $G$-action, if it meets all $G$-orbits orthogonally.

Equivalently we require that $G.\Sigma = M$ and that for each $x \in \Sigma$ and $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X(x)$ is orthogonal to $T_x \Sigma$.

We only remark here that each section is a totally geodesic submanifold and is given by $\exp(T_x(x, G)^\perp)$ if $x$ lies in a principal orbit.

If we put $N_G(\Sigma) := \{ g \in G : g.\Sigma = \Sigma \}$ and $Z_G(\Sigma) := \{ g \in G : g.s = s \text{ for all } s \in \Sigma \}$, then the quotient $W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$ turns out to be a discrete group acting properly on $\Sigma$. It is called the generalized Weyl group of the section $\Sigma$.

See [19] or [20] for more information on sections and their generalized Weyl groups.

2.4. Main Theorem. Let $M \times G \to M$ be a proper isometric right action of a Lie group $G$ on a smooth Riemannian manifold $M$, which admits a section $\Sigma$. Let us assume that

(1) For each $x \in \Sigma$ the slice representation $G_x \to O(T_x(G,x)^\perp)$ has a generalized Weyl group which is a reflection group (see section 3).

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^W(\Sigma)$$
between the space of horizontal $G$-invariant differential forms on $M$ and the space
of all differential forms on $\Sigma$ which are invariant under the action of the generalized
Weyl group $W(\Sigma)$ of the section $\Sigma$.

The proof of this theorem will take up the rest of this paper. According to Dadok
[10], remark after Proposition 6, for any polar representation of a connected compact
Lie group the generalized Weyl group $W(\Sigma)$ is a reflection group, so condition
(1) holds if we assume that:
(2) Each isotropy group $G_x$ is connected.

Proof of injectivity. Let $i : \Sigma \rightarrow M$ be the embedding of the section. We claim
that $i^* : \Omega^p_\text{hor}(M)^G \rightarrow \Omega^p(\Sigma)^{W(\Sigma)}$ is injective. Let $\omega \in \Omega^p_\text{hor}(M)^G$ with $i^* \omega = 0$.
For $x \in \Sigma$ we have $i_X \omega_x = 0$ for $X \in T_x \Sigma$ since $i^* \omega = 0$, and also for $X \in T_x(G,x)$
since $\omega$ is horizontal. Let $x \in \Sigma \cap M_{\text{reg}}$ be a regular point; then $T_x \Sigma = (T_x(G,x))^L$
and so $\omega_x = 0$. This holds along the whole orbit through $x$ since $\omega$ is $G$-invariant.
Thus $\omega|_{M_{\text{reg}}} = 0$, and since $M_{\text{reg}}$ is dense in $M$, $\omega = 0$.

So it remains to show that $i^*$ is surjective. This will be done in 4.2 below. $\square$

3. Representations

3.1. Invariant functions. Let $G$ be a reductive Lie group and let $\rho : G \rightarrow GL(V)$
be a representation in a finite dimensional real vector space $V$.

According to a classical theorem of Hilbert (as extended by Nagata [15], [16]), the
algebra of $G$-invariant polynomials $\mathbb{R}[V]^G$ on $V$ is finitely generated (in fact finitely
presented), so there are $G$-invariant homogeneous polynomials $f_1, \ldots, f_m$ on $V$ such
that each invariant polynomial $h \in \mathbb{R}[V]^G$ is of the form $h = q(f_1, \ldots, f_m)$ for a
polynomial $q \in \mathbb{R}[\mathbb{R}^m]$. Let $f = (f_1, \ldots, f_m) : V \rightarrow \mathbb{R}^m$; then this means that the
pullback homomorphism $f^* : \mathbb{R}[\mathbb{R}^m] \rightarrow \mathbb{R}[V]^G$ is surjective.

D. Luna proved in [14] that the pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \rightarrow
C^\infty(V, \mathbb{R})^G$ is also surjective onto the space of all smooth functions on $V$ which are
constant on the fibers of $f$. Note that the polynomial mapping $f$ in this case may
not separate the $G$-orbits.

G. Schwarz proved already in [23] that if $G$ is a compact Lie group, then the
pullback homomorphism $f^* : C^\infty(\mathbb{R}^m, \mathbb{R}) \rightarrow C^\infty(V, \mathbb{R})^G$ is actually surjective onto
the space of $G$-invariant smooth functions. This result implies in particular that $f$
separates the $G$-orbits.

3.2. Lemma. Let $\ell \in V^*$ be a linear functional on a finite dimensional vector
space $V$, and let $f \in C^\infty(V, \mathbb{R})$ be a smooth function which vanishes on the kernel
of $\ell$, so that $f|_{\ell^{-1}(0)} = 0$. Then there is a unique smooth function $g$ such that
$f = \ell g$.

Proof. Choose coordinates $x^1, \ldots, x^n$ on $V$ with $\ell = x^1$. Then $f(0, x^2, \ldots, x^n) = 0$
and we have $f(x^1, \ldots, x^n) = \int_0^1 \partial_1 f(tx^1, x^2, \ldots, x^n)dt.x^1 = g(x^1, \ldots, x^n).x^1$. $\square$

3.3. Lemma. Let $W$ be a finite reflection group acting on a finite dimensional
vector space $\Sigma$. Let $f = (f_1, \ldots, f_n) : \Sigma \rightarrow \mathbb{R}^n$ be the polynomial map whose
components $f_1, \ldots, f_n$ are a minimal set of homogeneous generators of the algebra
$\mathbb{R}[\Sigma]^W$ of $W$-invariant polynomials on $\Sigma$. Then the pullback homomorphism $f^* : \Omega^p(\mathbb{R}^n) \rightarrow \Omega^p(\Sigma)$
is surjective onto the space $\Omega^p(\Sigma)^W$ of $W$-invariant differential
forms on $\Sigma$. 
For polynomial differential forms and more general reflection groups this is the main theorem of Solomon [25]. We adapt his proof to our needs.

**Proof.** The polynomial generators \( f_i \) form a set of algebraically independent polynomials, \( n = \dim \Sigma \), and their degrees \( d_1, \ldots, d_n \) are uniquely determined up to order. We even have (see [12]):

\[
\begin{align*}
(1) & \quad d_1 \cdots d_n = |W|, \text{ the order of } W, \\
(2) & \quad d_1 + \cdots + d_n = n + N, \text{ where } N \text{ is the number of reflections in } W.
\end{align*}
\]

Let us consider the mapping \( f = (f_1, \ldots, f_n) : \Sigma \to \mathbb{R}^n \) and its Jacobian \( J(x) = \det(df(x)) \). Let \( x^1, \ldots, x^n \) be coordinate functions in \( \Sigma \). Then for each \( \sigma \in W \) we have

\[
J \cdot dx^1 \wedge \cdots \wedge dx^n = df_1 \wedge \cdots \wedge df_n = \sigma^*(df_1 \wedge \cdots \wedge df_n)
\]

\[
= (J \circ \sigma) \sigma^*(dx^1 \wedge \cdots \wedge dx^n) = (J \circ \sigma) \det(\sigma)(dx^1 \wedge \cdots \wedge dx^n),
\]

(3) \( J \circ \sigma = \det(\sigma^{-1})J. \)

The generators \( f_1, \ldots, f_n \) are algebraically independent over \( \mathbb{R} \), thus \( J \neq 0 \). Since \( J \) is a polynomial of degree \( (d_1-1) + \cdots + (d_n-1) = N \) (see (2)), the \( W \)-invariant set \( U = \Sigma \setminus J^{-1}(0) \) is open and dense in \( \Sigma \); by the inverse function theorem \( J \) is a local diffeomorphism on \( U \), and the \( 1 \)-forms \( df_1, \ldots, df_n \) are a coframe on \( U \).

Now let \( (\sigma_\alpha)_{\alpha=1, \ldots, N} \) be the set of reflections in \( W \), with reflection hyperplanes \( H_\alpha \). Let \( \ell_\alpha \in \Sigma^* \) be linear functionals with \( H_\alpha = \ell_\alpha^{-1}(0) \). If \( x \in H_\alpha \) we have \( J(x) = \det(\sigma_\alpha)J(\sigma_\alpha \cdot x) = -J(x) \), so that \( J|H_\alpha = 0 \) for each \( \alpha \), and by Lemma 3.2 we have

(4) \( J = c.\ell_1 \cdots \ell_N. \)

Since \( J \) is a polynomial of degree \( N \), \( c \) must be a constant. Repeating the last argument for an arbitrary function \( g \) and using (4), we get:

\[
(5) \quad \text{If } g \in C^\infty(\Sigma, \mathbb{R}) \text{ satisfies } g \circ \sigma = \det(\sigma^{-1})g \text{ for each } \sigma \in W, \text{ we have } g = J \cdot h \text{ for } h \in C^\infty(\Sigma, \mathbb{R})^W.
\]

After these preparations we turn to the assertion of the lemma. Let \( \omega \in \Omega^p(\Sigma)^W \).

Since the \( 1 \)-forms \( df_j \) form a coframe on \( U \), we have

\[
\omega|U = \sum_{j_1, \ldots, j_p} g_{j_1, \ldots, j_p} df_{j_1}|U \wedge \cdots \wedge df_{j_p}|U
\]

for \( g_{j_1, \ldots, j_p} \in C^\infty(U, \mathbb{R}) \). Since \( \omega \) and all \( df_j \) are \( W \)-invariant, we may replace \( g_{j_1, \ldots, j_p} \) by their averages over \( W \), or assume without loss that \( g_{j_1, \ldots, j_p} \in C^\infty(U, \mathbb{R})^W \).

Let us choose now a form index \( i_1 < \cdots < i_p \) with \( \{i_{p+1} < \cdots < i_n\} = \{1, \ldots, n\} \setminus \{i_1 < \cdots < i_p\} \). Then for some sign \( \varepsilon = \pm 1 \) we have

\[
\omega|U \wedge df_{i_{p+1}} \wedge \cdots \wedge df_{i_n} = \varepsilon g_{i_{i_1} \cdots i_p} df_1 \wedge \cdots \wedge df_n = \varepsilon g_{i_{i_1} \cdots i_p} J \cdot dx^1 \wedge \cdots \wedge dx^n,
\]

(6) \( \omega \wedge df_{i_{p+1}} \wedge \cdots \wedge df_{i_n} = \varepsilon k_{i_{i_1} \cdots i_p} dx^1 \wedge \cdots \wedge dx^n \)

for a function \( k_{i_{i_1} \cdots i_p} \in C^\infty(\Sigma, \mathbb{R}) \). Thus

(7) \( k_{i_{i_1} \cdots i_p}|U = g_{i_{i_1} \cdots i_p} \cdot J|U. \)

Since \( \omega \) and each \( df_j \) is \( W \)-invariant, from (6) we get \( k_{i_{i_1} \cdots i_p} \circ \sigma = \det(\sigma^{-1})k_{i_{i_1} \cdots i_p} \) for each \( \sigma \in W \). But then by (5) we have \( k_{i_{i_1} \cdots i_p} = \omega_{i_{i_1} \cdots i_p} J \) for unique \( \omega_{i_{i_1} \cdots i_p} \in C^\infty(\Sigma, \mathbb{R})^W \), and (7) then implies \( \omega_{i_{i_1} \cdots i_p}|U = g_{i_{i_1} \cdots i_p} \), so that the lemma follows since \( U \) is dense. \( \square \)
3.4. Question. Let \( \rho : G \to GL(V) \) be a representation of a compact Lie group in a finite dimensional vector space \( V \). Let \( f = (f_1, \ldots, f_m) : V \to \mathbb{R}^m \) be the polynomial mapping whose components \( f_i \) are a minimal set of homogeneous generators for the algebra \( \mathbb{R}[V]^G \) of invariant polynomials.

We consider the pullback homomorphism \( f^* : \Omega^p(\mathbb{R}^m) \to \Omega^p(V) \). Is it surjective onto the space \( \Omega^p_{\text{hor}}(V)^G \) of \( G \)-invariant horizontal smooth \( p \)-forms on \( V \)?

The proof of Theorem 3.7 below will show that the answer is yes for polar representations of compact Lie groups if the corresponding generalized Weyl group is a reflection group.

In general the answer is no. A counterexample is the following: Let the cyclic group \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \) of order \( n \), viewed as the group of \( n \)-th roots of unity, act on \( \mathbb{C} = \mathbb{R}^2 \) by complex multiplication. A generating system of polynomials consists of \( f_1 = |z|^2 \), \( f_2 = \text{Re}(z^n) \), \( f_3 = \text{Im}(z^n) \). But then each \( df_i \) vanishes at 0 and there is no chance to have the horizontal invariant volume form \( dx \wedge dy \) in \( f^*\Omega^{3}(\mathbb{R}^3) \).

3.5. Polar representations. Let \( G \) be a compact Lie group and let \( \rho : G \to GL(V) \) be an orthogonal representation in a finite dimensional real vector space \( V \) which admits a section \( \Sigma \). Then the section turns out to be a linear subspace and the representation is called a polar representation, following Dadok [10], who gave a complete classification of all polar representations of connected Lie groups. They were called variationally complete representations by Conlon [9] before.

3.6. Theorem (Terng [28], Theorem D or [19], 4.12). Let \( \rho : G \to GL(V) \) be a polar representation of a compact Lie group \( G \), with section \( \Sigma \) and generalized Weyl group \( W = W(\Sigma) \). Then the algebra \( \mathbb{R}[V]^G \) of \( G \)-invariant polynomials on \( V \) is isomorphic to the algebra \( \mathbb{R}[\Sigma]^W \) of \( W \)-invariant polynomials on the section \( \Sigma \), via the restriction mapping \( f \mapsto f|\Sigma \).

3.7. Theorem. Let \( \rho : G \to GL(V) \) be a polar representation of a compact Lie group \( G \), with section \( \Sigma \) and generalized Weyl group \( W = W(\Sigma) \). Let us suppose that \( W = W(\Sigma) \) is generated by reflections (a reflection group or Coxeter group). Then the pullback to \( \Sigma \) of differential forms induces an isomorphism

\[
\Omega^p_{\text{hor}}(V)^G \cong \Omega^p(\Sigma)^W(\Sigma).
\]

According to Dadok [10], remark after Proposition 6, for any polar representation of a connected compact Lie group the generalized Weyl group \( W(\Sigma) \) is a reflection group. This theorem is true for polynomial differential forms, and also for real analytic differential forms, by essentially the same proof.

Proof. Let \( i : \Sigma \to V \) be the embedding. By the first part of the proof of Theorem 2.4 the pullback mapping \( i^* : \Omega^p_{\text{hor}}(V)^G \to \Omega^p_{\text{hor}}(\Sigma)^W \) is injective, and we shall show that it is also surjective. Let \( f_1, \ldots, f_n \) be a minimal set of homogeneous generators of the algebra \( \mathbb{R}[\Sigma]^W \) of \( W \)-invariant polynomials on \( \Sigma \). Then by Lemma 3.3 each \( \omega \in \Omega^p(\Sigma)^W \) is of the form

\[
\omega = \sum_{j_1 < \cdots < j_p} \omega_{j_1 \cdots j_p} df_{j_1} \wedge \cdots \wedge df_{j_p},
\]

where \( \omega_{j_1 \cdots j_p} \in C^\infty(\Sigma, \mathbb{R})^W \). By Theorem 3.6 the algebra \( \mathbb{R}[V]^G \) of \( G \)-invariant polynomials on \( V \) is isomorphic to the algebra \( \mathbb{R}[\Sigma]^W \) of \( W \)-invariant polynomials on \( \Sigma \).
the section $\Sigma$, via the restriction mapping $i^*$. Choose polynomials $\tilde{f}_1, \ldots, \tilde{f}_n \in \mathbb{R}[V]^G$ with $\tilde{f}_i \circ i = f_i$ for all $i$. Put $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n): V \to \mathbb{R}^n$. Then we use the theorem of G. Schwarz (see 3.1) to find $h_{i_1, \ldots, i_p} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $h_{i_1, \ldots, i_p} \circ f_1 = \omega_{i_1, \ldots, i_p}$ and consider

$$\tilde{\omega} = \sum_{j_1 < \cdots < j_p} (h_{j_1, \ldots, j_p} \circ \tilde{f}) d\tilde{f}_{j_1} \wedge \cdots \wedge d\tilde{f}_{j_p},$$

which is in $\Omega^p_{nor}(V)^G$ and satisfies $i^*\tilde{\omega} = \omega$. \hfill \Box

**Sketch of another proof avoiding 3.3** (suggested by a referee). Let $R = C^\infty(V)^G = C^\infty(\Sigma)^W$ and let $\Omega^p_R$ be its module of Kähler $p$-forms (see Kunz [13] for the notion of Kähler forms). Also let $S = \mathbb{R}[V]^G = \mathbb{R}[\Sigma]^W$ (using 3.6). Then the canonical mapping $\Omega^p_R \to \Omega^p(\Sigma)^W$ is surjective. This follows for the canonical mapping from $\Omega^p_S$ into the space of forms with polynomial coefficients from the result of Solomon [25] by using 3.6 again as in the proof of 3.7; and it can be extended to smooth coefficients by Theorem 1.4 of Ronga [22], which says that equivariant stability and infinitesimal equivariant stability are equivalent, in a way which is similar to the argument of Proposition 6.8 of Schwarz [24]. So we see that the composition $\Omega^p_R \to \Omega^p(V)^G \to \Omega^p(\Sigma)^W$ is surjective, thus also the right hand side mapping has to be surjective.

**3.8. Corollary.** Let $\rho: G \to O(V, (\quad, \quad))$ be an orthogonal polar representation of a compact Lie group $G$, with section $\Sigma$ and generalized Weyl group $W = W(\Sigma)$. Let us suppose that $W = W(\Sigma)$ is generated by reflections (a reflection group or Coxeter group). Let $B \subset V$ be an open ball centered at 0.

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{nor}(B)^G \cong \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$ 

**Proof.** Check the proof of 3.7 or use the following argument. Suppose that $B = \{v \in V : |v| < 1\}$ and consider a smooth diffeomorphism $f: [0, 1) \to [0, \infty)$ with $f(t) = t$ near 0. Then $g(v) := \frac{f(|v|)}{|v|} v$ is a $G$-equivariant diffeomorphism $B \to V$ and by 3.7 we get:

$$\Omega^p_nor(B)^G \xrightarrow{(g^{-1})^*} \Omega^p_nor(V)^G \cong \Omega^p(\Sigma)^{W(\Sigma)} \xrightarrow{g^*} \Omega^p(\Sigma \cap B)^{W(\Sigma)}. \hfill \Box$$

**4. Proof of the main theorem**

Let us assume that we are in the situation of the main theorem 2.4, for the rest of this section.

**4.1.** For $x \in M$ let $S_x$ be a (normal) slice and $G_x$ the isotropy group, which acts on the slice. Then $G \times S_x$ is open in $M$ and $G$-equivariantly diffeomorphic to the associated bundle $G \to G/G_x$ via

$$G \times S_x \xrightarrow{q} G \times_{G_x} S_x \xrightarrow{\cong} G.S_x$$

$$\downarrow \quad \Downarrow r$$

$$G/G_x \xrightarrow{\cong} G.x$$
where $r$ is the projection of a tubular neighborhood. Since $q : G \times S_x \to G \times G_x S_x$ is a principal $G_x$-bundle with principal right action $(g, s) h = (gh, h^{-1} s)$, we have an isomorphism

$$q^* : \Omega(G \times G_x S_x) \to \Omega(G_x - \text{hor})(G \times S_x)^{G_x}.$$ 

Since $q$ is also $G$-equivariant for the left $G$-actions, the isomorphism $q^*$ maps the subalgebra $\Omega_{\text{hor}}^p(G.S_x)^G \cong \Omega_{\text{hor}}^p(G \times G_x S_x)^G$ of $\Omega(G \times G_x S_x)$ to the subalgebra $\Omega_{G_x - \text{hor}}^p(S_x)^{G_x}$ of $\Omega(G_x - \text{hor})(G \times S_x)^{G_x}$. So we have proved:

**Lemma.** In this situation there is a canonical isomorphism

$$\Omega_{\text{hor}}^p(G.S_x)^G \cong \Omega_{G_x - \text{hor}}^p(S_x)^{G_x}$$

which is given by pullback along the embedding $S_x \to G.S_x$.

**4.2. Rest of the proof of Theorem 2.4.** Now let us consider $\omega \in \Omega^p(\Sigma)^{W(\Sigma)}$. We want to construct a form $\tilde{\omega} \in \Omega_{\text{hor}}^p(M)^G$ with $i^* \tilde{\omega} = \omega$. This will finish the proof of Theorem 2.4.

Choose $x \in \Sigma$ and an open ball $B_x$ with center 0 in $T_x M$ such that the Riemannian exponential mapping $\exp_x : T_x M \to M$ is a diffeomorphism on $B_x$. We consider now the compact isotropy group $G_x$ and the slice representation $\rho_x : G_x \to O(V_x)$, where $V_x = \text{Nor}_x(G.x) = (T_x(G.x))^\perp \subset T_x M$ is the normal space to the orbit. This is a polar representation with section $T_x \Sigma$, and its generalized Weyl group is given by $W(T_x \Sigma) \cong N_G(\Sigma) \cap G_x/Z_G(\Sigma) = W(\Sigma)_x$ (see [19]); it is a Coxeter group by assumption (1) in 2.4. Then $\exp_x : B_x \cap V_x \to S_x$ is a diffeomorphism onto a slice and $\exp_x : B_x \cap T_x \Sigma \to \Sigma$ is a diffeomorphism onto an open neighborhood $\Sigma_x$ of $x$ in the section $\Sigma$.

Let us now consider the pullback $(\exp | B_x \cap T_x \Sigma)^* \omega \in \Omega^p(B_x \cap T_x \Sigma)^{W(T_x \Sigma)}$. By Corollary 3.8 there exists a unique form $\varphi^x \in \Omega_{G_x - \text{hor}}^p(B_x \cap V_x)^{G_x}$ such that $i^* \varphi^x = (\exp | B_x \cap T_x \Sigma)^* \omega$, where $i_x$ is the embedding. Then we have

$$((\exp | B_x \cap V_x)^{-1})^* \varphi^x \in \Omega_{G_x - \text{hor}}^p(S_x)^{G_x}$$

and by Lemma 4.1 this form corresponds uniquely to a differential form $\omega^x \in \Omega_{\text{hor}}^p(G.S_x)^G$ which satisfies $(i | \Sigma_x)^* \omega^x = \omega | \Sigma_x$, since the exponential mapping commutes with the respective restriction mappings. Now the intersection $G.S_x \cap \Sigma$ is the disjoint union of all the open sets $w_j(\Sigma_x)$ where we pick one $w_j$ in each left coset of the subgroup $W(\Sigma)_x$ in $W(\Sigma)$. If we choose $g_j \in N_G(\Sigma)$ projecting on $w_j$ for all $j$, then

$$(i | w_j(\Sigma_x))^* \omega^x = (\ell_{g_j} \circ i | \Sigma_x \circ w_j^{-1})^* \omega^x = (w_j^{-1})^* (i | \Sigma_x)^* \ell_{g_j}^* \omega^x = (w_j^{-1})^* (\omega | \Sigma_x) = \omega | w_j(\Sigma_x),$$

so that $(i | G.S_x \cap \Sigma)^* \omega^x = \omega | G.S_x \cap \Sigma$. We can do this for each point $x \in \Sigma$.

Using the method of Palais ([18], proof of 4.3.1) we may find a sequence of points $(x_n)_{n \in \mathbb{N}}$ in $\Sigma$ such that the $\pi(\Sigma_{x_n})$ form a locally finite open cover of the orbit space $M/G \cong \Sigma/W(\Sigma)$, and a smooth partition of unity $f_n$ consisting of $G$-invariant functions with $\text{supp}(f_n) \subset G.S_{x_n}$. Then $\omega := \sum_n f_n \omega^x \in \Omega^p(M)^G$ has the required property $i^* \omega = \omega$. \qed
5. Basic versus equivariant cohomology

5.1. Basic cohomology. For a Lie group $G$ and a smooth $G$-manifold $M$, by 2.2 we may consider the basic cohomology $H^p_G(M) = H^p(\Omega^*_\text{hor}(M))^G$, $d$.

The best known application of basic cohomology is the case of a compact connected Lie group $G$ acting on itself by left translations; see e.g. [11] and papers cited therein: By homotopy invariance and integration we get $H(G) = H_G(M)$, and the latter space turns out as the space $\Lambda(g^*)^G$ of $\text{ad}(g)$-invariant forms, using the inversion. This is the theorem of Chevalley and Eilenberg. Moreover, $\Lambda(g^*)^G = \Lambda(P)$, where $P$ is the graded subspace of primitive elements, using the Weil map and transgression, whose determination in all concrete cases by Borel and Hirzebruch is a beautiful part of modern mathematics.

In more general cases the determination of basic cohomology was more difficult. A replacement for it is equivariant cohomology, which comes in two guises:

5.2. Equivariant cohomology, Borel model. For a topological group and a topological $G$-space the equivariant cohomology was defined as follows; see [3]: Let $EG \to BG$ be the classifying $G$-bundle, and consider the associated bundle $EG \times_G M$ with standard fiber the $G$-space $M$. Then the equivariant cohomology is given by $H^p(EG \times_G M; \mathbb{R})$.

5.3. Equivariant cohomology, Cartan model. For a Lie group $G$ and a smooth $G$-manifold $M$ we consider the space $$(S^k g^* \otimes \Omega^p(M))^G$$
of all homogeneous polynomial mappings $\alpha : g \to \Omega^p(M)$ of degree $k$ from the Lie algebra $g$ of $G$ to the space of $p$-forms, which are $G$-equivariant: $\alpha(\text{Ad}(g^{-1})X) = \ell_g^* \alpha(X)$ for all $g \in G$. The mapping $$d_g : A^q_G(M) \to A^{q+1}_G(M),$$ $$A^q_G(M) := \bigoplus_{2k+p=q} (S^k g^* \otimes \Omega^p(M))^G,$$ $$(d_g \alpha)(X) := d(\alpha(X)) - i_{\xi_X} \alpha(X)$$satisfies $d_g \circ d_g = 0$ and the following result holds.

Theorem. Let $G$ be a compact connected Lie group and let $M$ be a smooth $G$-manifold. Then $$H^p(EG \times_G M; \mathbb{R}) = H^p(A^*_G(M), d_g).$$

This result is stated in [1] together with some arguments, and it is attributed to [5], [6] in chapter 7 of [2]. I was unable to find a satisfactory published proof.

5.4. Let $M$ be a smooth $G$-manifold. Then the obvious embedding $j(\omega) = 1 \otimes \omega$ gives a mapping of graded differential algebras $$j : \Omega^p_{\text{hor}}(M)^G \to (S^0 g^* \otimes \Omega^p(M))^G \to \bigoplus_k (S^k g^* \otimes \Omega^{p-2k}(M))^G = A^p_G(M).$$
On the other hand evaluation at \(0 \in \mathfrak{g}\) defines a homomorphism of graded differential algebras \(ev_0 : A^*_G(M) \to \Omega^*(M)^G\), and \(ev_0 \circ j\) is the embedding \(\Omega^*_\text{hor}(M)^G \to \Omega^*(M)^G\). Thus we get canonical homomorphisms in cohomology

\[
\begin{align*}
H^p(\Omega^*_\text{hor}(M)^G) & \xrightarrow{J^*} H^p(A^*_G(M), d_g) \quad \xrightarrow{d} H^p(\Omega^*(M)^G, d) \\
H^p_G(\text{basic}) & \quad \xrightarrow{d} H^p_G(M) \quad \xrightarrow{d} H^p(M)^G.
\end{align*}
\]

If \(G\) is compact and connected we have \(H^p(M)^G = H^p(M)\), by integration and homotopy invariance.

References


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