DEGREE OF IRRATIONALITY OF A PRODUCT OF TWO ELLIPTIC CURVES

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ABSTRACT. We consider the degree of irrationality \(d_r(S)\) of some algebraic surface \(S\). Firstly we give an estimate of \(d_r(S)\) for a surface \(S\) with a structure of a fiber space. Secondly we prove the existence of a nonsingular curve of genus 3 on \(E \times E\) for a certain elliptic curve \(E\) with complex multiplications. As a corollary, we obtain that \(d_r(E \times E) = 3\).

1. INTRODUCTION

Let \(V\) be an \(n\)-dimensional algebraic variety defined over a field \(k\), and let \(k(V)\) be the rational function field of \(V\). The degree of irrationality of \(V\) is defined to be the least number \(m\) such that \(m = [k(V) : k(x_1, \ldots, x_n)]\), where \(x_1, \ldots, x_n\) are algebraically independent elements of \(k(V)\) (cf. [6], [9]). By definition this number is a birational invariant and we denote it by \(d_r(V)\). In other words it is the minimal degree of a dominant rational map from \(V\) to the projective \(n\)-space. In the case when \(n = 1\), \(d_r(V)\) coincides with the gonality of a curve and has been studied mainly for plane curves (see, e.g., [3]).

In what follows we assume that \(k = \mathbb{C}\) and we work in the category of algebraic varieties over \(\mathbb{C}\). When \(n = 2\) and \(d_r(V) = 2\), some results are obtained in [8]. For an abelian variety \(A\), it is proved that \(d_r(A) \geq n + 1\) in [1]. Clearly we have that \(d_r(A) = 2\) if \(n = 1\). It seems to be important to determine the value \(d_r(A)\) when \(n = 2\), but only a few results have been obtained; for example, if \(A\) is a double covering of a Jacobian variety of a curve, then \(d_r(A) = 3\) (see [7]). In this paper we will give an estimate of \(d_r\) for a surface with a structure of a fiber space and prove the existence of a nonsingular curve of genus 3 on \(E \times E\), where \(E\) is a certain elliptic curve with complex multiplications. As a corollary, we obtain that \(d_r(E \times E) = 3\).

2. STATEMENT OF RESULTS

First we present an estimate of \(d_r\) for a surface with a structure of a fiber space.

**Proposition 1** (cf. [7]). Let \(S\) and \(C\) be a nonsingular projective surface and curve, respectively. Suppose that there is a surjective morphism \(f: S \to C\), whose general
fiber \( F \) is irreducible. Let \( g(F) \) denote the genus of \( F \). Then we have the following assertions:

1. If \( g(F) = 0 \), then \( d_r(S) = d_r(C) \).
2. If \( g(F) = 1 \) and \( f \) has a section, then \( d_r(S) \leq 2d_r(C) \).
3. If \( g(F) \geq 2 \) and \( d_r(F) = 2 \), then \( d_r(S) \leq 2d_r(C) \).
4. If \( g(F) = 3 \), \( d_r(F) \neq 2 \) and \( f \) has a section, then \( d_r(S) \leq 3d_r(C) \).

If we drop the assumption that \( f \) has a section in (2), then the conclusion does not hold true. For example let \( S \) be a hyperelliptic surface; then it has a structure of an elliptic fiber space with multiple singular fibers. We can shown that

\[
S \not= \text{elliptic fiber space} \quad \text{with multiple singular fibers.}
\]

We can shown that

\[
de_n(S) \not= \text{true.}
\]

For example let

\[
D \not= \text{a nonsingular curve of genus 3 which admits an elliptic involution}
\]

Under the situation above, suppose that at least one of \( a, b, c \) is an even number. Then there exist two elliptic curves

\[
E \not= \text{an integer.}
\]

Hence every \( \xi \) enjoys the condition. For the remaining case, letting \( k \) and \( l \neq 0 \) be rational integers, we have the following.

(i) If \( -m \equiv 1 \) (mod 8), then \( \xi = k + l\omega \) and \( \frac{1}{2} + l\omega \) are the suitable ones.

(ii) If \( -m \equiv 5 \) (mod 8), then \( \xi = k + 2\omega \) and \( \frac{1}{2} + l\omega \) are the suitable ones.

However we notice the following assertion.

Proposition 5. Suppose that \( \xi = \omega \) and \( m = 3, 11, 19, 43, 67 \) or 163. Then there exist no elliptic curves \( E_1 \) and \( E_2 \) satisfying \( (E_1, E_2) = 2 \) on \( E \times E \).

Remark 6. When \( m = 3 \) and \( \xi = \omega \), we consider the quotient of \( E \times E \) by the automorphism \( (z_1, z_2) \mapsto (\omega z_1, \omega z_2) \). Then the quotient space turns out to be a rational surface and hence \( d_r(E \times E) = 3 \) (cf. [7]).

Example 7. As an application of Theorem 2, we take an example from [2, (1.8)]. Let \( D \) be a nonsingular curve of genus 3 which admits an elliptic involution \( \pi: D \rightarrow E \). Let \( \xi_0 \in D \) be a branchpoint for \( \pi \) and embed \( D \rightarrow J = \text{Pic}^0(D) \) via \( p \mapsto \mathcal{O}_D(p - \xi_0) \). By using \( \xi_0 = \pi\xi_0 \), we identity \( E \rightarrow \text{Pic}^0(E) \) via \( x \mapsto \mathcal{O}_E(x - x_0) \). Then we have a natural injection \( \pi^*: E \rightarrow J \). We put \( A = J/\pi^*E \); then the map

\[
D \mapsto J \rightarrow A
\]

turns into an embedding. Hence we have \( d_r(A) = 3 \).
3. Proof

First we prove Proposition 1. Let us treat the case (1). Since $S$ is birationally equivalent to $C \times \mathbb{P}^1$, we have that $d_r(S) = d_r(C \times \mathbb{P}^1)$. Then we get $d_r(C \times \mathbb{P}^1) = d_r(C)$ (cf. [9]). Proofs of (2), (3) and (4) are done simultaneously. Let $\mathcal{K}_S$ denote the canonical divisor on $S$ and let $\Gamma$ be the section in (2) and (4). Let $\mathcal{F}$ be the sheaf on $S$ equal to $\mathcal{O}_S(2\Gamma)$, $\mathcal{O}_S(\mathcal{K}_S + \mathcal{F})$ and $\mathcal{O}_S(\mathcal{K}_S - \Gamma)$, corresponding to the cases (2), (3) and (4) respectively. Since $f_*\mathcal{F}$ is a coherent sheaf on $C$, we have a projective fiber space $\mathbb{P}(f_*\mathcal{F}) \to C$ associated with $f_*\mathcal{F}$ and a rational map $g: S \to \mathbb{P}(f_*\mathcal{F})$. Let $X$ be the image of $g$. Then we see that $X$ is a ruled surface over $C$. In the case (2) or (3), the degree of $g$ is 2, hence we conclude that $d_r(S) \leq 2d_r(C)$ by (1). On the other hand, in the case (4), since

$$\dim H^0(F, \mathcal{O}(K_F - F \cap \Gamma)) = h^0(F, \mathcal{O}(K_F - F \cap \Gamma)) = 2$$

for a general fiber $F$ and it is not hyperelliptic, the rational map $g$ has degree 3. Hence, similarly we infer that $d_r(S) \leq 3d_r(C)$.

Next we prove Theorem 2. Let $C$ be the nonsingular curve of genus 3. Since $C^2 = (C, C) = 4$, we see that $C$ is ample and $h^0(A, \mathcal{O}(C)) = 2$ from the Riemann-Roch theorem. The rational map defined by the complete linear system $|C|$ has 4 base points. By blowing-up these points, we get a morphism $f: \tilde{A} \to \mathbb{P}^1$. Clearly $f$ has 4 sections. As we mentioned in the Introduction, we have that $d_r(A) \geq 3$, hence it is sufficient to show that a general fiber is not hyperelliptic. Suppose that except for finitely many fibers every fiber is hyperelliptic. Then we have $d_r(A) = 2$ by (3), which is a contradiction. Hence a general fiber must be non-hyperelliptic, because in the moduli space of curves of genus 3, hyperelliptic ones consist of an analytic subspace. Thus by (4) we obtain $d_r(A) = 3$.

Before the proof of Theorem 3 we provide two lemmas. The next one may be well-known.

**Lemma 8.** Let $E$ be an elliptic curve on an abelian surface $A$. Then $h^0(A, \mathcal{O}(E)) = 1$.

**Proof.** From the exact sequence of sheaves

$$0 \to \mathcal{O}_A(-E) \to \mathcal{O}_A \to \mathcal{O}_E \to 0,$$

we get the long exact sequence of cohomology groups

$$0 \to H^1(A, \mathcal{O}(-E)) \to H^1(A, \mathcal{O}_A) \to H^1(E, \mathcal{O}_E) \to H^2(A, \mathcal{O}(-E)) \to H^2(A, \mathcal{O}_A) \to 0.$$

From this sequence and by the Serre duality theorem, we infer that $h^0(A, \mathcal{O}(E)) = \dim H^1(A, \mathcal{O}(-E)) = h^1(A, \mathcal{O}(-E))$. On the other hand, referring to [4, p. 571], we see that $\dim \ker r$ is the number of linearly independent holomorphic 1-forms on $A$ which vanish on $E$. Whence we have that $h^1(A, \mathcal{O}(-E)) \leq 1$, which proves the assertion.

**Lemma 9.** If there are two elliptic curves $E_1$ and $E_2$ satisfying $(E_1, E_2) = 2$ on an abelian surface $A$, then there is a nonsingular curve of genus 3 on $A$.

**Proof.** Putting $D = E_1 + E_2$, we see that $D$ is an ample divisor and hence $h^0(A, \mathcal{O}(D)) = 2$. By the above lemma the pencil $|D|$ has no fixed component. Hence by Bertini’s theorem its general member is an irreducible nonsingular curve of genus 3 (cf. [2, (1.4)]).
Now we proceed to the proof of Theorem 3. Let \( \varphi_{\alpha,\beta} : E \to E \times E \) be a morphism defined by \( \varphi_{\alpha,\beta}(z) = (\alpha z, \beta z) \), where \( \alpha \) and \( \beta \) \( \in \text{End}(E) \). Note that \( \text{End}(E) \) is generated by 1 and \( \alpha \) over \( \mathbf{Z} \). Put \( E_{\alpha,\beta} = \varphi_{\alpha,\beta}(E) \). By taking a suitable \( (\alpha, \beta, \gamma, \delta) \), we may obtain that \( (E_{\alpha,\beta}, E_{\gamma,\delta}) = 2 \). For example \( (E_{0,1}, E_{2,\lambda}) = 2 \) if we take \( \lambda \) as follows: In case \( a \) is even, let \( \lambda = a\xi \). On the contrary, in case \( a \) is odd, let \( \lambda = x + y(a\xi) \), where \( x \) and \( y \) \( \neq 0 \) are given as follows: If \( b \) and \( c \) are even, then let \( x \) be even and \( y \) be odd. If \( b \) or \( c \) is odd, then let \( x \) and \( y \) be odd. By simple calculations we see that the number of the elements of the set \( \{(2z, \lambda z) \in E_{2,\lambda} | 2z = 0 \text{ in } E \} \) is 2. Since \( E_{0,1} \) and \( E_{2,\lambda} \) meet transversally, we have that \( (E_{0,1}, E_{2,\lambda}) = 2 \). Using Lemma 9, we finish the proof of Theorem 3.

Next we prove Proposition 5. Since \( \text{End}(E_i) \) becomes a maximal order of \( K \) in this case, we make use of the results of [5]. Suppose that such curves \( E_i \) \( (i = 1, 2) \) exist. Then \( E_i \) is a translation of \( E_{\alpha_i,\beta_i} \) for some \( \alpha_i, \beta_i \in \text{End}(E) \) (cf. [5, Lemma 1]). Hence

\[
(E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}) = (E_1, E_2) = 2.
\]

Moreover, by [5, Corollary 1 on p. 6], we have that

\[
(E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}) = \frac{N(\alpha_1\beta_2 - \alpha_2\beta_1)}{N(\alpha_1, \beta_1)N(\alpha_2, \beta_2)},
\]

where \( N \) denotes the norm. Clearly we also have that

\[
(E_{\alpha_1,\beta_1}, E_{\alpha_2,\beta_2}) = 2.
\]

We can write \( \sigma_1\alpha_i = c_ia_i, \sigma_1\beta_i = c_ib_i + c_i\omega, \) where \( a_i, b_i, c_i \in \mathbf{Z} \) \( (i = 1, 2) \) and we may assume that \( (c_ia_i, c_ib_i + c_i\omega) \) form a canonical basis. Then we infer from the above that \( \gamma \gamma' = 2a_1a_2 \), where \( \gamma = (a_1b_2 - b_1a_2) + (a_1 - a_2)\omega \). Since \( 2 \) is a prime number in \( K \) and the class number of \( K \) is 1, we see that \( a_1 \) and \( a_2 \) are even numbers. Putting \( a_i = 2a'_i \), we obtain that \( \gamma \gamma' = 2a'_1a'_2 \), where \( \gamma' = (a'_1b'_2 - b'_1a'_2) + (a'_1 - a'_2)\omega \). We can repeat the same argument finitely many times, which gives rise to a contradiction.

Finally we mention a problem concerning \( d_r \).

**Problem 10.** Find the value \( d_r(A) \) for each abelian surface \( A \). Especially we ask whether the following assertions hold true:

1. Is there an abelian surface \( A \) satisfying \( d_r(A) \geq 4 \)? For example, is it true that \( d_r(E_1 \times E_2) = 4 \) if \( E_1 \) and \( E_2 \) are not isogenous?

2. If two abelian surfaces \( A_1 \) and \( A_2 \) are isogenous, then is it true that \( d_r(A_1) = d_r(A_2) \)?

REFERENCES


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