PRODUCTS OF $\omega^*$ IMAGES

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Abstract. Let $\omega^*$ be the Čech-Stone remainder $\beta\omega \setminus \omega$. We show that there exists a large class $O$ of images of $\omega^*$ such that whenever $S$ is a subset of $O$ of cardinality at most the continuum, then $\omega^* \times \prod S$ is again an image of $\omega^*$. The class $O$ contains all separable compact spaces, all compact spaces of weight at most $\omega_1$ and all perfectly normal compact spaces.

1. Introduction

For a Tychonoff space $X$, $\beta X$ represents the Čech-Stone compactification of $X$ and for $A \subset X$, $A^*$ represents the subspace $\text{cl}_X(A) \setminus X$. If $f : X \to Y$, then $\beta f : \beta X \to \beta Y$ is the Čech-Stone extension of $f$ and $f^* = \beta f \upharpoonright X^*$. The first infinite ordinal $\omega$ is given the discrete topology.

W. Just [Ju89] has proven that $\omega^* \times \omega^*$ is consistently not a continuous image of $\omega^*$. So it is consistent that there are two $\omega^*$ images whose product is not an $\omega^*$ image. This creates a limit to naive product results and a desire to find a general product result that includes many of the $\omega^*$ image results appearing in the literature. This is the theme of this paper. We will define a subclass $O$ of the class of all $\omega^*$ images and show (The Product Theorem) that if $S$ is a subset of $O$ of cardinality at most the continuum, then $\omega^* \times \prod S$ is an $\omega^*$ image. In the succeeding sections, we will show that $O$ contains all separable compact spaces, all compact spaces of weight at most $\omega_1$, all perfectly normal compact spaces, all zero-dimensional, orderable $\omega^*$ images, and all $\omega^*$ images of weight $< b$.

In this paper all spaces are assumed to be Hausdorff and all mappings between spaces are assumed to be continuous surjections. If $X$ is homeomorphic to $Y$, then this is denoted by $X \approx Y$. If $F$ is a family of functions with a common domain $X$ and ranges $Y_f$ for $f \in F$, then $\Delta F$ represents the diagonal function with domain $X$ and range $\prod_{f \in F} Y_f$ defined by $\Delta F(x) = (f(x))_{f \in F}$. A Boolean space is a zero-dimensional, compact space, and $CO(X)$ denotes the algebra of all clopen subsets of a space $X$. Thus, $CO(\omega^*)$ is isomorphic to the quotient algebra $\mathcal{P}(\omega)$ modulo the ideal of finite sets. The cardinality of the continuum is denoted by $c$. For an

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introduction to the theory of $\omega^*$ images we refer the reader to the article by van Mill [vM84].

2. Orthogonality and products

**Definition 2.1.** Two functions $f : X \to Y$ and $g : X \to Z$ are called **orthogonal** $(f \perp g)$ iff $f \Delta g : X \to Y \times Z$ is a surjection. If the sets are Boolean spaces and the functions are mappings, then the dual notion is as follows: If $A$ and $B$ are two subalgebras of a boolean algebra $C$, then $A$ and $B$ are called **independent** if for every non-zero $a \in A$ and for every non-zero $b \in B$, $a \land b \neq 0$.

**Definition 2.2.** $X$ is an **orthogonal $\omega^*$ image** if there exists a finite-to-one surjection $q : \omega \to \omega$ and a mapping $f : \omega^* \to X$ with $f \perp q^*$. If $X$ is a Boolean space, then the dual notion is as follows: A subalgebra $B$ of $CO(\omega^*)$ is called an **orthogonal subalgebra** if there exists a partition $\mathcal{R} = \{R_k : k < \omega\}$ of $\omega$ into finite sets such that for every non-empty $b \in B$, if $b = A^*$ where $A \subset \omega$, then $\{k < \omega : A \cap R_k \neq \emptyset\}$ is a co-finite subset of $\omega$.

Let us observe that a continuous image of an orthogonal $\omega^*$ image is again an orthogonal $\omega^*$ image.

In this paper, we will find it convenient to view $\omega$ in one of its alternate forms. Let $W = \{(k,j) \in \omega \times \omega : j \leq k\}$ be given the discrete topology (so $W^* \approx \omega^*$), let $p : W \to \omega$ be the projection $p(k,j) = k$, and for each $k < \omega$, let $W_k = p^{-1}(k)$.

As $p$ is a finite-to-one surjection, we have that $p^* : W^* \to \omega^*$ is an open surjection. Thus we have

$(\star) \quad \forall \text{ open } V \subset W^* \text{ with } p^*(V) = \omega^* \exists \text{ clopen } V' \subset V \text{ with } p^*(V') = \omega^*$.

**Henceforth** $W$, $W_k$, $W^*$, $p$, and $p^*$ will always have the above meanings.

**They will be frequently used in this paper.** The following demonstrates the canonical nature of $W$.

**Proposition 2.3.** If $X$ is an orthogonal $\omega^*$ image, then there exists a mapping $f : W^* \to X$ such that $f \perp p^*$.

**Proof.** Let $q$ be a finite-to-one surjection $q : \omega \to \omega$ and let $g$ be a mapping $g : \omega^* \to X$ such that $g \perp q^*$. Since $g \perp q^*$, for every $x \in X$ we have that $q^*(g^{-1}(x)) = \omega^*$. This means that for every $x \in X$ and for every $A \subset \omega$ such that $g^{-1}(x) \subset A^*$, $\{n < \omega : A \cap q^{-1}(n) \neq \emptyset\}$ is a co-finite subset of $\omega$. We will produce a finite-to-one surjection $\varphi : W \to \omega$ such that whenever $A \subset \omega$ satisfies that $\{n < \omega : A \cap q^{-1}(n) \neq \emptyset\}$ is co-finite in $\omega$, then $\{n < \omega : \varphi^{-1}(A) \cap W_n \neq \emptyset\}$ is co-finite in $\omega$. Putting $f = g \circ \varphi^*$, we see that $f \perp p^*$. To construct $\varphi$, first get an increasing function $\psi : \omega \to \omega$ such that for every $n < \omega$, $\psi(n) \geq |q^{-1}(n)|$. For every $i < \psi(0)$ let $\varphi_i : W_i \to \omega$ be arbitrary. For every $n \geq 0$ and for every $i$ with $\psi(n) \leq i < \psi(n+1)$ let $\varphi_i$ be a surjection from $W_i$ onto $q^{-1}(n)$. Define $\varphi : W \to \omega$ so that for every $i < \omega$, $\varphi | W_i = \varphi_i$. \hfill $\Box$

**Proposition 2.4.** The product of an $\omega^*$ image and an orthogonal $\omega^*$ image is an $\omega^*$ image.

**Proof.** This is an immediate consequence of the definition of orthogonal $\omega^*$ image.

It follows from Just’s result that $\omega^*$ need not be an orthogonal $\omega^*$ image.
Theorem 2.5 (The Product Theorem). The product of at most $\epsilon$ many orthogonal $\omega^*$ images is an orthogonal $\omega^*$ image.

Proof. Invoking Proposition 2.3, for every $s \in S \subset 2^\omega$ let $f_s : W^* \to X_s$ satisfy $f_s \perp p^*$. For every $k < \omega$ let $U_k = \{ t : t $ is a function with domain $2^k$ and range $W_k \}$. Let $U = \bigcup_{k<\omega} U_k$ have the discrete topology (so $U^* \approx \omega^*$). If $t \in U_k$, then define $r(t) = k$. As $r : U \to \omega$ is a finite-to-one surjection, $r^* : U^* \to \omega^*$ is a surjection. For every $s \in S$ define $p_s : U \to W$ by $p_s(t) = t(s \upharpoonright k)$ where $t \in U_k$, i.e., $p_s \upharpoonright U_k$ is the projection from $U_k$ onto the factor indexed by $s \upharpoonright k$. As $p_s$ is a finite-to-one surjection, $p_s^* : U^* \to \omega^*$ is a surjection. We claim that $r^* \perp \Delta\{ f_s \circ p_s^* : s \in S \}$. By compactness, it suffices to show that for any finite set $T \subset S, r^* \perp \Delta\{ f_s \circ p_s^* : s \in T \}$; for which, in turn, it suffices to show that for any collection of non-empty sets $\{ V_s, V_s' : s \in T \}$ where $V$ is clopen in $\omega^*$ and $V_s$ is open in $X_s$ for $s \in T$, that $r^*^{-1}(V) \cap \bigcap_{s \in T} (f_s \circ p_s^*)^{-1}(V_s) \neq \emptyset$. Let us check this. Since $f_s \perp p^*$, we have $p^*(f_s^{-1}(V_s)) = \omega^*$. So by (★), we can choose clopen $V_s' \subset W^*$ with $V_s' \subset f_s^{-1}(V_s)$ and $p^*(V_s') = \omega^*$. We will prove that $r^*^{-1}(V) \cap \bigcap_{s \in T} p_s^{-1}(V_s') \neq \emptyset$.

Let $A \subset \omega, A_s \subset W$ for $s \in T$ be subsets with $V = A^*, V_s' = A_s^*$. $A$ is infinite and $p(A_s)$ is co-finite for every $s \in T$ since $p^*(A_s) = \omega^*$. So $A \cap \bigcap_{s \in T} p(A_s)$ is infinite. Let $k \in A \cap \bigcap_{s \in T} p(A_s)$ be big enough to distinguish all members of $T$, i.e., if $s \not= s'$, $s,s' \in T$, then $s \upharpoonright k \not= s' \upharpoonright k$. For every such $k$, there is a mapping $t \in U_k$ such that for every $s \in T$, $t(s \upharpoonright k) \in A_s$. Since any $t$ like this is in $r^{-1}(A) \cap \bigcap_{s \in T} p_s^{-1}(A_s)$, the set $r^{-1}(A) \cap \bigcap_{s \in T} p_s^{-1}(A_s)$ is infinite. So $r^*^{-1}(V) \cap \bigcap_{s \in T} p_s^{-1}(V_s' \not= \emptyset)$.

We now strengthen a classical result.

Proposition 2.6. Every separable, compact space is an orthogonal $\omega^*$ image.

Proof. It suffices to show that $\beta\omega$ is an orthogonal $\omega^*$ image. Let $q : W \to \omega$ be the second projection $q(k,j) = j$. As $q$ is an infinite-to-one surjection, $q^* : W^* \to \beta\omega$ is a surjection. Since for every $j < \omega$, $\{ k < \omega : q^{-1}(j) \cap W_k \not= \emptyset \}$ is a co-finite subset of $\omega$, we see that $q^* \perp p^*$.

3. Weight $\omega_1$

Definition 3.1. A thick clopen subset $V \subset W^*$ is a clopen $V \subset W^*$ with $V = A^*$ where $A \subset W$ satisfies $\lim_{k \to \infty} |A \cap W_k| = \infty$. A thick zero-set $Z \subset W^*$ is a zero-set $Z \subset W^*$ with $Z = \bigcap_{k<\omega} V_k$ where $\{ V_k \}$ is a decreasing sequence of thick clopen subsets. If $G$ is a clopen subset of $W^*$, then a thick mapping $f : G \to X$ is a mapping $f : G \to X$ with $f^{-1}(x)$ a thick zero-set for every $x \in X$.

We now list some basic properties of thick sets.

T1. If $V \subset V'$ are clopen sets and $V$ is thick, then $V'$ is thick.
T2. If $\{ Z_k \}$ is a decreasing sequence of thick zero-sets, then $\bigcap_{k<\omega} Z_k$ is a thick zero-set.
T3. A thick zeroset contains a thick clopen set.
A thick clopen set contains two disjoint thick clopen sets.

T4. A thick mapping \( f : W^* \to X \) satisfies \( f \perp p^* \).

Let us prove the first part of T3. Let \( Z \subset W^* \) with \( Z = \bigcap_{j<\omega} V_j \) where \( \{V_j\} \) is a decreasing sequence of thick clopen sets. For \( j < \omega \) let \( A_j \subset W^* \) such that \( V_j = A_j^* \). Thus we have (a) \( j < k \) implies \( A_k \setminus A_j \) is finite and (b) for every \( j \), \( \lim_{k \to \infty} |A_j \cap W_k| = \infty \). By induction, using (a) and (b), define an increasing sequence \( \{r_n\} \subset \omega \) such that for every \( k \geq r_n \), \( A_n \cap W_k \subset \bigcap_{i<n} A_i \cap W_k \) and \( |A_n \cap W_k| \geq n \). For each \( n > 0 \) and for each \( i \) with \( r_n \leq i < r_{n+1} \) let \( F_i \) be a subset of \( A_n \cap W_i \) of cardinality \( n \). If \( F = \bigcup_{i<\omega} F_i \), then \( F^* \) is a thick clopen set with \( F^* \subset Z \).

We now give a topological proof of a strengthening of a result of Parovičenko [Pa63] using some ideas in Blaszczyk and Szymański [BS80].

**Proposition 3.2.** Let \( X \) be a compact metric space and let \( f : W^* \to X \) be a thick mapping. Let \( E \) and \( F \) be closed in \( X \) with \( X = E \cup F \). Then, there exists a clopen \( G \subset W^* \) such that \( f \upharpoonright G \) is a thick mapping onto \( E \) and \( f \upharpoonright (W^* \setminus G) \) is a thick mapping onto \( F \).

**Proof.** Let \( D \) be a countable, dense subset of \( E \cap F \). For every \( d \in D \) (by T3) choose disjoint, thick clopen subsets \( U_d \) and \( V_d \) of \( f^{-1}(d) \). Thus \( G_0 = f^{-1}(E \setminus F) \cup \bigcup_{d \in D} U_d \) and \( G_1 = f^{-1}(F \setminus E) \cup \bigcup_{d \in D} V_d \) are disjoint cozero sets of \( W^* \). Let \( G \) be clopen in \( W^* \) with \( G_0 \subset G \) and \( G_1 \subset (W^* \setminus G) \). Then \( f \upharpoonright G \) is a mapping onto \( E \). Let \( x \in X \) and let \( Z \) be a zero-dimensional, compact space of the same weight, it suffices to assume that \( X \) is Boolean. Via an embedding into the Cantor cube \( 2^{\omega_1} \) we may express \( X \) as an inverse limit space, \( X = \varprojlim \{X_\alpha, p^*_\alpha, \alpha < \beta < \omega_1\} \) such that:

(i) \( \beta \) limit implies \( X_\beta = \varprojlim \{X_\alpha, p^*_\alpha, \alpha < \gamma < \beta\} \) (the spectrum is continuous).

(ii) \( X_\alpha \) is a zero-dimensional, compact metric space.

(iii) \( X_{\alpha+1} = X^0_{\alpha+1} \cup X^1_{\alpha+1} \) where \( X^0_{\alpha+1} \) and \( X^1_{\alpha+1} \) are disjoint, clopen subsets of \( X_{\alpha+1} \) and for \( i = 0, 1 \), \( p^*_{\alpha+1} \upharpoonright X^i_{\alpha+1} \) is one-to-one (short projections are “simple”).

(iv) \( |X_0| = 1 \).

We will construct a system of thick mappings \( \{f_\alpha : W^* \to X_\alpha : \alpha < \omega_1\} \) such that \( f_\alpha = p^*_{\beta} \circ f_\beta \), \( \alpha < \beta < \omega_1 \). Since for every \( \alpha < \omega_1 \), \( f_\alpha \perp p^* \), by compactness we
will get that $f_{\omega_1} = \lim \{ f_\alpha : \alpha < \omega_1 \}$ satisfies $f_{\omega_1} \perp p^*$. 

**Limit Step.** If $\beta$ is a limit, then let $f_\beta = \lim \{ f_\alpha : \alpha < \beta \}$. Property T2 implies that $f_\beta$ is a thick mapping.

**Successor Step.** If $\beta = \alpha + 1$, then let $E = p_\alpha^{\omega+1}(X^{0}_\alpha)$ and $F = p_\alpha^{\omega+1}(X^{1}_\alpha)$. Apply Proposition 3.2 to get a clopen $G \subset W^*$ with $f_\alpha \mid G$ a thick mapping onto $E$ and $f_\alpha \mid (W^* \setminus G)$ a thick mapping onto $F$. Define $f_{\alpha+1} : W^* \to X_{\alpha+1}$ by

$$f_{\alpha+1}(z) = \begin{cases} (p_\alpha^{\omega+1} \mid X^{0}_\alpha)^{-1}(f_\alpha(z)) & \text{if } z \in G, \\ (p_\alpha^{\omega+1} \mid X^{1}_\alpha)^{-1}(f_\alpha(z)) & \text{if } z \notin G. \end{cases}$$

$f_{\alpha+1}$ is a thick mapping because

$$f_{\alpha+1}^{-1}(x) = \begin{cases} (f_\alpha \mid G)^{-1}(p_\alpha^{\omega+1}(x)) & \text{if } x \in X^{0}_{\alpha+1}, \\ (f_\alpha \mid (W^* \setminus G))^{-1}(p_\alpha^{\omega+1}(x)) & \text{if } x \in X^{1}_{\alpha+1}, \end{cases}$$

and both $f_\alpha \mid G$ and $f_\alpha \mid (W^* \setminus G)$ are thick mappings.

\[ \square \]

4. Perfectly Normal

**Proposition 4.1.** Let $X$, $Y$, $\overline{Y}$ and $Z$ be compact spaces and let $f : X \to Y$, $g : X \to Z$, $h : Y \to Y$, and $k : X \to \overline{Y}$ be mappings such that $f \perp g$, $f = h \circ k$, and $h$ is irreducible. Then, $k \perp g$.

**Proof.** Let $y \in Y$ and let $O_y$ be a neighbourhood of $y$. Irreducibility of $h$ implies that there exist non-empty, open $V \subset Y$ with $h^{-1}(V) \subset O_y$. Therefore, $k^{-1}(h^{-1}(V)) = f^{-1}(V) \subset k^{-1}(O_y)$. But $g(f^{-1}(V)) = Z$ since $f \perp g$. So $g(k^{-1}(O_y)) = Z$. Thus $g(k^{-1}(y)) = Z$.

\[ \square \]

We now strengthen a result of Przymusiński [Pr82]. For the reader’s convenience, we sketch his arguments and indicate how we apply Proposition 4.1. The following Lifting Lemma appears in his paper.

**Lemma 4.2.** Let $X$ be compact and perfectly normal, $Z$ a closed subspace of $X \times I$ and suppose that the restriction $\pi \mid Z : Z \to X$ of the projection $\pi : X \times I \to X$ is irreducible. If $f : \omega^* \to X$ is a continuous mapping of $\omega^*$ onto $X$, then there exists a continuous mapping $g : \omega^* \to Z$ of $\omega^*$ onto $Z$ such that $f = \pi \circ g$.

**Theorem 4.3.** Every perfectly normal compact space $X$ is an orthogonal $\omega^*$ image.

**Proof.** We express $X$ as the limit space of an inverse spectrum

$$X = \lim \{ X_\alpha, \pi_\alpha^\beta, \alpha < \beta < \kappa \}$$

with limit projections $\pi_\alpha : X \to X_\alpha$ such that:

(a) $\beta$ limit implies $X_\beta = \lim \{ X_\alpha, \pi_\alpha^\gamma, \alpha < \gamma < \beta \}$

(b) $w(X_0) \leq \omega_1$

(c) $\pi_0$ is irreducible (therefore all other projections are irreducible)

(d) $X_{\alpha+1}$ is a closed subset of $X_\alpha \times I$ and $\pi_\alpha^{\alpha+1}$ is the restriction to $X_{\alpha+1}$ of the projection of $X_\alpha \times I$ onto $X_\alpha$. 
Parts (b) and (c) are achieved by using the result of Šapirovski [Sa74] that every perfectly normal compact space admits a \(\pi\)-base of cardinality at most \(\omega_1\). By Theorem 3.3, let \(f_0 : W^* \to X_0\) satisfy \(f_0 \perp p^*\). By induction on \(\alpha \leq \kappa\), we construct \(f_\alpha : W^* \to X_\alpha\) where \(X_\alpha = X\) such that \(\beta < \alpha\) implies that \(\pi_\alpha \circ f_\alpha = f_\beta\). At successor stages use Lemma 4.2 to get \(f_\alpha+1\) and at limit stages let \(f_\alpha = \lim \{f_\beta : \beta < \alpha\}\). Thus we have \(f_\kappa : W^* \to X\) with \(f_0 = \pi_0 \circ f_\kappa\). Proposition 4.1 with \(X = W^*, Y = X_0, Z = \omega^*\) and \(f = f_0, g = p^*, h = \pi_0, k = f_\kappa\) implies that \(f_\kappa \perp p^*\).

5. Orderable

Theorem 5.1. Every zero-dimensional, orderable \(\omega^*\) image \(L\) is an orthogonal \(\omega^*\) image.

Proof. Let \(\prec\) be a compatible order for \(L\). Let \(L' = \{x \in L : x\) is a left neighbour of a jump in \(\prec\}\). Since \(L\) is Boolean, \(G = \{\{y \in L : y \leq x\} : x \in L'\}\) is a generating set for \(CO(L)\). It suffices to define \(\phi : L' \to \mathcal{P}(W)\) such that for every \(x, y \in L', x \prec y\) implies that (a) \(\phi(x) \setminus \phi(y)\) is finite and \(\phi(y) \setminus \phi(x)\) is infinite and (b) \([k < \omega : (\phi(y) \setminus \phi(x)) \cap W_k \neq \emptyset]\) is a co-finite subset of \(\omega\). Because then the function that sends \(\{y \in L : y \leq x\}\) to \(\phi(x)^*\) will extend to an embedding of \(CO(L)\) into \(CO(W^*)\) (by (a)) onto an orthogonal subalgebra of \(CO(W^*)\) (by (b)). Since \(L\) is an \(\omega^*\) image, let \(\psi : L' \to \mathcal{P}(\omega)\) satisfy that for every \(x, y \in L', x \prec y\) iff \(\psi(x) \setminus \psi(y)\) is finite. Finally, define \(\phi : L' \to \mathcal{P}(W)\) by \(\phi(x) = \{(k, j) : j \leq |\psi(x) \cap k|\}\).

Let \(Q\) denote the lexicographic ordered space \(2^{\omega^*}\). It is known (Bell [Be90]) that \(Q\) is an \(\omega^*\) image, so we get the following corollary.

Corollary 5.2. \(Q\) is an orthogonal \(\omega^*\) image.

Example 5.3. The Alexandroff one-point compactification of the discrete space \(\omega\) is an orthogonal \(\omega^*\) image because it is an image of \(Q\).

Corollary 5.4. Every first countable orderable compact space is an orthogonal \(\omega^*\) image.

Proof. This follows because Maurice [Ma64] has proved that every first countable orderable compact space is an image of \(Q\).

Question 5.5. Can zero-dimensionality be removed from the hypotheses of Theorem 5.1, i.e., is every orderable \(\omega^*\) image an orthogonal \(\omega^*\) image?

6. Weight \(\prec b\)

Recall that if \(f, g \in \omega^\omega\), then \(f \prec_* g\) means that eventually \(f(k) < g(k)\) and that \(b\) is by definition the least cardinality of an \(\prec_*\)-unbounded subset of \(\omega^\omega\).

Theorem 6.1. Every \(\omega^*\) image \(X\) of weight \(\prec b\) is an orthogonal \(\omega^*\) image.

Proof. It suffices to assume that \(X\) is Boolean. Let \(B\) be a subalgebra of \(CO(\omega^*)\) of cardinality \(\prec b\). For every \(b \in B\), choose \(M_b \subset \omega\) such that \(b = M_b^*\). For every \(b \in B\), define \(f_b \in \omega^\omega\) by \(f_b(k) = \min\{n \in M_b : n \geq k\}\). Let \(g \in \omega^\omega\) such that for every \(b \in B\), \(f_b \prec_* g\) and for every \(k < \omega, k < g(k)\). Let \(R = \{(k, j) : k < j < g(k)\}\) endowed with the discrete topology (therefore \(R^* \approx \omega^*\)), let for every \(k < \omega, R_k = \{(k, j) : k \leq j < g(k)\}\), and let \(R = \{R_k : k < \omega\}\). For every \(b \in B\), define \(A_b = \{(k, j) \in R : j \in M_b\}\). Then, \(C = \{A_b^* : b \in B\}\) is a subalgebra of \(CO(R^*)\),
B ∼ C by the natural map b ↦ A∗ b, and C is an orthogonal subalgebra of CO(R∗) because for every b ≠ ∅, \{k < ω : A∗ k ∩ Rk ≠ ∅\} is co-finite in ω.

A family \( A \) of subsets of ω is said to be strongly centered if whenever \( F \) is a finite subset of \( A \), then \( \bigcap F \) is infinite. Recall that \( p \) is the least cardinality of a strongly centered family \( A \) such that there does not exist an infinite \( X \subset \omega \) with the property that for every \( A \in A \), \( X \setminus A \) is finite. It is a basic fact that \( \omega_1 \leq p \leq c \).

We now strengthen the result of van Douwen and Przymusiński [DP80] that every compact space of weight < p is an \( ω^∗ \) image.

**Corollary 6.2.** Every compact space \( X \) of weight < p is an orthogonal \( ω^∗ \) image.

**Proof.** Since \( X \) is an \( ω^∗ \) image and \( p \leq b \) (cf. van Douwen [vD84]), we can apply Theorem 6.1.

It follows from this corollary that Martin’s Axiom implies that all compact spaces of weight less than \( c \) are orthogonal \( ω^∗ \) images.

In conclusion, we summarize our results relating to well-known classes of spaces.

**Summation 6.3.** Let \( D \) be the class of all compact spaces which have at least one of the following properties: separable, weight at most \( ω_1 \), perfectly normal, first countable orderable, and weight < p. If \( S \) is a subset of \( D \) of cardinality at most \( c \), then \( ω^∗ \times \prod S \) is an image of \( ω^∗ \).

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