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REPRESENTATIONS AT FIXED POINTS OF SMOOTH ACTIONS OF COMPACT CONNECTED LIE GROUPS

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Abstract. Let G be a compact connected Lie group acting smoothly on a connected closed manifold M with nonempty fixed point set F. In this paper, we study the relation between the cohomology of M or M_G and the equivalent representations of G at fixed points.

1. Introduction

Throughout this paper, we assume that Q is the rational field and G a compact connected Lie group acting smoothly on a connected closed manifold M with fixed point set F. Let M_G be the Borel construction associated with the G action on M. Let $T(M)$ denote the tangent bundle of M and $T_x(M)$ the tangent space at $x \in M$. For each $x \in F$, the induced G linear action on the tangent space $T_x(M)$ of M at $x \in F$ defines a real representation of G, which is denoted by Θ_x. Let $RO(G)$ and $RU(G)$ be the real and complex representation rings of G respectively. There is a complexification map $RO(G) \to RU(G)$, which is injective for a compact connected Lie group G. Denote the complexification of Θ_x also by Θ_x. Recall that M is totally nonhomologous to zero in M_G with coefficient in Q if the fibre inclusion $j: M \to M_G$ induces a surjection in cohomology $H^*(-;Q)$ ([5, p. 373]).

In this paper, we prove

Theorem 1.1. *Let G be a compact connected Lie group acting smoothly on a connected closed manifold M with nonempty fixed point set F. Then $\Theta_x = \Theta_y$ for any $x, y \in F$, if one of the following conditions is satisfied* :

(i) *$\widetilde{K}(M) \otimes Q$ is trivial, or*

(ii) *M is totally nonhomologous to zero in M_G with coefficient in Q, and $H^*(M;Q)$ is algebraically generated by some elements $\{x_i\}$ of odd degrees.*

Note that the Chern character ch: $K(M) \otimes Q \to \bigoplus_{i\geq 0} H^{2i}(M;Q)$ is an isomorphism ([7]). Thus condition (i) in the above theorem is equivalent to the condition that $H^{2i}(M;Z)$ is finite for all $0 < 2i \leq \dim(M)$.

Now let $T = (S^1)^r$ be a fixed maximal torus of a compact connected Lie group G. It is known that two representations of G are equivalent iff their restrictions on T are equivalent ([6, Corollary 1.8.3]). Thus we reduce the problem of equivalent representations of G to the case when G is a torus. It is well known that

$$RU((S^1)^r) = Z\{t_1, t_2, \ldots, t_r\},$$

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the finite Laurent series ring in t_i, where t_i is the 1-dimensional complex representation of the ith copy S^1 of $(S^1)^r$, given by

$$t_i(z)(w) = zw, \qquad z \in S^1, w \in C.$$

Let $I((S^1)^r)$ be the ideal of $RU((S^1)^r)$ generated by $1-t_1, 1-t_2, \ldots, 1-t_r$. In [4, Theorem VI], Bredon proved

Theorem. *Suppose the compact connected Lie group G acts smoothly on a connected manifold M with nonempty fixed point set F. Assume that*

$$\pi_{2i}(M) \quad \textit{is finite for all} \begin{cases} 1 \le i \le k-1 & \textit{for general } G, \\ 2 \le i \le k-1 & \textit{for semi-simple } G. \end{cases}$$

Then $\Theta_x - \Theta_y$ is in the ideal $(I(T))^k$ of $RU(T)$ for any fixed points x, y.

Note that the manifold M in Bredon's theorem is not necessarily closed. But we require M to be closed in our theorems, since we will use the fact that the K-theory is representable only in the category of finite CW-complexes, that is, $\widetilde{K}(X) \approx \widetilde{K}^0(X)$ if X is a finite CW-complex, where $\widetilde{K}^*(-)$ is the reduced cohomology represented by the well-known spectrum K ([9, pp. 216, 210]). By using the cohomology $H^*(M;Z)$, we will prove the following

Theorem 1.2. *Let G be a compact connected Lie group acting smoothly on a connected closed manifold M with nonempty fixed point set F. Then $\Theta_x - \Theta_y \in (I(T))^n$, if*

$$H^{2i}(M;Z) \quad \textit{is finite for all } 1 \le i \le n-1.$$

Moreover, if $T(M) \otimes C$ is stably trivial in $K(M^{(2n)}) \otimes Q$, then $\Theta_x - \Theta_y$ is in $(I(T))^{n+1}$, where $M^{(2n)}$, which contains at least one fixed point, is the $(2n)$-skeleton of a G-CW-structure of M.

Note that, as a CW-complex, the $(2n)$-G-CW-skeleton $M^{(2n)}$ might have cells of dimensions $> 2n$, since G is connected. Actually by [8], the G-space $M^{(k)}/M^{(k-1)}$ is a wedge of based G-spheres

$$G/H \times S^k/(G/H \times *)$$

which is $(k-1)$-connected. Here H is some closed isotropy subgroup of G.

As a specific example of applications of these theorems, we prove

Corollary 1.3. *Suppose G acts smoothly on a connected closed manifold M with nonempty fixed point set F. Suppose M is a rational homology sphere of dimension n. If n is odd, then $\Theta_x = \Theta_y$ for $x, y \in F$. If n is even, then there are at most two different representations Θ_x, $x \in F$, up to equivalency.*

2. Proofs of the theorems

Recall, if X is a G-space, then the equivariant complex K-theory $K_G(X)$ is formed from the free abelian group on the equivalence classes of G-complex vector bundles over X modulo the subgroup generated by $[\xi \oplus \eta] - [\xi] - [\eta]$. Its ring structure is induced by the tensor product of G-complex vector bundles. For a single point $*$, $K_G(*)$ is just the representation ring $RU(G)$.

Let $p_0 : E_G \to B_G$ be the universal principal G-bundle. Let $B_G^{(r)}$ be the r-skeleton of B_G, and $E_G^{(r)}$ the inverse image $p_0^{-1}(B_G^{(r)})$. For $G = S^1$, E_G can be taken to be the infinite sphere $S^\infty = \bigcup S^{2m+1}$, and B_G the infinite complex projective space CP^∞. Therefore we have $B_G^{(2k)} = CP(k) = B_G^{(2k+1)}$, and $E_G^{(2k)} = E_G^{(2k+1)} = S^{2k+1}$, when $G = S^1$. Note that any G vector bundle over E_G (resp. $E_G^{(r)}$) induces a vector bundle over B_G (resp. $B_G^{(r)}$). By [1, Proposition 1.6.1], this gives an isomorphism $K_G(E_G) \to K(B_G)$ (resp. $K_G(E_G^{(r)}) \to K(B_G^{(r)})$). Let

$$\alpha^{(r)} : RU(G) \to K_G(E_G^{(r)}) \ (\approx K(B_G^{(r)}))$$

be the homomorphism induced by the projection $E_G^{(r)} \to *$. By [1, Corollary 2.7.6, p. 105], if $G = S^1$, then the sequence

$$0 \to RU(G) \xrightarrow{\varphi} RU(G) \xrightarrow{\alpha^{(2n-1)}} K_G(E_G^{(2n-1)}) \to 0 \tag{1}$$

is exact. Here the injectivity of φ follows from the fact that φ is the multiplication by $(1-t)^n$ when $G = S^1$ ([5, p. 357]).

Let G act smoothly on M. Define the G action on $E_G \times M$ or $E_G^{(2m+1)} \times M$ to be the diagonal action. Then $M_G = (E_G \times M)/G$. Let $R^m(G) = (E_G^{(2m+1)} \times M)/G$. Note that the G action on M induces a G structure on the tangent bundle $T(M)$, and the projection

$$E_G \times M \to M \quad (\text{or } E_G^{(2m+1)} \times M \to M)$$

is G-equivariant. Then the G vector bundle $T(M)$ induces a G vector bundle over $E_G \times M$ (or $E_G^{(2m+1)} \times M$), pulling back by the above projection, thus defines a vector bundle $\overline{T}(M)$ over M_G (or a vector bundle $\overline{T}_m(M)$ over $R^m(G)$), which is called the tangent bundle along the fibres of the related fibre bundle ([2]). Obviously, $i^*(\overline{T}(M)) = \overline{T}_m(M)$, where $i : R^m(G) \to M_G$ is the inclusion. Also for $x \in F$, there exists a section ρ_x for the projection $p : R^m(M) \to CP(m)$. The point is, if we regard $\alpha^{(2n-1)}(\Theta_x)$ as an element of $K(B_G^{(2n-1)})$ ($\approx K_G(E_G^{(2n-1)})$), then

$$\alpha^{(2n-1)}(\Theta_x) = \rho_x^*(\overline{T}_m(M) \otimes C), \tag{2}$$

where $\rho_x^*(\overline{T}_m(M) \otimes C)$ is the bundle induced by ρ_x. The bundle $\overline{T}_m(M) \otimes C$ will provide us a global view for the local complex representations Θ_x. The following theorem is similar to [4, Theorem V].

Theorem 2.1. *Let $G = S^1$ act smoothly on a connected closed manifold M with nonempty fixed point set F. If $H^{2i}(M; Z)$ is finite for all $1 \le i \le n-1$, then*

$$\alpha^{(2n-1)}(\Theta_x - \Theta_y) = 0$$

in $K(CP(n-1))$, and $\Theta_x - \Theta_y$ is divisible by $(1-t)^n$. Moreover, if $T(M) \otimes C$ is stably trivial over $K(M^{(2n)}) \otimes Q$, then $\Theta_x - \Theta_y$ is divisible by $(1-t)^{n+1}$. Here $M^{(2n)}$, which contains at least one fixed point, is the $(2n)$-skeleton of a G-CW-structure of M.

Let K and H be the ring spectra corresponding to the nonconnective complex K-theory and the ordinary integral homology respectively. For a spectrum E, let $E_{(Q)}$ be the localization of E at the rational field Q in the Bousfield sense ([3]). Then $E_{(Q)}$ is a spectrum with $\pi_k(E_{(Q)}) \approx \pi_k(E) \otimes Q$. In particular, for the ring spectrum H,

$$H^*_{(Q)}(Y) \approx H^*(Y; Q) \approx H^*(Y; Z) \otimes Q,$$

where Y is a CW-complex. In general, we have

Lemma 2.2. *Let E be a spectrum and Y a finite CW-complex. Then*

$$E^*_{(Q)}(Y) \approx E^*(Y) \otimes Q.$$

Proof. Let $Y^{(n)}$ be the n-skeleton of Y. Since the cohomology represented by E satisfies the wedge axiom ([9, p. 146]) and the functor $\bigotimes Q$ commutes with finite products of abelian groups, the lemma is true if Y is a finite wedge of spheres $\{S^m_\alpha\}$ of the same dimension m. Note that both $E^*_Q(Y)$ and $E^*(Y) \otimes Q$ are vector spaces over Q. This means we can do the induction from lower-dimensional skeletons of Y to the higher skeletons, by using the exact sequences associated with $E^*_Q(-)$ and $E^*(-) \otimes Q$ for the pair $(Y^{(n)}, Y^{(n-1)})$. Then the lemma follows. $\square$

Proof of Theorem 2.1. The proof here is similar to that of [11, Theorem 1.1]. Consider the Leray-Serre spectral sequences $\{E^{p,q}_r(i); d^{(i)}_r\}$ with local coefficients (which are actually constant) given by $H^*_{(Q)}(M)$ and $H^*_{(Q)}(\mathrm{pt})$, $i = 1, 2$, converging to $H^*_{(Q)}(R^k(M))$ and $H^*_{(Q)}(CP(k))$ respectively ([9, p. 350] or [10, p. 630]), with

$$E^{p,q}_2(1) = H^p(CP(k); H^q_{(Q)}(M)),$$
$$E^{p,q}_2(2) = H^p(CP(k); H^q_{(Q)}(\mathrm{pt})).$$

Also consider the morphism

$$p^* : E^{p,q}_r(2) \to E^{p,q}_r(1)$$

of related spectral sequences induced by the projection $p : R^k(M) \to CP(k)$. Since

$$H^i_{(Q)}(M) = 0 \quad \text{if } i \text{ is even and } 2 \leq i \leq 2n-2,$$

we see at stage 2 that the morphism p^* is an isomorphism if $p+q$ is even and $0 \leq p+q \leq 2n-1$. Now the spectral sequence $E^{p,q}_r(2)$ collapses and all nontrivial elements on stage 2 survive to infinity. Thus the images of p^* are all permanent cocycles. Since the projection p has a section ρ_x, the nontrivial images of p^* also survive to infinity when $0 \leq p+q \leq 2n-2$. Therefore the morphism $p^* : E^{p,q}_r(2) \to E^{p,q}_r(1)$ is an isomorphism for all $r \geq 2$ if $p+q$ is even and $0 \leq p+q \leq 2n-1$, which induces an isomorphism

$$p^* : H^i_{(Q)}(CP(k)) \to H^i_{(Q)}(R^k(M))$$

for i even and $0 \leq i \leq 2n-1$.

Next we consider the Atiyah-Hirzebruch-Whitehead spectral sequences $\{E^{p,q}_r(i), d^{(i)}_r\}$ ([9, p. 340] or [10, p. 630]), $i = 3, 4$, built up from the CW-skeleton filtrations of $R^k(M)$ and $CP(k)$, and converging to $K^*_{(Q)}(R^k(M))$ and $K^*_{(Q)}(CP(k))$ respectively, with

$$E^{p,q}_2(3) = H^p(R^k(M); K^q_{(Q)}(\mathrm{pt})) = H^p_{(Q)}(R^k(M); K^q(\mathrm{pt})),$$
$$E^{p,q}_2(4) = H^p(CP(k); K^q_{(Q)}(\mathrm{pt})) = H^p_{(Q)}(CP(k); K^q(\mathrm{pt})).$$

Let

$$p^* : E^{p,q}_r(4) \to E^{p,q}_r(3)$$

be the morphism of related spectral sequences induced by the projection p. Then, at stage 2, p^* is an isomorphism if p is even and $0 \leq p \leq 2n-2$. Since the spectral sequence $\{E^{p,q}_r(4), d^{(4)}_r\}$ collapses and the projection p has a section ρ_x, we see that

$p^*: E_r^{p,q}(4) \to E_r^{p,q}(3)$ is an isomorphism for $r \geq 2$ if p is even and $0 \leq p \leq 2n-2$. Thus

$$p^*: K^0_{(Q)}(CP(k)) \to K^0_{(Q)}(R^k(M))$$

is an isomorphism up to the elements of filtrations $> 2n-1$, that is,

$$p^*: K^0_{(Q)}(CP(k))/F_{2n} \to K^0_{(Q)}(R^k(M))/G_{2n}$$

is an isomorphism, where F_{2n}, G_{2n} are subgroups of elements of filtrations $> 2n-1$ of related groups. Let $\eta: K \to K_{(Q)}$ be the Bousfield localization and $\eta^*: K^*(X) \to K^*_{(Q)}(X)$ the induced homomorphism. Note that $K^0(X) \approx K(X)$ if X is a finite CW-complex. Thus we may regard η^* to be defined on $K(X)$. Choose $k > 2n+1$ and assume in $K^0_{(Q)}(R^k(M))$

$$\eta^*(\overline{T}_k(M) \otimes C) = p^*(\xi) + a,$$

where $a \in K^0_{(Q)}(R^k(M))$ is an element of filtration $> (2n-1)$, and $\xi \in K^0_{(Q)}(CP(k))$.

Let $j: B_G^{(2n-1)} \to CP(k)$ be the inclusion. Consider the homomorphism $\alpha^{(2n-1)}: RU(S^1) \to K(B_G^{(2n-1)})$. Since by (2), $\alpha^{(2n-1)}(\Theta_x) = \rho_x^*(\overline{T}_k(M) \otimes C)$, we have

$$\begin{aligned}\eta^*\alpha^{(2n-1)}(\Theta_x) &= \eta^* j^* \rho_x^*(\overline{T}_k(M) \otimes C) = j^*\rho_x^*\eta^*(\overline{T}_k(M) \otimes C)\\ &= j^*\rho_x^*(p^*(\xi) + a) = j^*(\xi),\end{aligned}$$

where the last equality is due to the fact that the element a is of filtration $> (2n-1)$, thus $j^*\rho_x^*(a) = 0$. Consequently, $\eta^*\alpha^{(2n-1)}(\Theta_x)$ is independent of the choices of $x \in F$, and $\eta^*\alpha^{(2n-1)}(\Theta_x - \Theta_y) = 0$ for any $x, y \in F$.

Note that $B_G^{(2n-1)}$ is $CP(n-1)$, since $G = S^1$. By Lemma 2.2 and the structure of $K^0(CP(n-1))$, we see that $\eta^*: K^0(CP(n-1)) \to K^0_{(Q)}(CP(n-1))$ is injective. Thus $\alpha^{(2n-1)}(\Theta_x - \Theta_y) = 0$ for any $x, y \in F$. Therefore

$$\Theta_x - \Theta_y \in \ker(\alpha^{(2n-1)}) = I(S^1)^n,$$

which implies that $\Theta_x - \Theta_y$ is divisible by $(1-t)^n$. This completes the proof for the first statement.

We now consider the last statement. First, we have the exact sequence

$$\widetilde{K}^0_{(Q)}(M^{(2n)}) \xleftarrow{f^*} \widetilde{K}^0_{(Q)}(R^k(M^{(2n)})) \xleftarrow{g^*} K^0_{(Q)}(R^k(M^{(2n)}), M^{(2n)}),$$

where $f: M^{(2n)} \to R^k(M^{(2n)})$ and $g: R^k(M^{(2n)}) \to (R^k(M^{(2n)}), M^{(2n)})$ are the inclusion and the projection respectively. Let $\lambda - m$ be the class in $\widetilde{K}^0_{(Q)}(R^k(M^{(2n)}))$ which corresponds to $i^*\eta^*(\overline{T}_k(M) \otimes C)$, where $i: R^k(M^{(2n)}) \to R^k(M)$ is the inclusion and m is the complex dimension of $\overline{T}_k(M) \otimes C$. Then $f^*(\lambda - m)$ is zero by the assumed condition. Thus by the exactness,

$$\lambda - m = g^*(\zeta)$$

for some $\zeta \in K^0_{(Q)}(R^k(M^{(2n)}), M^{(2n)})$.

Similar to what we did for the first statement, we consider the Leray-Serre spectral sequences $\{E_r^{p,q}(i); d_r^{(i)}\}$ with coefficients given by $H^*_{(Q)}(M^{(2n)})$ and $H^*_{(Q)}(\mathrm{pt})$, converging to $H^*_{(Q)}(R^k(M^{2n}), M^{(2n)})$ and $\widetilde{H}^*_{(Q)}(CP(k))$ for $i = 5, 6$ respectively, with

$$E_2^{p,q}(5) = \widetilde{H}^p(CP(k); H^q_{(Q)}(M^{(2n)})),$$
$$E_2^{p,q}(6) = \widetilde{H}^p(CP(k); H^q_{(Q)}(\mathrm{pt})).$$

Let $(p_1')^* : E_r^{p,q}(6) \to E_r^{p,q}(5)$ be the morphism of related spectral sequences induced by p_1', where $p_1' : (R^k(M^{(2n)}), M^{(2n)}) \to (CP(k), *)$ is the projection induced by the bundle projection $p_1 : R^k(M^{(2n)}) \to CP(k)$. Since $M^{(2n)}$ contains at least one fixed point x, p_1' has a section ρ_x. By the fact that

$$H^i_{(Q)}(M^{(2n)}) = 0 \quad \text{if } i \text{ is even and } 2 \le i \le 2n-2,$$

at stage 2, we see $(p_1')^*$ is an isomorphism if $p+q$ is even and $0 \le p+q \le 2n$. Now the spectral sequence $E_r^{p,q}(6)$ collapses and all nontrivial elements on stage 2 survive to infinity. Thus the images of $(p_1')^*$ are permanent cocycles, and the nontrivial images of $(p_1')^*$ survive to infinity, for p_1' has a section ρ_x. This implies that $(p_1')^* : E_r^{p,q}(6) \to E_r^{p,q}(5)$ is an isomorphism for $p+q$ even and $0 \le p+q \le 2n$, and $r \ge 2$. Thus

$$(p_1')^* : \widetilde{H}^i_{(Q)}(CP(k)) \to H^i_{(Q)}(R^k(M^{(2n)}), M^{(2n)})$$

are isomorphisms if i is even and $0 \le i \le 2n$.

Next consider the Atiyah-Hirzebruch-Whitehead spectral sequences

$$\{E_r^{p,q}(i), d_r^{(i)}\}, \qquad i = 7, 8,$$

built up by the CW-skeleton filtrations of $(R^k(M^{(2n)}), M^{(2n)})$ and $(CP(k), *)$, and converging to $K^*_{(Q)}(R^k(M^{(2n)}), M^{(2n)})$ and $\widetilde{K}^*_{(Q)}(CP(k))$ respectively, with

$$\begin{aligned} E_2^{p,q}(7) &= H^p(R^k(M^{(2n)}), M^{(2n)}; K^q_{(Q)}(\mathrm{pt})) \\ &= H^p_{(Q)}(R^k(N^{(2n)}), M^{(2n)}; K^q(\mathrm{pt})), \end{aligned}$$

$$E_2^{p,q}(8) = \widetilde{H}^p(CP(k); K^q_{(Q)}(\mathrm{pt})) = \widetilde{H}^p_{(Q)}(CP(k); K^q(\mathrm{pt})).$$

Note that at stage 2, $(p_1')^* : E_2^{p,q}(8) \to E_2^{p,q}(7)$ is an isomorphism if p is even and $0 \le p \le 2n$. Similar to what we did in the first statement for the spectral sequences $\{E_r^{p,q}(i); d_r^{(i)}\}$ with $i = 3, 4$, we see

$$(p_1')^* : \widetilde{K}^0_{(Q)}(CP(k)) \to K^0_{(Q)}(R^k(M^{(2n)}), M^{(2n)})$$

is an isomorphism up to filtrations $> 2n$. Therefore we may assume $\zeta = (p_1')^*(c) + a$ in $K^0_{(Q)}(R^k(M^{(2n)}), M^{(2n)})$, where $c \in \widetilde{K}^0_{(Q)}(CP(k))$, and the element a is of filtration $> 2n$. Thus in $\widetilde{K}^0_{(Q)}(R^k(M^{(2n)}))$

$$i^*\eta^*(\overline{T}_k(M) \otimes C) - m = g^*(p_1')^*(c) + g^*(a).$$

Let $h_x : B_G^{(2n)} \to R^k(M^{(2n)})$ be the CW-approximation of the composition

$$B_G^{(2n)} \xrightarrow{j} CP(k) \xrightarrow{\rho_x} R^k(M).$$

Then

$$\begin{aligned}\eta^*\alpha^{(2n)}(\Theta_x) &= \eta^* j^* \rho_x^*(\overline{T}_k(M)\otimes C) = j^*\rho_x^*\eta^*(\overline{T}_k(M)\otimes C)\\ &= h_x^* i^* \eta^*(\overline{T}_k(M)\otimes C) = h_x^* g^*(p_1')^*(c) + h_x^* g^*(a) + m\\ &= h_x^* g^*(p_1')^*(c) + m = h_x^* p_1^* j_0^*(c) + m = j^* j_0^*(c) + m\end{aligned}$$

from the commutative diagram

$$\begin{array}{ccccc} (R^k(M^{(2n)}),\, M^{(2n)}) & \xleftarrow{g} & R^k(M^{(2n)}) & \xrightarrow{i} & R^k(M) \\ \downarrow p_1' & & \downarrow p_1 & & \downarrow p \\ (CP(k),\, *) & \xleftarrow{j_0} & CP(k) & \xrightarrow{1} & CP(k) \\ & & \uparrow j & & \\ & & B_G^{(2n)} & & \end{array}$$

where j_0 is the ordinary projection $CP(k) \to (CP(k), *)$. Here the fifth equality is due to the fact that the element a is of filtration $> 2n$. The sixth equality is from the fact $p_1' g = j_0 p_1$. The last equality follows from the fact that ih_x is homotopic to $\rho_x j$, thus pih_x $(= p_1 h_x)$ is homotopic to $p\rho_x j$ $(= j)$. This shows $\eta^*\alpha^{(2n)}(\Theta_x)$ is independent of the choices of $x \in F$ and $\eta^*\alpha^{(2n)}(\Theta_x - \Theta_y) = 0$ for any $x, y \in F$. Since $B_G^{(2n)} = CP(n)$ and $\eta^* : K^*(CP(n)) \to K^*_{(Q)}(CP(n))$ is injective, we have $\alpha^{(2n)}(\Theta_x - \Theta_y) = 0$. The last statement follows from the fact that $\ker(\alpha^{(2n)}) = \ker(\alpha^{(2n+1)})$. □

Proof of Theorem 1.2. The proof is similar to that of [4, Theorem VI]. By considering a fixed maximal torus T of G, we may reduce G to the case when $G = (S^1)^r$. Consider the map $S^1 \to (S^1)^r$ given by $z \to (z^{n_1}, z^{n_2}, \ldots, z^{n_r})$, which induces a homomorphism $RU((S^1)^r) \to RU(S^1)$ given by $t_i \to t^{n_i}$, where $n_1, n_2, \ldots, n_r$ are integers. Suppose

$$\Theta_x - \Theta_y = P(t_1, t_2, \ldots, t_r) \in RU((S^1)^r).$$

Then, by Theorem 2.1, $P(t^{n_1}, t^{n_2}, \ldots, t^{n_r})$ is divisible by $(1-t)^n$ (or $(1-t)^{n+1}$ when $T(M)\otimes C$ is stably trivial in $K(M^{(2n)})\otimes Q$) for any integers $n_1, n_2, \ldots, n_r$. An argument on elementary algebra, as claimed in [4], shows this is equivalent to $P(t_1, t_2, \ldots, t_r) \in (I((S^1)^r))^n$ (resp. $(I((S^1)^r))^{n+1}$). □

Proof of Theorem 1.1. Note that in condition (ii), M is totally nonhomologous to zero in M_G with coefficient in Q implies that M is totally nonhomologous to zero in M_{S^1} with coefficient in Q for any circle subgroup of G. Then similar to the proof of Theorem 1.2, we may assume $G = S^1$ for both cases (i) and (ii). By the exact sequence (1), it suffices to prove $\alpha^{(2k+1)}(\Theta_x - \Theta_y) = 0$ for all $k > 0$ and $x, y \in F$.

For (i), we consider the Leray-Serre spectral sequences $\{E_r^{p,q}(i), d_r^{(i)}\}$, $i = 9, 10$, with

$$\begin{aligned} E_2^{p,q}(9) &= H^p(CP(k), K^q_{(Q)}(M)),\\ E_2^{p,q}(10) &= H^p(CP(k), K^q_{(Q)}(\mathrm{pt})), \end{aligned}$$

converging to $K^0_{(Q)}(R^k(M))$ and $K^0_{(Q)}(CP(k))$ respectively. Note that

$$E_2^{p,q}(9) = H^p(CP(k), K^q_{(Q)}(M)) = H^p(CP(k), K^q(M) \otimes Q),$$

and the morphism $p^* : E_r^{p,q}(10) \to E_r^{p,q}(9)$ is an isomorphism at $r = 2$ if $p + q$ is even. With a similar argument as for the spectral sequences $\{E_r^{p,q}(i), d_r^{(i)}\}$, $i = 3, 4$, in the proof of Theorem 2.1, we see that

$$p^* : K^0_{(Q)}(CP(k)) \to K^0_{(Q)}(R^k(M))$$

is an isomorphism. Thus we may assume

$$\eta^*(\overline{T}_k(M) \otimes C) = p^*(\xi),$$

where $\xi \in K^0_{(Q)}(CP(k))$. Then, similar to the proof of Theorem 2.1,

$$\eta^* \alpha^{(2k+1)}(\Theta_x) = \rho_x^* \eta^*(\overline{T}_k(M) \otimes C) = \rho_x^* p^*(\xi) = \xi \in K^0_{(Q)}(CP(k)),$$

which is independent of the choices of $x \in F$. Therefore $\alpha^{(2k+1)}(\Theta_x - \Theta_y) = 0$ for any $x, y \in F$. Thus $\Theta_x = \Theta_y$ by (1).

Consider statement (ii). Since M is totally nonhomologous to zero in M_G with coefficient in Q implies that M is totally nonhomologous to zero in $R^k(M)$ with coefficient in Q for any $k \geq 0$, we see that $H^*(R^k(M); Q)$ is generated by some $\{1, c_i\}$ and some products of two or more c_i as a module over $H^*(CP(k); Q)$ for any $k > 0$, where c_i is of odd degree. Consider the homomorphism $\rho_x^* : H^*(R^k(M); Q) \to H^*(CP(k); Q)$. Then we have $\rho_x^*(c_i) = 0$, since the degree of c_i is odd. Thus ρ_x^* is independent of the choices of $x \in F$.

Now let X be a finite CW-complex and

$$\mathrm{ch} : K^0_{(Q)}(X) = K^0(X) \otimes Q \to H^{**}(X; Q)$$

the Chern character, where $H^{**}(X) = \bigoplus_{i=0}^{\infty} H^{2i}(X; Q)$. Then ch is an isomorphism ([7]) and we have the following commutative diagram:

$$\begin{array}{ccc} K^0_{(Q)}(R^k(M)) & \xrightarrow{ch} & H^{**}(R^k(M));\ Q) \\ \downarrow \rho_x^* & & \downarrow \rho_x^* \\ K^0_{(Q)}(CP^k)) & \xrightarrow{ch} & H^{**}(CP(k);\ Q) \end{array} \tag{3}$$

Since $\rho_x^* : H^{**}(R^k(M); Q) \to H^{**}(CP(k); Q)$ is independent of the choices of $x \in F$, the map $\rho_x^* : K^0_{(Q)}(R^k(M)) \to K^0_{(Q)}(CP(k))$ is independent of the choices of $x \in F$ by diagram (3). Thus $\rho_x^*(\overline{T}_k(M) \otimes C) \in K^0(CP(k))$ is independent of the choices of $x \in F$ by the commutative diagram

$$\begin{array}{ccc} K^0(R^k(M)) & \xrightarrow{\eta^*} & K^0_{(Q)}(R^k(M)) \\ \downarrow \rho_x^* & & \downarrow \rho_x^* \\ K^0(CP(k)) & \xrightarrow{\eta^*} & K^0_{(Q)}(CP(k)) \end{array} \tag{4}$$

where the η^* in the bottom row is injective, and the proof for (ii) follows. □

Proof of Corollary 1.3. If n is odd, then, by using the Atiyah-Hirzebruch-Whitehead spectral sequence with $E_2^{p,q} = \widetilde{H}^p(M; K^q_{(Q)}(\mathrm{pt}))$ converging to $\widetilde{K}^*_{(Q)}(M)$, we have $\widetilde{K}^0_{(Q)}(M) = 0$. This means $\widetilde{K}^0(M) \otimes Q = 0$ by Lemma 2.2, and $\Theta_x = \Theta_y$ by Theorem 1.1(i).

Now let n be even. Similar to the proof of Theorem 1.2, we may assume $G = S^1$. Consider the Leray-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ with $E_2^{p,q} = H^p(CP^\infty; H^q_{(Q)}(M))$, converging to $H^*_{(Q)}(M_G)$. Obviously, this spectral sequence collapses. Thus $H^*_{(Q)}(M_G)$ is a free $H^*_{(Q)}(CP^\infty)$ module with a basis $\{1, c\}$. Since we are working on the coefficient Q, we may require $c^2 \in p^* H^*_{(Q)}(CP^\infty)$. Actually, if

$$c^2 = p^*(a)c + p^*(b^2),$$

then we can replace c by $c' = c - \frac{1}{2}p^*(a)$ and see $(c')^2 \in p^* H^*_{(Q)}(CP^\infty)$. Let $\rho_x^*(c) = b_x$. Then c^2 is in the image of p^* implies $(b_x)^2 = (b_y)^2$ for $x, y \in F$. Thus $\rho_x^*(c) = \rho_y^*(c)$ or $-\rho_y^*(c)$.

Now the Leray-Serre spectral sequence associated with $R^k(M)$ collapses, and $H^*_{(Q)}(R^k(M))$ is a free $H^*_{(Q)}(CP(k))$ module with a basis $\{1, c'\}$. By the map of Leray-Serre spectral sequences induced by the inclusion $j : R^k(M) \to M_G$, we may require $c' = j^*(c)$. Thus by diagrams (3) and (4) again, if $\rho_x^*(c) = \rho_y^*(c)$ in $H^*(CP^\infty; Q)$, then $\alpha^{(2k+1)}(\Theta_x - \Theta_y) = 0$ for any $k \geq 0$. This means $\Theta_x = \Theta_y$. Since we have at most two different morphisms

$$\rho_x^* : H^*(M_G; Q) \to H^*(CP^\infty; Q),$$

there are at most two representations Θ_x for $x \in F$ up to equivalency. □

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References

1. M. F. Atiyah *K-theory*, Benjamin, New York, 1967. MR **36:**7130
2. A. Borel and Hirzebruch, *On the characteristic classes of the homogeneous spaces*, Amer. J. Math. **80** (1958), 458–538. MR **21:**1586
3. A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281. MR **80m:**55006
4. G. E. Bredon, *Representations at fixed points of smooth actions of compact groups*, Ann. of Math. (2) **89** (1969), 512–532. MR **39:**7628
5. ———, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972. MR **54:**1265
6. Wu Yi Hsiang, *Cohomology theory of topological transformation groups*, Springer-Verlag, New York, Heidelberg, and Berlin, 1975. MR **54:**11363
7. Max Karoubi, *K-theory, An introduction*, Springer-Verlag, Berlin, Heidelberg, and New York, 1978. MR **58:**7605
8. J. P. May, *Equivariant homotopy and homology theory*, Contemp. Math., vol. 12, Amer. Math. Soc., Providence, RI, 1982, 209–217. MR **83m:**55011
9. R. M. Swizter, *Algebraic topology —homotopy and homology*, Springer-Verlag, New York, Heidelberg, and Berlin, 1975. MR **52:**6695

10. G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, Heidelberg, and Berlin, 1978. MR **80b:**55001
11. Huajian Yang, *Representations at fixed points of smooth actions of finite groups*, (to appear). CMP 95:03

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