

GENERALIZED RATIONAL IDENTITIES OF SUBNORMAL SUBGROUPS OF SKEW FIELDS

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ABSTRACT. Let D be a skew field with infinite center K such that $[D : K] = \infty$, and let H be a non-central subnormal subgroup of the multiplicative group $D^* = D \setminus \{0\}$ of D . Then there are no non-trivial generalized rational identities of H . This generalizes a theorem proved by Makar-Limanov.

Several authors (e.g. [5], [6]) have studied subnormal subgroups of multiplicative groups of skew fields which satisfy some rational identity. In this note we shall study general rational identities of non-central subnormal subgroups of skew fields.

Let D be a skew field with center K and $K\langle X \rangle$ the free K -algebra on a finite set $X = \{x_1, x_2, \dots, x_n\}$. We denote by $D\langle X \rangle = D *_k K\langle X \rangle$ the free product of D and $K\langle X \rangle$ over K and by $D(X)$ the universal skew field of fractions of $D\langle X \rangle$. Let $d = (d_i)$ be an element of D^n and $\alpha_d: D\langle X \rangle \rightarrow D$ the D -ring homomorphism defined by $\alpha_d(x_i) = d_i$, $i = 1, 2, \dots, n$. We denote by Σ_d the set of all matrices over $D\langle X \rangle$ which are mapped by α_d to invertible matrices over D . Let Σ_d^{-1} be the set of all entries of inverses A^{-1} over $D(X)$ for all $A \in \Sigma_d$. Then Σ_d^{-1} is a ring and it contains $D\langle X \rangle$ as a subring. Moreover, there is a D -ring homomorphism $\beta_d: \Sigma_d^{-1} \rightarrow D$ which extends α_d and satisfies that any element of Σ_d^{-1} not in the kernel of β_d has an inverse in Σ_d^{-1} (see [4, Chapter 7]). Let $f = f(x_i)$ be an element of $D(X)$. If f belongs to Σ_d^{-1} , we say f is defined at (d_i) and write $f(d_i)$ instead of $\beta_d(f)$.

The purpose of this note is to prove the following theorem which is a generalization of a theorem of Makar-Limanov [6, Theorem].

Theorem 1. *Let D be a skew field with infinite center K such that the dimension of D over K is infinite, let H be a non-central subnormal subgroup of the multiplicative group $D^* = D \setminus \{0\}$ of D and let $f(x_i)$ be a non-zero element of $D(X)$. Then there is an element $(h_i) \in H^n$ such that $f(x_i)$ is defined at (h_i) and $f(h_i) \neq 0$.*

Let R be a ring and A an $m \times m$ matrix over R . We say that A is non-full if there are matrices P, Q over R such that $A = PQ$ and P is $m \times r$, Q is $r \times m$, with $r < m$; in the contrary case A is full. We denote by e_j the column vector with 1 in the j th place and 0's elsewhere and by A_j^* the matrix obtained by replacing the 1st column of A by $e_j \in {}^m R$. We denote by I_k the $k \times k$ identity matrix over R . Let B and C be matrices over R . If there are identity matrices I_p and I_q such that the diagonal sums $B \oplus I_p$ and $C \oplus I_q$ are associated, then B and C are said

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to be stably associated (see [4, p. 26]). If a and b are invertible elements of R , we denote by (a, b) the multiplicative commutator $a^{-1}b^{-1}ab$ of a and b . We denote by $R[t]$ the polynomial ring over R in a central indeterminate t , $R[[t]]$ the ring of formal power series, and $R((t))$ the ring of formal Laurent series. There is a natural embedding of $R[[t]]$ into $R((t))$, so we shall think of $R[[t]]$ as a subring of $R((t))$. If R is a semifir, we denote by $U(R)$ the universal skew field of fractions of R . Let R and S be K -algebras. We denote by $R *_K S$ the free product of R and S over K . Let $Y = \{y_1, y_2, \dots, y_n\}$ be a finite set and $K\langle X, Y \rangle$ the free K -algebra on the set $X \cup Y = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$. We write $D\langle X, Y \rangle$ for $D *_K K\langle X, Y \rangle$ and denote by $D(X, Y)$ the universal skew field of fractions of $D\langle X, Y \rangle$. We denote by $D\langle X, X^{-1}, Y, Y^{-1} \rangle$ (resp. $D\langle X, Y, Y^{-1} \rangle$) the D -subring of $D(X, Y)$ generated by $X \cup X^{-1} \cup Y \cup Y^{-1} = \{x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}, y_1, y_2, \dots, y_n, y_1^{-1}, y_2^{-1}, \dots, y_n^{-1}\}$ (resp. $X \cup Y \cup Y^{-1}$).

To prove Theorem 1, we need several lemmas. We begin with the following.

Lemma 2. *Let R and S be rings and let $\varphi: R \rightarrow S[[t]]$ be a ring homomorphism. Then there is a ring homomorphism $\tilde{\varphi}: R[[t]] \rightarrow S[[t]]$ such that $\tilde{\varphi}(\sum_{i \geq 0} r_i t^i) = \sum_{i \geq 0} \varphi(r_i) t^i$, where $r_i \in R$ for all i .*

Proof. Note that the sum $\sum_{i \geq 0} \varphi(r_i) t^i$ is defined in $S[[t]]$, as is every element of the form $\sum_{i \geq 0} a_i t^i$ with $a_i \in S[[t]]$. The fact that $\tilde{\varphi}$ is a ring map is then immediate from the corresponding property of φ .

Lemma 3. *Let F be a free group on the set $X \cup Y = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$, and $K[F]$ the group algebra over K . Then $D *_K K[F]$ is a fir and there is the natural isomorphism $\varphi: U(D *_K K[F]) \cong D(X, Y)$. In particular, φ induces the isomorphism $D *_K K[S] \cong D\langle X, Y, Y^{-1} \rangle$, where S denotes a subsemigroup of F generated by $X \cup Y \cup Y^{-1} \subset F$ and $K[S]$ the semigroup algebra.*

Proof. The result follows from [3, Theorem 5.3.2] and [2, Lemma 8].

Lemma 4. *Let h be a non-central element of D and set $\bar{y}_i = y_i^{-1} h^{-1} y_i x_i y_i^{-1} h y_i \in D\langle X, Y, Y^{-1} \rangle$, $i = 1, 2, \dots, n$. Then the D -ring homomorphism τ of $D\langle X \rangle$ into $D\langle X, Y, Y^{-1} \rangle$ defined by $\tau(x_i) = \bar{y}_i - x_i$, $i = 1, 2, \dots, n$, is injective.*

Proof. First we show that the D -ring homomorphism $\alpha: D\langle X, Y \rangle \rightarrow D\langle X, Y, Y^{-1} \rangle$ defined by $\alpha(x_i) = x_i$ and $\alpha(y_i) = \bar{y}_i$, $i = 1, 2, \dots, n$, is injective. Let $\Delta = \{d_1 = 1, d_2 = h^{-1}, d_3, \dots\}$ be a K -basis of D and P the K -basis of $D\langle X \rangle$ consisting of the monomials $d_{i_0} x_{i_1} d_{i_1} x_{i_2} \dots x_{i_m} d_{i_m}$ where $d_{ij} \in \Delta$, $x_{ij} \in X$. If α is not injective, let us take a non-zero element $f(x_i, y_i)$ of $\ker \alpha$ such that the Y -degree is minimal and write

$$f(x_i, y_i) = \sum_{j,k} p_{jk}(x_i) y_j q_{jk}(x_i, y_i) + r(x_i)$$

where $p_{jk}(x_i) \in P$ with $p_{js}(x_i) \neq p_{jt}(x_i)$ for $s \neq t$, $q_{jk}(x_i, y_i) \in D\langle X, Y \rangle$, $r(x_i) \in D\langle X \rangle$. Without loss of generality we may assume $p_{11}(x_i) y_1 q_{11}(x_i, y_i) \neq 0$. Then we have

$$\alpha(f(x_i, y_i)) = \sum_{j,k} p_{jk}(x_i) \bar{y}_j q_{jk}(x_i, \bar{y}_i) + r(x_i) = 0.$$

Since each $p_{jk}(x_i) \bar{y}_j q_{jk}(x_i, \bar{y}_i)$ can be written as a K -linear combination of the monomials of the form $p_{jk}(x_i) y_j^{-1} h^{-1} y_j x_j u$, where $u \in D\langle X, Y, Y^{-1} \rangle$, by [1, Corollary

8.1] and Lemma 3 we obtain $r(x_i) = 0$. By [1, Corollary 8.1], $p_{jk}(x_i)y_j^{-1}h^{-1}y_jx_j$'s are right $D\langle X, Y, Y^{-1} \rangle$ -linearly independent, so we obtain $y_1^{-1}hy_1q_{11}(x_i, \bar{y}_i) = 0$, and hence $q_{11}(x_i, \bar{y}_i) = 0$, which contradicts the minimality of the Y -degree of $f(x_i, y_i)$. Thus α is injective. Now let $\beta: D\langle X \rangle \rightarrow D\langle X, Y \rangle$ be a D -ring homomorphism defined by $\beta(x_i) = y_i - x_i, i = 1, 2, \dots, n$, and $\gamma: D\langle X, Y \rangle \rightarrow D\langle X \rangle$ a D -ring homomorphism defined by $\gamma(x_i) = x_i, \gamma(y_i) = 2x_i, i = 1, 2, \dots, n$. Then $\gamma\beta = 1$ on $D\langle X \rangle$. Hence β is injective. Clearly, $\tau = \alpha\beta$, thus τ is injective. This completes the proof.

Lemma 5. *Let h be a non-central element of D and $f(x_i)$ a non-zero element of $D\langle X \rangle$ such that $f(x_i)$ is defined at $(1) = (1, 1, \dots, 1) \in D^n$. Then $f(x_i)$ is defined at $((x_i, y_i^{-1}hy_i)) = (x_i^{-1}y_i^{-1}h^{-1}y_ix_iy_i^{-1}hy_i) \in D\langle X, Y \rangle^n$ and $f((x_i, y_i^{-1}hy_i)) \neq 0$.*

Proof. We may assume that $f(x_i) \notin D$ and $f(x_i)$ is the first component of the solution u of a matrix equation $Au = e_j$, where $A = A(x_i)$ is an $m \times m$ matrix over $D\langle X \rangle$ such that $A(1)$ is invertible over D and $e_j \in {}^mR$. Since $A(1)$ is invertible and $(x_i, y_i^{-1}hy_i) = 1$ for $x_i = y_i = 1, i = 1, 2, \dots, n$, $A((x_i, y_i^{-1}hy_i))$ is a full matrix over $D\langle X, X^{-1}, Y, Y^{-1} \rangle$, hence invertible over $D\langle X, Y \rangle$ by Lemma 3. Thus $f(x_i)$ is defined at $((x_i, y_i^{-1}hy_i)) \in D\langle X, Y \rangle$. Suppose $f((x_i, y_i^{-1}hy_i)) = 0$. We write $A_j^* = A_j^*(x_i)$, the matrix obtained by replacing the 1st column of $A(x_i)$ by $e_j \in {}^mR$. By Cramer's rule [4, Proposition 1.3, p. 384], $A_j^*((x_i, y_i^{-1}hy_i))$ is a singular matrix over $D\langle X, Y \rangle$. Then, by Lemma 3, $A_j^*((x_i, y_i^{-1}hy_i))$ is a non-full matrix over $D\langle X, X^{-1}, Y, Y^{-1} \rangle$. Hence $A_j^*((1 + x_it, y_i^{-1}hy_i))$ is a singular matrix over $D\langle X, Y \rangle((t))$. Let us abbreviate $y_i^{-1}h^{-1}y_ix_iy_i^{-1}hy_i$ as \bar{y}_i for $i = 1, 2, \dots, n$. We have the following representation in $D\langle X, Y \rangle((t))$

$$\begin{aligned} (1 + x_it, y_i^{-1}hy_i) &= (1 + x_it)^{-1}y_i^{-1}h^{-1}y_i(1 + x_it)y_i^{-1}hy_i \\ &= 1 + g_{i1}t + g_{i2}t^2 + \dots + g_{ip}t^p + \dots \end{aligned}$$

where $g_{i1} = \bar{y}_i - x_i, g_{ip} \in D\langle X, Y, Y^{-1} \rangle$. Since $A(1)$ is invertible, it follows that $A((1 + x_it, y_i^{-1}hy_i)) = A(1 + g_{i1}t + g_{i2}t^2 + \dots + g_{ip}t^p + \dots)$ is an invertible matrix over $D\langle X, Y \rangle((t))$. Thus $f(x_i)$ is defined at $((1 + x_it, y_i^{-1}hy_i)) \in D\langle X, Y \rangle((t))^n$ and $f((1 + x_it, y_i^{-1}hy_i)) = 0$. On the other hand, by [2, Lemma 7] $f(1 + x_it) \in D\langle X \rangle((t))$ has the following form:

$$f(1 + x_it) = f_0 + f_k(x_i)t^k + f_{k+1}(x_i)t^{k+1} + \dots$$

where $f_0 \in D, k \geq 1$ and $f_s(x_i) \in D\langle X \rangle$ is homogeneous of X -degree s , for $s = k, k+1, k+2, \dots$ and $f_k(x_i) \neq 0$. By Lemma 2 we have a D -ring homomorphism $\varphi: D\langle X \rangle[[t]] \rightarrow D\langle X, Y, Y^{-1} \rangle[[t]]$ such that $\varphi(t) = t$ and $\varphi(x_i) = g_{i1} + g_{i2}t + \dots + g_{ip}t^{p-1} + \dots, i = 1, 2, \dots, n$. Then we have $\varphi(1 + x_it) = (1 + x_it, y_i^{-1}hy_i), i = 1, 2, \dots, n$, and

$$\begin{aligned} f((1 + x_it, y_i^{-1}hy_i)) &= \varphi(f(1 + x_it)) \\ (1) \qquad \qquad \qquad &= f_0 + \varphi(f_k(x_i))t^k + \varphi(f_{k+1}(x_i))t^{k+1} + \dots \end{aligned}$$

The coefficient of t^k of the right-hand side of (1) is $f_k(\bar{y}_i - x_i)$. By Lemma 4 $f_k(\bar{y}_i - x_i) \neq 0$, hence $f((1 + x_it, y_i^{-1}hy_i)) \neq 0$ in $D\langle X, Y \rangle((t))$, a contradiction. This completes the proof.

Lemma 6. *Let D be a skew field with infinite center K and with the dimension of D over K infinite, and let H be a non-central subnormal subgroup of $D^* = D \setminus \{0\}$. Then any full matrix over $D\langle X \rangle$ is invertible for some choice of X in H .*

Proof. Let $A(x_i)$ be an $m \times m$ full matrix over $D\langle X \rangle$ and H a r -subnormal subgroup of D^* , that is, there is a chain $H = H_r \triangleleft H_{r-1} \triangleleft \cdots \triangleleft H_0 = D^*$. We shall use induction on r . We denote by $\text{diag}(x_1, x_2, \dots, x_n)$ the $n \times n$ matrix such that the (i, i) -entry is x_i and 0's elsewhere. Since $A(x_i) \oplus \text{diag}(x_1, x_2, \dots, x_n)$ is a full matrix over $D\langle X \rangle$, by Cohn's specialization lemma [4, Lemma 5.9.5] there is an element $(d_i) \in D^n$ such that $A(d_i) \oplus \text{diag}(d_1, d_2, \dots, d_n)$ is invertible. Clearly $(d_i) \in H_0^n$ and $A(d_i)$ is invertible. Thus the lemma holds for the case $r = 0$. Now let s be the supremum of the rank of $A(x_i)$ as its arguments range over $H = H_r$, and let h be a non-central element of H_r . Let $p(x_i)$ be a non-zero entry of $A(x_i)$. Then $p(x_i) \in D\langle X \rangle$ and so is defined at $(1) \in D^n$, and hence by Lemma 5, $p((x_i, y_i^{-1}hy_i)) \neq 0$. By Higman's trick (see [4, p. 272]), the 1×1 matrix $(p((x_i, y_i^{-1}hy_i)))$ is stably associated to a full matrix $P(x_i, y_i)$ over $D\langle X, Y \rangle$. By induction hypothesis there is an element $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) \in H_{r-1}^{2n}$ such that $P(a_i, b_i)$ is invertible. Hence $p((a_i, b_i^{-1}hb_i)) \neq 0$ and $(a_i, b_i^{-1}hb_i) \in H_r^n$. Thus we may assume the rank of $A(u_i)$ is $s \geq 1$ for $(u_i) \in H_r^n$. We have to show that $s = m$. So let us assume that $s < m$. Applying the D -automorphism of $D\langle X \rangle$ defined by $x_i \rightarrow u_i x_i$, $i = 2, \dots, n$, we may assume that the rank of $A(1)$ is s and using elementary transformations we may take the principal $s \times s$ minor of $A(1)$ to be invertible. Thus we can write

$$A(x_i) = \begin{pmatrix} B_1(x_i) & B_2(x_i) \\ B_3(x_i) & B_4(x_i) \end{pmatrix}$$

where $B_1(x_i)$ is $s \times s$ and $B_1(1)$ is invertible. Then we obtain

$$\begin{aligned} & \begin{pmatrix} 1 & B_1(x_i)^{-1}B_2(x_i) \\ 0 & B_4(x_i) - B_3(x_i)B_1(x_i)^{-1}B_2(x_i) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -B_3(x_i) & 1 \end{pmatrix} \begin{pmatrix} B_1(x_i)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_1(x_i) & B_2(x_i) \\ B_3(x_i) & B_4(x_i) \end{pmatrix}. \end{aligned}$$

Since $A(x_i)$ is invertible in $D\langle X \rangle$, $B_4(x_i) - B_3(x_i)B_1(x_i)^{-1}B_2(x_i) \neq 0$. Let $q(x_i) \in D\langle X \rangle$ be a non-zero entry of $B_4(x_i) - B_3(x_i)B_1(x_i)^{-1}B_2(x_i)$. Clearly $q(x_i)$ is defined at $(1) \in D^n$, hence by Lemma 5, $q(x_i)$ is defined at $((x_i, y_i^{-1}hy_i)) \in D\langle X, Y \rangle^n$ and $q((x_i, y_i^{-1}hy_i)) \neq 0$. Thus there is a full matrix $Q = Q(x_i, y_i)$ over $D\langle X, Y \rangle$ such that $q((x_i, y_i^{-1}hy_i))$ is the first component of the solution u of a matrix equation $Qu = e_j$ and $Q_j^* = Q_j^*(x_i, y_i)$ is full. Clearly $B_1((x_i, y_i^{-1}hy_i))$ is a full matrix over $D\langle X, X^{-1}, Y, Y^{-1} \rangle$; by Higman's trick, it is stably associated to a full matrix $B'_1(x_i, y_i)$ over $D\langle X, Y \rangle$. By induction hypothesis there is an element $(v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n) \in H_{r-1}^{2n}$ such that $B'_1(v_i, w_i) \oplus Q(v_i, w_i) \oplus Q_j^*(v_i, w_i)$ is invertible. Hence $q((v_i, w_i^{-1}hw_i)) \neq 0$ and the rank of $A((v_i, w_i^{-1}hw_i))$ is greater than or equal to $s + 1$. Since $((v_i, w_i^{-1}hw_i)) \in H_r^n$, this contradicts the definition of s and the result follows.

Proof of Theorem 1. There is a full matrix $A(x_i)$ over $D\langle X \rangle$ such that $f(x_i)$ is the first component of the solution u of a matrix equation $Au = e_j$ and $A_j^* = A_j^*(x_i)$ is full. Clearly $A(x_i) \oplus A_j^*(x_i)$ is full, so by Lemma 6 there is an element $(h_i) \in H^n$

such that $A(h_i) \oplus A_j^*(h_i)$ is invertible. Thus $f(x_i)$ is defined at $(h_i) \in H^n$ and $f(h_i) \neq 0$ by Cramer's rule. This completes the proof.

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