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ON CLOSE TO LINEAR COCYCLES

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ABSTRACT. If we have a flow $(X, \mathbb{Z}^m)$ and a cocycle h on this flow, $h : X \times \mathbb{Z}^m \to \mathbb{R}^m$, then h is called *close to linear* if h can be written as the direct sum of a linear (constant) cocycle and a cocycle in the closure of the coboundaries. Many of the desirable consequences of linearity hold for such cocycles and, in fact, a close to linear cocycle is cohomologous to a cocycle which is norm close to a linear one. Furthermore in the uniquely ergodic case all cocycles are close to linear. We also establish that a close to linear cocycle which is covering is cohomologous to one with the special property that it can be extended by piecewise linearity to an invertible cocycle from $X \times \mathbb{R}^m$ to itself. This implies that a suspension obtained from a close to linear cocycle is isomorphic to a time change of the suspension obtained from the identity cocycle.

1. The space of cocycles

This paper is one of a sequence ([2], [1], [3], [4]) designed to understand the structure of the space of continuous cocycles and the suspension flows they can be used to produce. Here we identify and study a particularly well-behaved class which we call "close to linear".

Let X be a compact metric space and let $\mathbb{Z}^m$ denote the integer lattice in $\mathbb{R}^m$, m-dimensional Euclidean space. We will assume that $\mathbb{Z}^m$ acts as a group of commuting homeomorphisms on X, that is, we have a *flow* $(X, \mathbb{Z}^m)$. A *cocycle* for such a flow is a continuous map $h : X \times \mathbb{Z}^m \to \mathbb{R}^m$ such that for all $x \in X$, $a, b \in \mathbb{Z}^m$ we have $h(x, a + b) = h(x, a) + h(ax, b)$ where ax denotes the action of a on x. This relationship is called the *cocycle equation.* Observe that the range of h could be $\mathbb{R}^n$ for any $n \geq 1$ and, indeed, by looking at the coordinate functions which are also cocycles we could do analysis by taking $n = 1$. (This viewpoint is exploited in [4].) However, as we will see below, the case $n = m$ is the appropriate environment in which to investigate the construction of $\mathbb{R}^m$ flows using cocyles and we will restrict ourselves to that situation in this paper.

Let $\mathcal{C}$ denote the set of cocycles on $(X, \mathbb{Z}^m)$. Clearly $\mathcal{C}$ is a vector space over $\mathbb{R}$. Using the norm $|t| = \sum_{i=1}^m |t_i|$ for $t = (t_1, \dots, t_m) \in \mathbb{R}^m$,

$$\|h\| = \sup\left\{\frac{|h(x, a)|}{|a|} : x \in X \text{ and } a \in \mathbb{Z}^m\right\}$$

defines a norm on $\mathcal{C}$. Using the cocycle equation it is not hard to show that

$$\|h\| = \sup\{|h(x, e_j)| : x \in X \text{ and } 1 \leq j \leq m\}$$

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where $e_1, \ldots, e_m$ is the standard basis for $\mathbb{R}^m$. With this norm $\mathcal{C}$ turns out to be a separable Banach space.

There are several easy ways of obtaining cocycles. If $T \in \mathcal{L}$, the linear operators from $\mathbb{R}^m$ to itself, then $h(x,a) = T(a)$ defines a cocycle in $\mathcal{C}$. Conversely if $h \in \mathcal{C}$ and $h(x,a) = h(y,a)$ for all $x, y \in X$ and $a \in \mathbb{Z}^m$, then the map $a \to h(x,a)$ is linear. Consequently $\mathcal{L}$ is a closed subspace of $\mathcal{C}$ which is called the space of *constant cocyles.*

The second easy way to produce cocycles is as follows. Suppose f is a continuous function from X into $\mathbb{R}^m$. Define $h \in \mathcal{C}$ by setting $h(x,a) = f(ax) - f(x)$. Such a cocyle is called a *coboundary* and the coboundaries, $\mathcal{B}$, form another subspace of $\mathcal{C}$. If two cocycles differ by a coboundary we will say they are *cohomologous.* These ideas are important because cohomologous cocycles have essentially the same properties.

Form the closed subspace $\mathcal{D} = \mathcal{L} + \overline{\mathcal{B}}$. It is the subspace $\mathcal{D}$ which is the subject of study of this paper. We refer to elements of $\mathcal{D}$ as "close to linear". In essence $\mathcal{D}$ consists of cocycles cohomologous to linear cocycles (which are essentially trivial) and limits of sequences of such cocycles. We want to understand the set $\mathcal{D}$; it may be rather rich (see §2), but we will establish that the members of $\mathcal{D}$ are all "well-behaved" cocycles (see §4). Of course in terms of the cocycle norm, a linear cocycle plus a coboundary may be very far away from a linear one. However, we will show that many consequences of linearity persist in $\mathcal{D}$. Furthermore it will turn out that every element of $\mathcal{D}$ is cohomologous to a cocycle which *is* close to a linear one in the norm sense. These ideas will justify the terminology.

Cocycles are important tools in the construction of certain special $\mathbb{R}^m$ flows, namely the $\mathbb{R}^m$ suspensions. So assume we have a flow $(X, \mathbb{Z}^m)$ and a cocycle h. Form the space $X \times \mathbb{R}^m$ and note that there is a trivial $\mathbb{R}^m$ action on this space given by $(x,t)s = (x, t+s)$. Also for each $a \in \mathbb{Z}^m$, define a homeomorphism $T_a : X \times \mathbb{R}^m \to X \times \mathbb{R}^m$ by $T_a(x,t) = (ax, t - h(x,a))$. It is obvious that each T_a commutes with the $\mathbb{R}^m$ action and, in fact, the group of these maps gives a $\mathbb{Z}^m$ action on $X \times \mathbb{R}^m$ because h satisfies the cocycle equation. We now form the quotient space

$$X_h = X \times \mathbb{R}^m / \{T_a : a \in \mathbb{Z}^m\}$$

and let π be the canonical projection from $X \times \mathbb{R}^m$ to X_h. We thus obtain an $\mathbb{R}^m$ flow $(X_h, \mathbb{R}^m)$ which we call the $\mathbb{R}^m$ *suspension of* X *given by* h. In the case $m = 1$ this construction is just the usual flow under a function since in one dimension every cocycle is given by $h(x,n) = \sum_{i=0}^{n-1} f(ix)$ for $n > 0$ and a similar formula for $n < 0$, where f is a continuous function. It is easy to check that X_h is a compact Hausdorff space in the case where $f > 0$, but even in the general one-dimensional case, it is not clear that X_h is well behaved in a topological sense or that X is embedded as a global section in the flow. The relationship between properties of h and the corresponding suspension flow $(X_h, \mathbb{R}^m)$ was investigated in detail in [1]. The main results we will need from that paper are restated here for completeness.

Definition 1. A cocycle $h \in \mathcal{C}$ is called *covering* if

(a) X_h is a Hausdorff space,
(b) π is a local homeomorphism.

If in addition π is one-to-one on $X \times \{0\}$, then h is called *embedding.*

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Note that when h is an embedding cocycle, X is naturally embedded in X_h as a global section.

Theorem 1. *Suppose $(X, \mathbb{Z}^m)$ has a free dense orbit. A cocycle h is covering if and only if $|h(x,a)| \to \infty$ uniformly in x as $|a| \to \infty$.*

This theorem is used extensively in what follows. We thus will impose the *Standing Assumption* that $(X, \mathbb{Z}^m)$ has a free dense orbit.

Theorem 2. *If $h : X \times \mathbb{Z}^m \to \mathbb{R}^m$ is covering, then X_h is a compact metric space. Moreover, there are constants M_1 and M_2 such that $M_1|a| \leq |h(x,a)| \leq M_2|a|$ for all $a \in \mathbb{Z}^m$ with $|a|$ sufficiently large.*

In fact if h is a covering cocycle into a space $\mathbb{R}^n$ and X_h turns out to be compact, then n must be equal to m. This justifies our restriction to $n = m$.

It is thus clear that the covering cocycles from $X \times \mathbb{Z}^m$ to $\mathbb{R}^m$ play a key role in constructing well-behaved $\mathbb{R}^m$ suspension actions. Furthermore the suspension flows are important in understanding general minimal $\mathbb{R}^m$ actions. (See [3].)

As an easy consequence of Theorem 1, we have:

Remark 1. If $L \in \mathcal{L}$, then L is covering if and only if L is invertible as a linear map.

Remark 2. If two covering cocycles are cohomologous, then the corresponding suspensions are isomorphic as flows. Any cocycle cohomologous to a covering cocycle is covering.

Note that by Remark 2, $\mathcal{L} + \mathcal{B}$ generates the same suspension flows as $\mathcal{L}$ itself. We want to understand what new suspension flows come from the closure points of $\mathcal{L} + \mathcal{B}$ in $\mathcal{C}$.

Covering cocycles may exhibit a stronger property which is closely related to the structure of the phase space and orbits of the corresponding suspension. This involves extending h from $X \times \mathbb{Z}^m$ to $X \times \mathbb{R}^m$.

Let $I^m = [0,1]^m$ be the unit cube in $\mathbb{R}^m$. By the *standard triangulation* of I^m we mean the complex K with $|K| = I^m$ consisting of the m-simplices

$$\mathcal{S}_\sigma = \{0, e_{\sigma 1}, e_{\sigma 1} + e_{\sigma 2}, \ldots, e_{\sigma 1} + \ldots + e_{\sigma m}\}$$

and all their faces generated by all permutations σ of $(1, \ldots, m)$. It is easy to see that

$$|\mathcal{S}_\sigma| = \{t \in I^m : 1 \geq t_{\sigma 1} \geq t_{\sigma 2} \geq \ldots \geq t_{\sigma m} \geq 0\}.$$

Now for each $x \in X$, $h(x, \cdot)$ can be extended to a map $H(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$. To do this give $\mathbb{R}^m$ a simplicial structure with $K + \mathbb{Z}^m$ and view $h(x, \cdot)$ as a vertex map into $\mathbb{R}^m$. Then let $H(x, \cdot)$ be the piecewise-linear extension of $h(x, \cdot)$ on $|K + \mathbb{Z}^m| = \mathbb{R}^m$ to $\mathbb{R}^m$. It is clear that H is a jointly continuous map and $H(x, t + a) = h(x, a) + H(ax, t)$ where $t \in \mathbb{R}^m$ and $a \in \mathbb{Z}^m$.

If $H(x, \cdot)$ is one-to-one and onto for all $x \in X$ the cocycle h is *invertible* in the sense of [2] where they were called "cocycles for suspensions". If h is invertible, then there is an orbit-preserving homeomorphism from X_h onto X_{id} (the suspension corresponding to the identity map on $\mathbb{Z}^m$). (Flows with this property are called *conjugate*.) So the space X_h is topologically independent of h and the dynamics differ only in a time change—the orbits are the same as those of X_{id}. In fact h is invertible in the sense that $H(x, \theta(x,t)) = t$ where $\theta(x,t)$ is the required

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(continuous) time change. Note that in view of Remark 2 all this holds if h is merely cohomologous to an invertible cocycle.

We first establish that H being onto holds for all covering cocycles.

Proposition 1. *If h is covering, then $H(x,\cdot)$ is onto $\mathbb{R}^m$ for all $x \in X$.*

Proof. Fix x and denote $H(x,\cdot)$ by H. If h is covering, H can be extended continuously to an m-dimensional sphere by defining $H(x,\infty)=\infty$. Now if H is not onto, then H can be thought of as a map from S_r^m to $\mathbb{R}^m$ (denoted by H again) where S_r^m is the m-sphere of radius r and r is chosen so large that $|H(x,t)-H(x,s)| \geq 1$ for $|t-s| \geq r$. (That this can be done is a consequence of the fact that $|h(x,a)| \geq M_1|a|$ for a sufficiently large.) By the Borsuk-Ulam theorem, we now have a pair of antipodal points s and $-s$ say, with $H(x,s)=H(x,-s)$. But $|s-(-s)|=2r$ which is a contradiction. $\square$

Thus if h is covering and H is one-to-one at each $x \in X$, then h is invertible. Furthermore the extension of h is piecewise-linear in the sense of the discussion above. If h is covering and H is one-to-one for all $x \in X$, then we call h a *piecewise-linear invertible cocycle* (P-L invertible).

The following sections will bring these concepts together in a natural way for $\mathcal{D}=\mathcal{L}+\overline{\mathcal{B}}$. In §2 we will note that $\mathcal{D}$ is normally a rich subspace of $\mathcal{C}$ worthy of study in its own right. In §3 we will establish that an element of $\mathcal{D}$ has a close relationship to a linear cocycle motivating the term "close to linear" for its elements. We also establish that if $(X,\mathbb{Z}^m)$ is uniquely ergodic (i.e. there is only one probability measure on X which is invariant under $\mathbb{Z}^m$), then in fact $\mathcal{D}$ is all of $\mathcal{C}$ so that in this case all the results apply to all cocycles. Finally in §4 we will show that each covering cocycle in $\mathcal{D}$ is cohomologous to a P-L invertible cocycle so that the suspensions obtained from members of $\mathcal{D}$ are all conjugate to the identity suspension.

2. The subspace $\mathcal{D}=\mathcal{L}\oplus\overline{\mathcal{B}}$

We first establish that $\mathcal{D}$ is actually a direct sum.

Proposition 2. $\mathcal{D}=\mathcal{L}\oplus\overline{\mathcal{B}}$.

Proof. Suppose $L \in \mathcal{L}\cap\overline{\mathcal{B}}$ and $L \neq 0$. Choose a unit vector v such that $Lv \neq 0$ and find a sequence $\{a_k\}$ in $\mathbb{Z}^m$ with $|a_k-kv| \leq m$. In this case $|La_k| \geq Bk$ for some $B>0$ and k large enough. Now choose $f \in \mathcal{B}$ such that $\|L-f\| < B/2$. Then $(B/2)|a_k| \geq |La_k - f(x,a_k)| \geq |La_k| - |f(x,a_k)| \geq Bk - C$ where C is some bound for f. Thus $C \geq Bk-(B/2)|a_k| = (B/2)k+(B/2)(k|v|-|a_k|) \geq (B/2)k-(B/2)m$. Letting $k\to\infty$ gives a contradiction. $\square$

In view of Proposition 2, if $h \in \mathcal{D}$ we can associate with h a fixed $L_h \in \mathcal{L}$ and $f_h \in \overline{\mathcal{B}}$ so that h has the (unique) decomposition $h = L_h + f_h$. We immediately obtain $\overline{\mathcal{B}} = \{h \in \mathcal{D} : L_h = 0\}$.

The following lemma is helpful in dealing with elements of $\overline{\mathcal{B}}$.

Lemma 1. *If $f \in \overline{\mathcal{B}}$, then $|f(x,a)|/|a| \to 0$ uniformly in x as $|a|\to\infty$.*

Proof. Given $\varepsilon>0$, let $g \in \mathcal{B}$ with $\|g-f\| < \varepsilon/2$. As g is bounded, choose $A>0$ so that $|g(x,a)|/|a| < \varepsilon/2$ when $|a| \geq A$. Now $|f(x,a)| \leq |f(x,a)-g(x,a)|+|g(x,a)|$ and so $|f(x,a)|/|a| < \varepsilon$ whenever $|a| \geq A$. $\square$

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Corollary 1. *If $h \in \mathcal{D}$, then $|h(x,a) - L_h(a)|/|a| \to 0$ uniformly in x as $|a| \to \infty$.*

We now turn to the situation for covering cocycles.

Proposition 3. *If h is covering, then $h + \overline{\mathcal{B}}$ consists of covering cocycles.*

Proof. As h is covering, $|h(x,a)| \geq M_1|a|$ for some $M_1 > 0$ and a with $|a|$ sufficiently large. If $f \in \overline{\mathcal{B}}$, then $|h(x,a) + f(x,a)|/|a| \geq |h(x,a)|/|a| - |f(x,a)|/|a| \geq M_1/2$ for $|a|$ sufficiently large by Lemma 1. □

Corollary 2. *If $h \in \mathcal{D}$, then h is covering if and only if L_h is invertible.*

Proof. If h is covering, then $|h(x,a)|/|a| \geq M_1$ for $|a|$ large enough and L_h must be invertible by Corollary 1. The converse follows from the proposition and Remark 1. □

Theorem 3. *Suppose $\mathcal{B}$ is not closed. Then $\mathcal{C}$ contains non-trivial covering cocycles* (*i.e. covering cocycles not cohomologous to constant ones*) *and these are dense in $\mathcal{D}$.*

Proof. Suppose $f \in \overline{\mathcal{B}} - \mathcal{B}$ and $L \in \mathcal{L}$ is invertible. Then $L + f$ is covering by Corollary 2. If $L+f$ is cohomologous to a constant cocycle M say, then $L+f = M+g$ where g is a coboundary; so by Proposition 2, $f = g \in \mathcal{B}$, which is a contradiction. Now if $h \in \mathcal{D}$, then $h = L_h + f_h$. Then $L' + f_h$ can be made as close to h as we like with L' invertible since the invertible linear maps are dense in $\mathcal{L}$. Now if $f_h \in \mathcal{B}$ and we choose $f \in \overline{\mathcal{B}} - \mathcal{B}$ with $\|f\|$ sufficiently small, then $f_h + f \notin \mathcal{B}$ and $L' + f_h + f$ is arbitrarily close to h. □

We show in [4] that under our standing assumption of a free dense orbit, $\overline{\mathcal{B}} \neq \mathcal{B}$ for real-valued cocycles. Of course this applies to the situation here also by considering the coordinate maps. Thus, in fact, the supposition that $\mathcal{B}$ is not closed is unnecessary in the statement of the theorem provided $(X, \mathbb{Z}^m)$ has a free dense orbit.

Proposition 4. *The covering cocycles are open in $\mathcal{C}$.*

Proof. This is an easy consequence of the definition of the norm and the fact that a cocycle is covering if and only if $|h(x,a)| \geq M_1|a|$ for some $M_1 > 0$ and $|a|$ sufficiently large. □

Corollary 3. *The covering cocycles form an open dense set in $\mathcal{D}$.*

3. Linearisation in $\mathcal{D}$

The next result indicates a precise sense in which $h \in \mathcal{D}$ is "close to linear".

Theorem 4. *Let $h \in \mathcal{C}$. The following are equivalent:*

(a) *$h \in \mathcal{D}$.*
(b) *There is a linear cocycle $L \in \mathcal{L}$ such that we can find a sequence of cocycles $\{h_n\}$ all cohomologous to h with $h_n \to L$.*
(c) *The distance between the sets $\mathcal{L}$ and $h + \mathcal{B}$ is 0.*

Proof. (a)⇒(b). If $h \in \mathcal{D}$, let $f_n \in \mathcal{B}$ be a sequence with $f_n \to f_h$. Now $h - f_n \to L_h$ and $h - f_n$ is cohomologous to h.

(b)⇒(c). Obvious.

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(c)$\Rightarrow$(a). If $d(h+\mathcal{B},\mathcal{L})=0$, then there exist $f_n\in\mathcal{B}$ and $L_n\in\mathcal{L}$ such that $\|h+f_n-L_n\|\le 1/n$. Thus f_n-L_n is a sequence in $\mathcal{D}$ converging to $-h$. So $h\in\overline{\mathcal{D}}=\mathcal{D}$. $\square$

Given $h\in\mathcal{C}$, there is a natural way of constructing cohomologous cocycles which average the values of h over a central portion of each orbit. Choose ℓ to be a positive integer and define $P_\ell(x)=(1/\ell^m)\sum h(x,a)$ where the sum is taken over all $a\in\mathbb{Z}^m$ such that $1\le a_j\le\ell$ for all $1\le j\le m$. Since P_ℓ is continuous, we can use it to define a coboundary and set $h_\ell(x,a)=P_\ell(ax)-P_\ell(x)+h(x,a)$. Thus h_ℓ is cohomologous to h. Now

$$\begin{aligned} h_\ell(x,e_i) &= h(x,e_i)+\frac{1}{\ell^m}\sum\{h(e_ix,a)-h(x,a):1\le a_j\le\ell,1\le j\le m\}\\ &= h(x,e_i)+\frac{1}{\ell^m}\sum\{h(ax,e_i)-h(x,e_i):1\le a_j\le\ell,1\le j\le m\}\\ &= \frac{1}{\ell^m}\sum\{h(ax,e_i):1\le a_j\le\ell,1\le j\le m\}. \end{aligned}$$

This argument to construct the cohomologous cocycles was shown to us by H Furstenberg.

In the one-dimensional case it can be shown that the sign of $\int h(x,1)d\mu$ is the same for every invariant measure μ, say positive. (See Theorem 1.12 and its proof in [1].) If $h_\ell(x,1)$ is not eventually strictly positive, the above expansion for h_ℓ can be used to produce an invariant measure with non-positive integral. Thus in the one-dimensional case a covering cocycle generated by an unrestricted continuous function is cohomologous to one generated by a strictly positive or negative function. Thus for real suspensions one can restrict oneself to that simpler situation without loss of generality.

We can now characterise $\mathcal{D}$ in terms of the integrals of h. Define an operator $A_\ell^x: C(X)\to\mathbb{R}$ by $A_\ell^x(f)=\frac{1}{\ell^m}\sum f(ax)$ where the sum is over all a with $1\le a_i\le\ell$.

Lemma 2. *$\int_X f(x)d\mu=0$ for all invariant probability measures μ if and only if $A_\ell^x(f)\to 0$ uniformly in x as $\ell\to\infty$.*

Proof. Suppose that $A_\ell^x(f)$ does not converge to 0 uniformly in x as $\ell\to\infty$. We then have $\varepsilon>0$, $\ell_k\to\infty$, $\{x_k\}$ with $|A_{\ell_k}^{x_k}(f)|\ge\varepsilon$ for all k. Since these linear operators are all in the unit ball, we can suppose $A_{\ell_k}^{x_k}$ converges weakly to a linear operator θ, say. It is easy to check that θ is invariant in the sense that $\theta(ag)=\theta(g)$ ($a\in\mathbb{Z}^m$) where $ag(x)=g(ax)$. So θ is represented by an invariant probability measure μ and $\int_X f(x)d\mu\ne 0$.

The converse is immediate because $A_\ell^x(f)$ is an ergodic average of f. $\square$

Theorem 5. *Let $h\in\mathcal{C}$ and μ be an invariant probability measure on $(X,\mathbb{Z}^m)$, and define the linear map $L_\mu(a)=\int_X h(x,a)d\mu$. Then $h\in\mathcal{D}$ if and only if the maps L_μ are all the same, i.e. the integral of h does not depend on the choice of μ.*

Proof. Let $h\in\mathcal{D}$ and fix $a\in\mathbb{Z}^m$. By Corollary 1

$$\left|\frac{1}{k}h(x,ka)-L_h(a)\right|\to 0$$

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uniformly in x as $k \to \infty$. But by the ergodic theorem applied to the integer action induced by a

$$\frac{1}{k}h(x, ka) = \frac{1}{k}\sum_{j=0}^{k-1} h(jax, a)$$

must converge a.e. μ to an integrable function whose integral is $\int_X h(x,a)d\mu$. Thus

$$L_h(a) = \int_X L_h(a)d\mu = \int_X h(x,a)d\mu$$

and so $L_\mu = L_h$ for any μ.

Conversely suppose $\int_X h(x,a)d\mu = L(a)$ where L is independent of μ. Then by Lemma 2 and the above definition of h_ℓ

$$(h_\ell(x, e_i) - Le_i)_j = A_\ell^x((h(\cdot, e_i) - Le_i)_j) \to 0$$

uniformly in x for any $i, j = 1, \ldots, m$. Thus $h_\ell \to L$, and so $h \in \mathcal{D}$ by Theorem 4. □

Corollary 4. *If $(X, \mathbb{Z}^m)$ is uniquely ergodic, then $\mathcal{C} = \mathcal{D}$.*

The question of when $\mathcal{C} = \mathcal{D}$ in general turns out to be a deeper issue. Note that if $m = 1$, unique ergodicity is both necessary and sufficient by Theorem 5 since every continuous function generates a cocycle. The general case is addressed in [4].

4. P-L invertible cocycles and $\mathcal{D}$

We will establish that the P-L invertible cocycles form an open set in $\mathcal{C}$. The results of §3 then enable us to establish a close relationship between the elements of $\mathcal{D}$ and P-L invertible cocycles.

Lemma 3. *Let $N > 0$ be an integer and $V = \mathbb{Z}^m \cap [-N, N]^m$. Suppose $f : V \to \mathbb{R}^m$ and let $F : [-N, N]^m \to \mathbb{R}^m$ be the piecewise-linear extension of f. Suppose F is one-to-one. There is $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^m$ and $\sup_{v \in V} |f(v) - g(v)| < \varepsilon$, then G is one-to-one where G is the piecewise-linear extension of g to $[-N, N]^m$.*

Proof. Let Ξ be the collection of simplices $\{v + \mathcal{S} : \mathcal{S}$ is in the standard triangulation of $[0,1]^m$ and $v \in \mathbb{Z}^m \cap [-N, N-1]^m\}$. If $\mathcal{S} \in \Xi$, then F is an invertible linear map on $\mathcal{S}$ and so G will be invertible on $\mathcal{S}$ also for g close enough to f. If $\mathcal{S}_1, \mathcal{S}_2 \in \Xi$ and $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint, we can find a linear function $\gamma : \mathbb{R}^m \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\gamma(F(w)) > c$ for $w \in \mathcal{S}_1$ and $\gamma(F(w)) < c$ for $w \in \mathcal{S}_2$. If g is close enough to f we can ensure $\gamma(G(w)) > c$ for $w \in \mathcal{S}_1$ and $\gamma(G(w)) < c$ for $w \in \mathcal{S}_2$. Thus $G(\mathcal{S}_1) \cap G(\mathcal{S}_2) = \phi$.

Now suppose $\mathcal{S}_1$, $\mathcal{S}_2 \in \Xi$ with a common face $\mathcal{S}'$ and assume $\mathcal{S}'$ is a largest common face in terms of dimension. Let $v_1, \ldots, v_p$ be the vertices of $\mathcal{S}'$ and then $f(v_1), \ldots, f(v_p)$ are the vertices of $F(\mathcal{S}')$ which is a largest common face of $F(\mathcal{S}_1)$ and $F(\mathcal{S}_2)$. (Note that $F(\mathcal{S}_1)$, $F(\mathcal{S}_2)$ and $F(\mathcal{S}')$ are all simplices of the same dimension as $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}'$ respectively as F is one-to-one.) Now let $\mathcal{H}$ be a hyperplane separating $F(\mathcal{S}_1)$ and $F(\mathcal{S}_2)$ and containing $F(\mathcal{S}')$.

We can choose $b_{p+1}, \ldots, b_m \in \mathcal{H}$ such that if

$$\gamma(w) = \det\big(w, f(v_2) - f(v_1), \ldots, b_m - f(v_1)\big),$$

then

$$\mathcal{H} = \{w : \gamma(w) = c\}$$

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where $c = \gamma\big(f(v_1)\big)$. Now for vertices of $\mathcal{S}_1$ not in $\mathcal{S}'$, γ will be less than c (say), and then for vertices of $\mathcal{S}_2$ not in $\mathcal{S}'$, it will be greater than c. Now for g close enough to f, the function $\det\big(w, g(v_2) - g(v_1), \dots, b_m - g(v_1)\big)$ will have the same property because the determinant depends continuously on its arguments. Thus $G(\mathcal{S}_1) \cap G(\mathcal{S}_2) = G(\mathcal{S}')$.

Since Ξ is finite, this completes the proof. $\square$

Theorem 6. *The P-L invertible cocycles are open in $\mathcal{C}$.*

Proof. Let h be P-L invertible and $M_1 > 0$ such that $|h(x,a)| \geq M_1|a|$ for $|a|$ sufficiently large. (M_1 exists as h is covering.) Let $g \in \mathcal{C}$ and $\|h - g\| < M_1/2$. Note that $\|g\| < M_1/2 + \|h\| = \alpha/m$, say. Using the cocycle equation, if $t \in \mathbb{R}^m$ and $a \in \mathbb{Z}^m$ is the integer part of t, then $|G(x,t) - g(x,a)| \leq m\|g\|$ for all $x \in X$. Now there is an integer $N > 0$ (independent of g) such that $|G(x,t)| \geq \alpha$ for any $x \in X$ and $|t| \geq N$. If not, we can find a sequence $g_n \in \mathcal{C}$ satisfying $\|g_n - h\| < M_1/2, t_n \in \mathbb{R}^m$ with $|t_n| \to \infty$ and $x_n \in X$ such that $|G_n(x_n, t_n)| < \alpha$. Thus if a_n is the integer part of t_n, then $|g_n(x_n, a_n)| < m\|g_n\| + \alpha < 2\alpha$. But now

$$\begin{aligned} M_1|a_n|/2 &\geq |h(x_n, a_n) - g_n(x_n, a_n)| \\ &\geq |h(x_n, a_n)| - |g_n(x_n, a_n)| \geq M_1|a_n| - m\|g_n\| - \alpha. \end{aligned}$$

So $2\alpha > \alpha + m\|g_n\| \geq M_1|a_n|/2$ which is a contradiction as clearly $|a_n| \to \infty$.

Now suppose g is not P-L invertible. An application of the cocycle equation shows that we can find $x \in X$, $s \in I^m$ and $t \in \mathbb{R}^m$ with $G(x,s) = G(x,t)$. As $|G(x,s)| \leq m\|g\| < \alpha$, we know that $t \in [-N, N]^m$.

Using this N, the lemma determines ε_x for $h(x,\cdot)$ restricted to V for each $x \in X$. There is $\delta_x > 0$ such that $d(x,y) < \delta_x$ implies $|h(x,a) - h(y,a)| < \varepsilon_x/2$ for $a \in V$. Cover X with balls of radius δ_x about each x and select a finite subcover. Thus we have $x_1, x_2, \dots, x_n$, say, with

$$X = \bigcup_{i=1}^{n} \{y : d(y, x_i) < \delta_{x_i}\}.$$

Let $\varepsilon = \min\{\varepsilon_{x_1}, \varepsilon_{x_2}, \dots, \varepsilon_{x_n}, M_1\}$ and suppose that $\|g - h\| < \varepsilon/2mN$. Now fix $x \in X$ and choose x_i with $d(x, x_i) < \delta_{x_i}$. Since $H(x_i, \cdot)$ is invertible, it is enough to show that $|g(x,a) - h(x_i,a)| < \varepsilon_{x_i}$ for $a \in V$. But $|g(x,a) - h(x_i,a)| \leq |g(x,a) - h(x,a)| + |h(x,a) - h(x_i,a)| \leq \|g - h\|\,|a| + \varepsilon_{x_i}/2 < \varepsilon_{x_i}$ for $a \in V$ since $|a| \leq mN$. $\square$

Corollary 5. *Suppose $h \in \mathcal{D}$ and h is covering. Then h is cohomologous to a P-L invertible cocycle.*

Proof. This is immediate from Theorem 4, Corollary 2 and the theorem above since $L \in \mathcal{L}$ is P-L invertible precisely if L is invertible in the usual sense. $\square$

Corollary 6. *Suppose $h \in \mathcal{D}$ and h is covering. Then the suspension $(X_h, \mathbb{R}^m)$ is conjugate to the identity suspension $(X_{\mathrm{id}}, \mathbb{R}^m)$, i.e. $(X_h, \mathbb{R}^m)$ is isomorphic to a time change of $(X_{\mathrm{id}}, \mathbb{R}^m)$.*

Proof. An invertible cocyle is what was termed a "cocycle for a suspension" in [2], so this follows from the above corollary and Theorem 2.2 of [2]. $\square$

Corollary 7. *If $(X, \mathbb{Z}^m)$ is uniquely ergodic, then every $\mathbb{R}^m$ suspension of it given by a covering cocycle is isomorphic to a time change of the identity suspension.*

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In the case $m = 1$, we have already noted that up to coboundaries we can assume $f > 0$ ($f < 0$). In this case, of course, h is increasing (decreasing) and so P-L invertible. Thus the conclusions of the above theorem and corollaries hold for $m = 1$ without any assumption on the size of $\mathcal{D}$.

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