

E_∞ -RING STRUCTURES FOR TATE SPECTRA

J. E. McCLURE

(Communicated by Thomas Goodwillie)

1. INTRODUCTION

Let G be a compact Lie group and k_G a G spectrum (as defined in [3, Section I.2]). Greenlees and May ([2]) have defined an associated G -spectrum $t(k_G)$ called the *Tate spectrum* of k_G . They observe that if k_G is a ring G -spectrum, then there is an induced ring G -spectrum structure on $t(k_G)$, and that if k_G is homotopy-commutative, then $t(k_G)$ will also be homotopy-commutative (see [2, Proposition 3.5]). It is therefore natural to ask whether an equivariant E_∞ ring structure on k_G induces an equivariant E_∞ ring structure on $t(k_G)$ (we will recall the definition in a moment). We offer both positive and negative answers to this question.

On the positive side, we show that $t(k_G)$ inherits a structure which is somewhat weaker than an equivariant E_∞ ring structure, but which should be adequate for most practical purposes. To explain this we first recall that, given a G -universe U , there is an equivariant operad $\mathcal{L}(U)$ whose j th space consists of the (nonequivariant) linear isometries from $U^{\oplus j}$ to U . From now on we fix a complete G -universe U . By definition, an *equivariant E_∞ operad* \mathcal{C} is an equivariant operad which is equivariantly equivalent to $\mathcal{L}(U)$, and an *equivariant E_∞ ring structure* is an action of an equivariant E_∞ operad (see [3, Definition VII.2.1]). Next let us define an *E'_∞ operad* to be a nonequivariant E_∞ operad provided with trivial G -action. For example, if \mathcal{C} is an equivariant E_∞ operad, then its G -fixed points form an E'_∞ operad \mathcal{C}^G by [3, Example VII.1.4]. We define an *equivariant E'_∞ ring structure* to be an action of an E'_∞ operad (in other words, it is an action of a nonequivariant E_∞ operad through G -maps). Since any action by \mathcal{C} restricts to an action by \mathcal{C}^G , we see that an equivariant E_∞ ring structure always includes an equivariant E'_∞ ring structure.

The reason why E'_∞ ring structures are interesting is that if k_G is an E'_∞ ring spectrum, then the fixed-point spectra $(k_G)^H$ have (nonequivariant) E_∞ ring structures which are consistent as H varies (see Remark VII.2.5 of [3]); this is likely to be the property one needs for applications.

Our positive result is:

Theorem 1. *If k_G is an equivariant E'_∞ ring spectrum, then so is $t(k_G)$; in particular all fixed-point spectra $(t(k_G))^H$ are nonequivariant E_∞ ring spectra.*

Received by the editors June 24, 1994 and, in revised form, November 10, 1994.

1991 *Mathematics Subject Classification.* Primary 55P91.

The author was partially supported by National Science Foundation grant 9207731-DMS.

©1996 American Mathematical Society

The proof of Theorem 1 will show that the diagram in Proposition 3.5 of [2] is a diagram of equivariant E'_∞ ring spectra.

To state our negative result we need to recall the definition of $t(k_G)$. Let EG be a contractible free G -CW complex and let $\tilde{E}G$ denote the G -space defined by the cofiber sequence

$$EG_+ \rightarrow S^0 \rightarrow \tilde{E}G$$

(here $+$ denotes a disjoint basepoint). Let $F(EG_+, k_G)$ be the function spectrum of maps from EG_+ to k_G ([3, Definition I.3.2]). Then $t(k_G)$ is defined to be the G -spectrum

$$F(EG_+, k_G) \wedge \tilde{E}G.$$

Let us write ι for the natural map $S^0 \rightarrow \tilde{E}G$.

Theorem 2. *Let G be a finite group and let k_G be any G -spectrum. Suppose that $t(k_G)$ has an equivariant E_∞ ring structure whose unit factors (up to equivariant homotopy) through $\sum_G^\infty \iota$. Then $t(k_G)$ must be equivariantly contractible.*

This implies that if k_G is a ring G -spectrum for which $t(k_G)$ is not equivariantly contractible, then $t(k_G)$ cannot have an equivariant E_∞ ring structure whose underlying ring G -spectrum structure is compatible with that of k_G under the natural map $k_G \rightarrow t(k_G)$. In particular, the underlying ring G -spectrum structure of $t(k_G)$ cannot be that defined in [3, Proposition 3.5]. Thus it seems that there is no natural way to give $t(k_G)$ an equivariant E_∞ ring structure.

I would like to thank Mike Hopkins for suggesting this problem to me and the referee for clarifying a point in the proof.

2. PROOF OF THEOREM 1

Theorem 1 is an immediate consequence of the following two lemmas, of which the second is well-known. Let us recall from [3, Definition VII.2.7] that, given an equivariant operad \mathcal{C} , a \mathcal{C}_0 space is an action of \mathcal{C} in the category of based G -spaces; that is, it is a based G -space X with based G -maps

$$(\mathcal{C}_j)_+ \wedge_{\Sigma_j} X^{(j)} \rightarrow X$$

(here $^{(j)}$ denotes j -fold smash product) satisfying the same compatibility conditions that are used to define an equivariant \mathcal{C} -space. In particular, this definition makes sense if \mathcal{C} is a nonequivariant operad provided with the trivial G -action; it then says that \mathcal{C} acts on X through G -maps.

Lemma 3. *There is a nonequivariant E_∞ operad \mathcal{D} for which $\tilde{E}G$ is an equivariant \mathcal{D}_0 space.*

Lemma 4. *Let \mathcal{C} be any equivariant operad.*

(a) *If k_G is a \mathcal{C} ring spectrum (that is, if it has an equivariant action of \mathcal{C}), then so is $F(Y_+, k_G)$ for any G -space Y .*

(b) *If h_G is a \mathcal{C} ring spectrum and X is a \mathcal{C}_0 -space, then $h_G \wedge X$ is a \mathcal{C} ring spectrum.*

Proof of Theorem 1. Suppose that k_G has an action of an E'_∞ operad \mathcal{C}' . Let $\mathcal{C} = \mathcal{C}' \times \mathcal{D}$, where \mathcal{D} is the operad of Lemma 3. Then \mathcal{C} is an E'_∞ operad, and it acts on k_G (via the projection $\mathcal{C}' \times \mathcal{D} \rightarrow \mathcal{C}'$) and on $\tilde{E}G$ (via the projection

$\mathcal{C}' \times \mathcal{D} \rightarrow \mathcal{D}$). Now Lemma 4(a) implies that \mathcal{C} acts on $F(EG_+, k_G)$, and the theorem follows from Lemma 4(b) if we take h_G to be $F(EG_+, k_G)$ and X to be \widetilde{EG} . \square

Proof of Lemma 4. In each case, we specify the structural maps which constitute the \mathcal{C} -action; the fact that they satisfy the necessary compatibility relations is a straightforward application of the methods of [3, Sections VI.1–VI.3].

For part (a) the structural map

$$\xi_j : \mathcal{C}_j \times F(Y_+, k_G)^{(j)} \rightarrow F(Y_+, k_G)$$

is the adjoint of the composite

$$\begin{aligned} Y_+ \wedge \mathcal{C}_j \times F(Y_+, k_G)^{(j)} &\xrightarrow{\Delta \wedge 1} (Y_+)^{(j)} \wedge \mathcal{C}_j \times F(Y_+, k_G)^{(j)} \\ &\xrightarrow{\cong} \mathcal{C}_j \times \left((Y_+)^{(j)} \wedge F(Y_+, k_G)^{(j)} \right) \xrightarrow{1 \times e} \mathcal{C}_j \times k_G^{(j)} \xrightarrow{\xi'_j} k_G; \end{aligned}$$

here Δ is the diagonal map of Y , the isomorphism is that of [3, Proposition VI.1.5], e is the evaluation map, and ξ'_j is the structural map of k_G .

For part (b) the structural map

$$\xi_j : \mathcal{C}_j \times (h_G \wedge X)^{(j)} \rightarrow h_G \wedge X$$

is the composite

$$\mathcal{C}_j \times (h_G \wedge X)^{(j)} = \mathcal{C}_j \times (h_G^{(j)} \wedge X^{(j)}) \xrightarrow{\delta} (\mathcal{C}_j \times h_G^{(j)}) \wedge (\mathcal{C}_{j+} \wedge X^{(j)}) \xrightarrow{\xi'_j \wedge \xi''_j} h_G \wedge X,$$

where δ is the map given in Definition VI.3.5 of [3] and ξ'_j, ξ''_j are the structural maps for h_G and X . \square

Proof of Lemma 3. First let us observe that \widetilde{EG} is nonequivariantly contractible and that for any nontrivial subgroup H of G the H -fixed set $(\widetilde{EG})^H$ is exactly S^0 ; the same is true for $(\widetilde{EG})^{(j)}$ since the smash product of spaces commutes with H -fixed sets.

Let Map_*^G denote based G -maps. Restriction to the G -fixed set gives a map

$$\phi : \text{Map}_*^G(\widetilde{EG}^{(j)}, \widetilde{EG}) \rightarrow \text{Map}_*(S^0, S^0)$$

which we claim is a weak equivalence. Assuming this for the moment, let \mathcal{D}'_j be the space $\phi^{-1}(\text{id})$. Then the spaces \mathcal{D}'_j with the evident composition operations γ form an operad \mathcal{D}' and \widetilde{EG} is a \mathcal{D}'_0 -space. The only thing preventing \mathcal{D}' from being a nonequivariant E_∞ operad is that the action of Σ_j on \mathcal{D}'_j may not be free. To remedy this let \mathcal{C} be any nonequivariant E_∞ operad and define \mathcal{D} to be $\mathcal{D}' \times \mathcal{C}$, acting on \widetilde{EG} via the projection $\mathcal{D}' \times \mathcal{C} \rightarrow \mathcal{D}'$.

It only remains to prove the claim that ϕ is a weak equivalence. First we observe that the reduced diagonal map

$$\Delta : \widetilde{EG} \rightarrow \widetilde{EG}^{(j)}$$

is a weak equivalence on each fixed-point set, and is therefore a G -homotopy equivalence by the equivariant Whitehead theorem. It follows that

$$\Delta^* : \text{Map}_*^G(\widetilde{EG}, \widetilde{EG}) \rightarrow \text{Map}_*^G(\widetilde{EG}^{(j)}, \widetilde{EG})$$

is a homotopy equivalence, so it suffices to verify the claim when $j = 1$.

To handle this case, we map the cofiber sequence

$$EG_+ \rightarrow S^0 \rightarrow \tilde{E}G$$

into $\tilde{E}G$ to get a fiber sequence

$$\text{Map}_*^G(\tilde{E}G, \tilde{E}G) \rightarrow \text{Map}_*^G(S^0, \tilde{E}G) \rightarrow \text{Map}_*^G(EG_+, \tilde{E}G).$$

The middle term is equal to S^0 , so it suffices to show that the third term is weakly contractible. For this we recall that the functor $\text{Map}_*^G(EG_+, -)$ takes G -maps which are nonequivariant weak equivalences to weak equivalences (for example, this follows from [1, XI.5.6] since $\text{Map}_*^G(EG_+, -)$ is a special case of the holim construction). Since $\tilde{E}G$ is nonequivariantly contractible, we see that $\text{Map}_*^G(EG_+, \tilde{E}G)$ is weakly contractible and we are done. \square

3. PROOF OF THEOREM 2

As motivation for the proof of Theorem 2, we first explain why the operad \mathcal{D}' constructed in the proof of Lemma 3 is not equivalent to the linear isometries operad $\mathcal{L}U$. Let $G = Z/2$ for simplicity and consider the $G \times \Sigma_2$ -spaces $\mathcal{L}U_2$ and \mathcal{D}'_2 (recall that \mathcal{D}'_2 has trivial G -action). Let H be the diagonal copy of $Z/2$ in $G \times \Sigma_2 = Z/2 \times Z/2$. We claim that $\mathcal{L}U_2$ has H fixed points but \mathcal{D}'_2 has none; this certainly implies that $\mathcal{L}U_2$ and \mathcal{D}'_2 are not $G \times \Sigma_2$ -equivalent. To see that $\mathcal{L}U_2$ has H -fixed points we need only show that there is an H -equivariant linear isometry from $U \oplus U$ to U ; but this is obvious since as an H -representation $U \oplus U$ is a complete H -universe, and is therefore H -isomorphic to U . (We note for later use that $(\mathcal{L}U_2)^H$ is in fact contractible by [3, Lemma II.1.5].) On the other hand, if \mathcal{D}'_2 had an H -fixed point, then there would be a $G \times \Sigma_2$ -equivariant map

$$\tilde{E}G^{(2)} \rightarrow \tilde{E}G$$

(with Σ_2 acting trivially on the target) which extends the identity map of S^0 , and passing to H -fixed points would give a (nonequivariant) map $(\tilde{E}G^{(2)})^H \rightarrow S^0$ which extends the identity map of S^0 . But this is impossible since $(\tilde{E}G^{(2)})^H$ is contractible: there is a (nonequivariant) homeomorphism

$$\tilde{E}G \rightarrow (\tilde{E}G^{(2)})^H$$

which takes x to $x \wedge gx$, where g is the generator of G .

The proof of Theorem 2 is a variant of the same idea. For simplicity, we begin with the case $G = Z/2$. Suppose that $t(k_G)$ has an equivariant E_∞ ring structure whose unit η factors through $\Sigma_G^\infty \iota$. Then there is a G -homotopy commutative diagram of G -spectra

$$\begin{array}{ccc} \mathcal{L}U_2 \times_{\Sigma_2} (S_G^0)^{(2)} & \xrightarrow{1 \times \Sigma_G^\infty \iota^{(2)}} & \mathcal{L}U_2 \times_{\Sigma_2} \left(\Sigma_G^\infty \tilde{E}G \right)^{(2)} & \longrightarrow & \mathcal{L}U_2 \times_{\Sigma_2} t(k_G)^{(2)} \\ \downarrow \xi_2 & & & & \downarrow \xi'_2 \\ S_G^0 & \xrightarrow{\eta} & & & t(k_G) \end{array}$$

where ξ_2 and ξ'_2 are the structural maps for S_G^0 and $t(k_G)$. Next we recall that the upper-left corner of this diagram is an equivariant suspension spectrum, so that

we may pass to the adjoint to get a G -homotopy commutative diagram of spaces. More precisely, [3, Proposition VI.5.3] gives an isomorphism

$$\mathcal{L}U_2 \times_{\Sigma_2} (S^0)^{(2)} \cong \Sigma_G^\infty(\mathcal{L}U_{2+} \wedge_{\Sigma_2} (S^0)^{(2)})$$

which carries ξ_2 to the composite

$$\Sigma_G^\infty(\mathcal{L}U_{2+} \wedge (S^0)^{(2)}) = \Sigma_G^\infty(\mathcal{L}U_2/\Sigma_2)_+ \xrightarrow{\Sigma_G^\infty \pi} \Sigma_G^\infty S^0;$$

here π is the evident projection $(\mathcal{L}U_2/\Sigma_2)_+ \rightarrow S^0$. Thus the adjoint of the diagram above has the form

$$\begin{array}{ccc} \mathcal{L}U_{2+} \wedge_{\Sigma_2} (S^0)^{(2)} & \xrightarrow{1 \wedge_{\Sigma_2} t^{(2)}} & \mathcal{L}U_{2+} \wedge \tilde{E}G^{(2)} \longrightarrow \Omega_G^\infty(\mathcal{L}U_2 \times_{\Sigma_2} t(k_G)^{(2)}) \\ \downarrow = & & \\ (\mathcal{L}U_2/\Sigma_2)_+ & & \downarrow \Omega_G^\infty \xi'_2 \\ \downarrow \pi & & \\ S^0 & \xrightarrow{\tilde{\eta}} & \Omega_G^\infty t(k_G) \end{array}$$

For our purposes, the important thing about this diagram is that $\tilde{\eta} \circ \pi$ factors, up to G -homotopy, through $\mathcal{L}U_{2+} \wedge \tilde{E}G^{(2)}$. Precomposing with the projection

$$\mathcal{L}U_{2+} \wedge (S^0)^{(2)} \rightarrow \mathcal{L}U_{2+} \wedge_{\Sigma_2} (S^0)^{(2)},$$

we see that the composite

$$(1) \quad \mathcal{L}U_{2+} \wedge (S^0)^{(2)} = (\mathcal{L}U_2)_+ \xrightarrow{\pi} S^0 \xrightarrow{\tilde{\eta}} \Omega_G^\infty t(k_G)$$

(where we have again written π for the evident projection) factors up to $G \times \Sigma_2$ -homotopy through $\mathcal{L}U_{2+} \wedge \tilde{E}G^{(2)}$. Now let H be the diagonal copy of $Z/2$ in $G \times \Sigma_2$. Passing to the H -fixed points of (1) (and noting that the H -fixed points of $\Omega_G^\infty t(k_G)$ are the same as the G -fixed points since Σ_2 acts trivially), we see that the composite

$$(2) \quad (\mathcal{L}U_2^H)_+ \xrightarrow{\pi^H} S^0 \xrightarrow{\tilde{\eta}^G} (\Omega_G^\infty t(k_G))^G$$

factors up to (nonequivariant) homotopy through

$$\mathcal{L}U_{2+}^H \wedge (\tilde{E}G^{(2)})^H.$$

But we have shown in the first paragraph of this section that $(\tilde{E}G^{(2)})^H$ is contractible, so composite (2) is (nonequivariantly) homotopy trivial. We also showed in the first paragraph that $\mathcal{L}U_2^H$ is contractible, so π^H is an equivalence, and we conclude that

$$\tilde{\eta}^G : S^0 \rightarrow (\Omega_G^\infty t(k_G))^G$$

is homotopy trivial. This means that $\tilde{\eta}$ is G -homotopy trivial, and passing to the adjoint, we see that η itself is G -homotopy trivial. But η is the unit of the equivariant E_∞ ring $t(k_G)$, so $t(k_G)$ must be equivariantly contractible, as was to be shown.

So far we have assumed that G is $Z/2$. When G is finite of order n the action of G on itself by multiplication induces a homomorphism $\rho : G \rightarrow \Sigma_n$, and one can repeat the argument given above with $\mathcal{L}U_2$ replaced by $\mathcal{L}U_n$, Σ_2 replaced by Σ_n , and H replaced by the subgroup of $G \times \Sigma_n$ consisting of elements of the form $(g, \rho(g))$. \square

REFERENCES

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972. MR **51**:1825
2. J. P. C. Greenlees and J. P. May, *Generalized Tate, Borel and co Borel cohomology*, preprint.
3. L. G. Lewis, J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Math., vol. 1213, Springer-Verlag, Berlin and New York, 1986. MR **88e**:55002

DEPARTMENT OF MATHEMATICS, MATHEMATICAL SCIENCES BUILDING, PURDUE UNIVERSITY,
WEST LAFAYETTE, INDIANA 47907-1395