

GENERALIZED CYCLIC COHOMOLOGY ASSOCIATED WITH DEFORMED COMMUTATORS

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ABSTRACT. The generalized cyclic cohomology is introduced which is associated with q -deformed commutators $xy - qyx$. Some formulas related to the trace of the product of q -deformed commutators are established. The Chern character of odd dimension associated with q -deformed commutators is studied.

1. INTRODUCTION

In non-commutative differential geometry [1], the Chern character of a p -summable Fredholm module is expressed by the trace of some product of quantized differentials $df = [F, f]$, where $[F, f]$ is the commutator $Ff - fF$ and F is a self-adjoint idempotent operator. In the odd dimension case, the Chern character ch_{2n-1} is expressed as

$$\text{tr}(\omega(a^0, a^1) \cdots \omega(a^{2n-2}, a^{2n-1}) - \omega(a^{2n-1}, a^0) \cdots \omega(a^{2n-3}, a^{2n-2}))$$

where $\omega(x, y) = p(xy) - p(x)p(y)$ is the curvature of some mapping p (see also [2], [6]).

Let \mathcal{A} be an algebra over \mathbb{C} , and let C^n be the space of $n + 1$ -linear functions on \mathcal{A} . The basic operations in the cyclic cohomology are b', b, t , etc. (cf. [1], [5]), where b is the Hochschild boundary operation. Define $C_\lambda^n = \{f \in C^n : tf = f\}$ and $Af = (1 + t + \cdots + t^n)f$, $f \in C^n$. Let $pf = \sum_{j=0}^n (n-j)t^j f$, $f \in C^n$. Define $S = bpb'$. The restriction of $2\pi iS$ at $\mathcal{Z}_\lambda^n = \{f \in C_\lambda^n : bf = 0\}$ coincides with A. Connes' S operator (cf. [1], [12]).

Related to the Chern character, in the previous papers [11], [12], the author studied the cyclic cohomology associated with the product of commutators. Let X and Y be two subalgebras (or subgroups) of a unital algebra \mathcal{A} . Let $C^{m,n}$ be the space of multi-linear functions (or functions) $f(x^0, \dots, x^m; y^0, \dots, y^n)$, $x^i \in X$ and $y^j \in Y$. Let b_x, b'_x, t_x, A_x, S_x and b_y, b'_y, t_y, A_y, S_y be the operators b, b', t, A, S with respect to the x 's and y 's respectively. Let $C_\lambda^{n,n} = \{f \in C^{n,n} : t_x f = t_y f =$

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$f\}$. Suppose that there is a trace ideal J in \mathcal{A} with trace τ . Assume that there is a natural number k such that $[x^1, y^1] \cdots [x^k, y^k] \in J$, for $x^i \in X$ and $y^j \in Y$. Define

$$(1) \quad \phi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau[x^0, y^0] \cdots [x^n, y^n], \quad \text{for } n \geq k - 1$$

and

$$(2) \quad \psi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau x^0 y^0 [x^1, y^1] \cdots [x^n, y^n], \quad \text{for } n \geq k.$$

Then $A_x \phi_n = A_y \phi_n$ and it is denoted by $A\phi_n$.

Theorem A [11]. *For $n \geq k$, there is a $\Theta_n \in C_\lambda^{n,n}$ such that $A\phi_n = b_x b_y \Theta_n + \hat{\phi}_{n+1}$ where*

$$\hat{\phi}_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} S_x^m S_y^m \phi_p, \quad p = k, k-1,$$

and there is a $\tilde{\Theta}_n \in C_\lambda^{n,n}$ such that $A\phi_n = b_x b_y \tilde{\Theta}_n + \tilde{\phi}_{n+1}$ where

$$\tilde{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} S_x^m S_y^m A\phi_p, \quad \text{for } p = k+1, k,$$

and the functions Θ_n and $\tilde{\Theta}_n$ are expressed by ψ_p, \dots, ψ_n .

By means of Theorem A (in the case of $k = 1$) and the analytic model, the author proved that the function trace $[(\bar{\lambda}^0 - S^*)^{-1}, (\mu^0 - S)^{-1}] [(\bar{\lambda}_1 - S^*)^{-1}, (\mu_1 - S)^{-1}]$, $\lambda^i, \mu^i \in \text{sp}(S)$, is a complete unitary invariant for some subnormal operator S with trace class commutator $[S^*, S]$ (cf. [14], [15]).

For the unbounded operator case, the author (cf. [9], [10]) also studied the almost unperturbed Schrödinger pair of operators u and v which are self-adjoint operators on a Hilbert space \mathcal{H} satisfying the condition that

$$e^{ius} e^{ivt} - e^{ist} e^{ivt} e^{ius} \in \mathcal{L}^1(\mathcal{H}), \quad s, t \in \mathbb{R},$$

where $\mathcal{L}^1(\mathcal{H})$ is the trace class of operators on \mathcal{H} . If we denote e^{ius} and e^{ivt} by x and y respectively, then $q(x, y) = e^{ist}$ is a complex number determined by x and y . Therefore instead of the commutator we have to study the trace class q -deformed commutator $\{x, y\} \stackrel{\text{def}}{=} xy - q(x, y)yx$. By the way, now-a-day the study of the q -deformed (or q -twisted) commutators becomes an interesting subject (cf. [3], [4], [7], etc.). In [9] and [10], the author studied the form of cyclic one-cocycles associated with q -deformed commutators, which has some application for establishing the theory of principal distribution and others.

The first aim of the present paper is to generalize Theorem A to the q -deformed commutators case. Suppose that there is a function $q(x, y)$, $x \in X$, $y \in Y$, satisfying $q(1, y) = q(x, 1) = 1$ and $q(x^1 x^2, y^1 y^2) = \prod_{i,j=1}^2 q(x^i, y^j)$. Assume that there is a natural number k such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$, $x^i \in X$, $y^j \in Y$. We define new functions ϕ_n and ψ_n by changing the commutator to a q -deformed commutator in (1) and (2). In §4 of the present paper, we give Theorem 1 on these functions ϕ_n and ψ_n which is a generalization of Theorem A in this q -deformed commutator case.

This generalization provides a possibility to study some complete unitary invariants for some unbounded hyponormal operators or pseudo-differential operators $u + iv$ where the pair of u and v is an almost unperturbed Schrödinger pair of operators.

In Theorem 1, the formulas are established for the functions ϕ_n, \dots on the “manifolds”

$$M^{n,n} = \{ (x^0, \dots, x^n; y^0, \dots, y^n) \in X^n \times Y^n : \prod_{i,j=1}^n q(x^i, y^j) = 1 \}.$$

Off these manifolds $M^{m,m}$, the q -deformed commutator case is quite different from the commutator case. The second aim of the present paper is to study the structure (see Theorem 2 of §5) of the Chern character ch_{2n-1} of $2n-1$ dimension associated with the q -deformed commutator off the manifold $M^{m,m}$. In the lower dimension cases, it is calculated in the corollary of Theorem 2 that the Chern characters ch_1 and ch_3 are boundaries of some cyclic cochain off $M^{n,n}$. Further study will be needed to answer the question whether or not all the Chern characters ch_m of odd dimensions associated with q -deformed commutators are boundaries of some cyclic cochains off the manifold $M^{m,m}$. Theorem 2 may provide a basis for this study. We have to point out that the Chern characters of odd dimension have not been fully studied neither on the manifold $M^{n,n}$ (for the q -deformed commutator case), nor for the commutator case.

All of these studies we mentioned above are based on some new tools, the operations $\delta_x, \delta'_x, \tau_x$ and $\delta_y, \delta'_y, \tau_y$, which are the generalizations of b_x, b'_x, t_x and b_y, b'_y, t_y respectively. The formulation of this study is given in §2, which is a set of modified definitions of Hochschild cohomology and cyclic cohomology. The setting is that of a semidirect product of groups, i.e. a group X acting on a group Y by automorphisms q_x . This returns to the ordinary case if q_x acts trivially, i.e. $q_x y = y$ for all x, y . Although in §2 the concept of a q -deformed commutator is not needed, the modified cyclic cohomology operations are introduced for obtaining the formulas of Lemma 1 in §3. These are formulas connecting ϕ_n, ψ_n , etc., which are the trace of products of some q -deformed commutators $\{x, y\} = xy - q_x(y)x$, $x \in X, y \in Y$. These formulas are necessary for establishing Theorems 1 and 2. It was not easy to find out those definitions in §2. Although in Theorem 1 and the proof of Theorem 2, $q_x(y)$ is simply $q(x, y)y$, the formulation adopted in §2 and §3 is for general automorphisms q_x for two reasons. First, even if we restrict ourselves on the simpler case $q_x(y) = q(x, y)y$, we cannot simplify either notation or formulas in §2 and §3. The more important reason is that the present formulation may set a basis for further study. As a matter of fact, in [13], the author studied a special case about the perturbation of some partial differential operators in which $q_x(y)$ is not $q(x, y)y$ but the Chern character of dimension one is the boundary of a zero cyclic cochain off a lower dimensional manifold. That case is not covered in §5 of the present paper. Therefore the setting for general $q_x(\cdot)$ in §2 may provide a tool to calculate certain Chern characters of odd dimension associated with q -deformed commutators in which $q_x(\cdot)$ is not $q(x, y)y$.

In the statement of Theorem 2 and its Corollary, only cyclic cohomology is involved, but in their proof, the formulas in generalized cyclic cohomology are involved.

This paper is only an introduction to a circle of ideas whose natural continuation will be explained in subsequent papers.

2. BASIC DEFINITIONS

Let X and Y be two groups. Let 1 be the identity of X and Y . Suppose that for every $x \in X$ there is an automorphism $q_x : Y \rightarrow Y$ satisfying $q_{x_1 x_2} = q_{x_1} q_{x_2}$, $q_1 =$ identity mapping. For $m \geq 0$ and $n \geq 0$, let $C^{m,n} = C^{m,n}(X, Y)$ be the space of functions

$$f_{m,n}(x^0, \dots, x^m; y^0, \dots, y^n), \quad x^i \in X, y^j \in Y.$$

Define δ'_x and $\delta_x : C^{m,n} \rightarrow C^{m+1,n}$ in the following. For $f \in C^{m,n}$, $x = (x^0, \dots, x^{m+1})$ and $y = (y^0, \dots, y^n)$,

$$\begin{aligned} (\delta'_x f)(x; y) &= \sum_{j=0}^{m-n} (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; y) \\ &\quad + \sum_{j=m-n+1}^m (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; \\ &\quad q_{x^{m-n+1}}(y^0), \dots, q_{x^j}(y^{j-m+n-1}), y^{j-m+n}, \dots, y^n) \end{aligned}$$

and

$$\begin{aligned} (\delta_x f)(x; y) &= (\delta'_x f)(x; y) \\ &\quad + (-1)^{m+1} f(x^{m+1} x^0, x^1, \dots, x^m; q_{x^{m-n+1}}(y^0), \dots, q_{x^{m+1}}(y^n)), \end{aligned}$$

if $m \geq n$; and

$$\begin{aligned} (\delta'_x f)(x; y) &= \sum_{j=0}^m (-1)^j f(x^0, \dots, x^j x^{j+1}, \dots, x^{m+1}; \\ &\quad y^0, \dots, y^{n-m-2}, q_{x^0}(y^{n-m-1}), \dots, q_{x^j}(y^{n-m+j-1}), y^{n-m+j}, \dots, y^n) \end{aligned}$$

and

$$\begin{aligned} (\delta_x f)(x; y) &= (\delta'_x f)(x; y) + (-1)^{m+1} f(x^{m+1} x^0, \dots, x^m; \\ &\quad q_{x^{m+1}}(y^0), \dots, q_{x^{m+1}}(y^{n-m-2}), q_{x^{m+1} x^0}(y^{n-m-1}), q_{x^1}(y^{n-m}), \dots, q_{x^{m+1}}(y^n)), \end{aligned}$$

if $0 \leq m < n$. Define δ'_y and $\delta_y : C^{m,n} \rightarrow C^{m,n+1}$ as follows. For $f \in C^{m,n}$, $x = (x^0, \dots, x^m)$ and $y = (y^0, \dots, y^{n+1})$,

$$\begin{aligned} (\delta'_y f)(x; y) &= \sum_{j=0}^{n-m-1} (-1)^m f(x; y^0, \dots, y^j y^{j+1}, \dots, y^{n+1}) \\ &\quad + \sum_{j=n-m}^n (-1)^j f(x; y^0, \dots, y^{n-m-1}, q_{x^0}^{-1}(y^{n-m}), \dots, q_{x^{j-n+m}}^{-1}(y^j) y^{j+1}, \dots, y^{n+1}) \end{aligned}$$

and

$$\begin{aligned} (\delta_y f)(x; y) &= (\delta'_y f)(x; y) \\ &\quad + (-1)^{n-1} f(x; y^{n+1} y^0, \dots, y^{n-m-1}, q_{x^0}^{-1}(y^{n-m}), \dots, q_{x^m}^{-1}(y^n)), \end{aligned}$$

if $m < n$; and

$$(\delta'_y f)(x; y) = \sum_{j=0}^n (-1)^j f(x; q_{x^{m-n}}^{-1}(y^0), \dots, q_{x^{j+m-n}}^{-1}(y^j) y^{j+1}, y^{j+2}, \dots, y^{n+1})$$

and

$$(\delta_y f)(x; y) = (\delta'_y f)(x; y) + (-1)^{n+1} f(x; q_{x^0 \dots x^{m-n}}^{-1}(y^{n+1}) q_{x^{m-n}}^{-1}(y^0), q_{x^{m-n+1}}^{-1}(y^1), \dots, q_{x^m}^{-1}(y^n))$$

if $m \geq n \geq 0$. It can be verified through calculation that $\delta_x'^2 = \delta_x^2 = 0$ and $\delta_y'^2 = \delta_y^2 = 0$. If $q_x = \text{identity}$, then $\delta_x = b_x$ and $\delta_y = b_y$ in [11]. These δ_x and δ_y are the generalized Hochschild boundary operations in some sense.

Define $\tau_x : C^{m,n} \rightarrow C^{m,n}$ and $\tau_y : C^{m,n} \rightarrow C^{m,n}$ as follows. For $f \in C^{m,n}$, $x = (x^0, \dots, x^m)$ and $y = (y^0, \dots, y^n)$,

$$(\tau_x f)(x; y) = (-1)^m f(x^m, x^0, \dots, x^{m-1}; q_{x^{m-n}}(y^0), \dots, q_{x^m}(y^n)),$$

$$(\tau_y f)(x; y) = (-1)^n f(x; q_{x^0 \dots x^{m-n}}^{-1}(y^n), q_{x^{m-n+1}}^{-1}(y^0), \dots, q_{x^m}^{-1}(y^{n-1}))$$

if $m \geq n$; and

$$(\tau_x f)(x; y) = (-1)^m f(x^m, x^0, \dots, x^{m-1}; q_{x^m}(y^0), \dots, q_{x^0}(y^{n-m}), \dots, q_{x^m}(y^n)),$$

$$(\tau_y f)(x; y) = (-1)^n f(x; y^n, y^0, \dots, y^{n-m-2}, q_{x^0}^{-1}(y^{n-m-1}), \dots, q_{x^m}^{-1}(y^{n-1}))$$

if $m < n$. If $q_x = \text{identity}$, then $\tau_x = t_x$ and $\tau_y = t_y$ in [12]. Through complicated calculation it can be verified that the operations $\delta_x, \delta'_x, \tau_x$ commute with $\delta_y, \delta'_y, \tau_y$. We also have

$$(3) \quad \delta'_x(1 - \tau_x) = (1 - \tau_x)\delta_x, \quad \delta'_y(1 - \tau_y) = (1 - \tau_y)\delta_y.$$

These formulas are the generalizations of the formula $b'(1 - t) = (1 - t)b$ in [5]. Define

$$\alpha_x f = (1 + \tau_x + \dots + \tau_x^m) f, \quad \alpha_y f = (1 + \tau_y + \dots + \tau_y^n) f$$

for $f \in C^{m,n}$ which are the generalizations of operators A_x and A_y in [12]. Then

$$(4) \quad \alpha_x \delta'_x = \delta_x \alpha_x, \quad \alpha_y \delta'_y = \delta_y \alpha_y.$$

For $n \geq 0$ let $p_n(z) = \sum_{j=0}^n (n - j) z^j$. Define $\pi_x f = p_m(\tau_x) f$ and $\pi_y f = p_n(\tau_y) f$ for $f \in C^{m,n}$. Define $\sigma_x = \delta_x \pi_x \delta'_x$ and $\sigma_y = \delta_y \pi_y \delta'_y$. These are generalizations of Connes' operators S_x and S_y in [12]. Then σ_x commutes with σ_y . From (3) it is easy to see that $\delta'_x \delta_x \alpha_x = -(1 - \tau_x) \sigma_x$, $\delta'_y \delta_y \alpha_y = -(1 - \tau_y) \sigma_y$. From this, we may prove that σ_x commutes $\delta_x \alpha_x$ and σ_y commutes $\delta_y \alpha_y$.

3. TRACE OF PRODUCT OF DEFORMED COMMUTATORS

Suppose X and Y are also subgroups of an algebra \mathcal{A} over \mathbb{C} . Define

$$\{x, y\} = xy - q_x(y)x, \quad x \in X, y \in Y,$$

where q_x satisfies the conditions in §2. This $\{x, y\}$ is called the q -deformed commutator of x and y . Suppose that there is a trace ideal J of \mathcal{A} with a trace τ on J , i.e. τ is a linear functional on J satisfying $\tau(ab) = \tau(ba)$ for $b \in J$ and $a \in \mathcal{A}$. Assume that there is a natural number k such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$. For $n \geq k$, define $\psi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau x^0 y^0 \{x^1, y^1\} \cdots \{x^n, y^n\}$, for $x^j \in X, y^j \in Y$. Then $\psi_n \in C^{n,n}$. Define functions

$$\begin{aligned} \xi_n(x; y^0, \dots, y^{n-1}) &= \psi_n(x; 1, y^0, \dots, y^n), & n \geq k, \\ \eta_n(x^0, \dots, x^{n-1}; y) &= \psi(1, x^0, \dots, x^n; y), & n \geq k, \\ \phi_n(x; y) &= \psi_{n+1}(1, x^0, \dots, x^n; 1, y^0, \dots, y^n), & n \geq k - 1, \end{aligned}$$

where $x = (x^0, \dots, x^n)$ and $y = (y^0, \dots, y^n)$. The following lemma gives the basic relations between ψ_n, ξ_n, η_n , and ϕ_{n-1} .

Lemma 1. For $n \geq k$,

$$\begin{aligned} (5) \quad & \xi_{n+1} = \delta_x \psi_n, & \eta_{n+1} &= -\tau_x \delta_y \psi_n, \\ (6) \quad & (1 - \tau_x) \xi_n = \delta'_x \phi_{n-1}, & (1 - \tau_y) \eta_n &= \delta'_y \phi_{n-1}, \\ (7) \quad & (1 - \tau_y^{-1}) \xi_n = \delta_x \phi_{n-1}, & (1 - \tau_x^{-1}) \eta_n &= \delta_y \phi_{n-1}, \\ (8) \quad & (1 - \tau_y) \psi_n = \delta'_y \xi_n - \tau_y \phi_n, & (1 - \tau_x) \psi_n &= \delta'_x \eta_n + \phi_n. \end{aligned}$$

Proof. We only give the proofs of those formulas in which ξ 's are involved. The others can be proved similarly. The basic formulas for deformed commutators are

$$(9) \quad \{x_1 x_2, y\} = x_1 \{x_2, y\} + \{x_1, q_{x_2}(y)\} x_2$$

and

$$(10) \quad \{x, y_1 y_2\} = \{x, y_1\} y_2 + q_x(y_1) \{x, y_2\},$$

for $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. To prove (5), by means of (9), we have

$$\begin{aligned} \xi_{n+1}(x; y) &= \tau x^0 x^1 y^0 \{x^2, y^1\} \cdots \{x^{n+1}, y^n\} - \tau x^0 q_{x^1}(y^0) x^1 \{x^2, y^1\} \cdots \{x^{n+1}, y^n\} \\ &= \psi_n(x^0 x^1, x^2, \dots, x^{n+1}; y) - \psi_n(x^0, x^1 x^2, \dots, x^{n+1}; q_{x^1}(y^0), y^1, \dots, y^n) \\ &\quad + \tau x^0 q_{x^1}(y^0) \{x^1, q_{x^2}(y)\} x^2 \{x^3, y^2\} \cdots \{x^{n+1}, y^n\}, \end{aligned}$$

for $x = (x^0, \dots, x^{n+1})$ and $y = (y^0, \dots, y^n)$. Continuing this process, we may prove (5).

To prove (6), by means of (9), we have

$$\xi_n(x; y) = \phi_{n-1}(x^0 x^1, \dots, x^n; y) - \tau \{x^0, q_{x^1}(y^0)\} x^1 \{x^2, y^1\} \cdots \{x^n, y^{n-1}\}.$$

The last term of the right-hand side of the above formula equals

$$\begin{aligned}
 & -\phi_{n-1}(x^0, x^1x^2, \dots, x^n; q_{x^1}(y^0), y^1, \dots, y^{n-1}) \\
 & \quad + \tau\{x^0, q_{x^1}(y^0)\}\{x^1, q_{x^1}(y^1)\}x^2\{x^3, y^2\} \dots
 \end{aligned}$$

Continuing this process, we may prove (6). By the definitions and (9), it is easy to verify that

$$\tau_x \xi_n - \tau_y^{-1} \xi_n = \delta_x \phi_{n-1} - \delta'_x \phi_{n-1}.$$

Therefore (7) follows from (6). To prove (8), by means of (10), we observe that

$$\begin{aligned}
 \psi_n(x; y) - \phi_n(x; y) &= \tau x^0 \{x^1, y^1\} \dots \{x^n, y^n\} q_{x^0}(y^0) \\
 &= \xi_n(x; y^1, \dots, y^{n-1}, y^n q_{x^0}(y^0)) \\
 & \quad - \tau x^0 \{x^1, y^1\} \dots \{x^{n-1}, y^{n-1}\} q_{x^n}(y^n) \{x^n, q_{x^0}(y^0)\}.
 \end{aligned}$$

Continuing this process we may prove that $\psi_n(x; y) - \phi_n(x; y)$ equals

$$\begin{aligned}
 & \sum_{j=2}^{n+1} (-1)^{n+1-j} \xi_n(x; y_1, \dots, y_{j-1} q_{x^j}(y^j), q_{x^{j+1}}(y^{j+1}), \dots, q_{x^{n+1}}(y^{n+1})) \\
 & \quad + (-1)^n \psi_n(x; q_{x^1}(y^1), \dots, q_{x^{n+1}}(y^{n+1})),
 \end{aligned}$$

where $x^{n+1} = x^0, y^{n+1} = y^0$. Thus $\psi_n - \phi_n = -\tau_y^{-1} \delta'_y \xi_n + \tau_y^{-1} \psi_n$ which proves (8). □

4. A BASIC THEOREM

Let us consider a very important special case. Suppose that there is a function $q(\cdot, \cdot)$ on $X \times Y$ satisfying

$$(11) \quad q(x^1x^2; y^1y^2) = \prod_{i,j=1}^2 q(x^i, y^j), \quad x^i \in X, y^j \in Y,$$

and

$$(12) \quad q(1, y) = q(x, 1) = 1, \quad x \in X, y \in Y.$$

Let $Q = \{q(x, y) : x \in X, y \in Y\} \subset \mathbb{C}$; then Q is a group. Let $\tilde{Y} = \{cy : c \in Q, y \in Y\}$. Define $q_x(cy) = c^2 q(x, y)y, c \in Q, y \in Y$. Then $q_x : \tilde{Y} \rightarrow \tilde{Y}$ is an isomorphism satisfying condition (1). Let $\tilde{C}^{m,n} = \tilde{C}^{m,n}(X, \tilde{Y})$ be the space of the restriction of functions f in $C^{m,n}(X, \tilde{Y})$ satisfying the condition that

$$f(x; c^0y^0, \dots, c^ny^n) = \prod_{j=0}^n c^j f(x; y^0, \dots, y^n)$$

for $c^j \in Q, y^j \in Y$. For example, $\psi_n \in \tilde{C}^{m,n}$. Then $\tilde{C}^{m,n}$ is invariant with respect to the set $D = \{\delta'_x, \delta_x, \tau_x, \delta'_y, \delta_y, \tau_y\}$ of operations defined in $C^{m,n}(X, \tilde{Y})$.

Let us consider the restriction on $\tilde{C}^{m,n}$ of those operations in D only. Then these operations in D possess all the properties described in §2.

Let $M^{m,n} = \{(x^0, \dots, x^m; y^0, \dots, y^n) : x^i \in X, y^j \in Y, \prod_{i=0}^m \prod_{j=0}^n q(x^i, y^j) = 1\}$ and $\hat{C}^{m,n}$ be the restriction of the functions in $\tilde{C}^{m,n}$ on $M^{m,n}$. An important property of $\hat{C}^{m,n}$ is that $\tau_x \alpha_x f = \alpha_x f, \tau_y \alpha_y f = \alpha_y f, f \in \hat{C}^{m,n}$. Define $C_\lambda^{m,n} = \{f \in \hat{C}^{m,n} : \tau_x f = \tau_y f = f\}$. It is easy to see that $\alpha_x \phi_n(x; y) = \alpha_y \phi_n(x, y)$ for $(x, y) \in M^{n,n}$. Define $\alpha \phi_n = \alpha_x \phi_n = \alpha_y \phi_n$ on $M^{n,n}$. Then $\alpha \phi_n \in C_\lambda^{m,n}$. By the same method in [12], using Lemma 1 and the formulas in §2 and §3, we may prove the following theorem and omit the proof.

Theorem 1. *For $n \geq k$, there is a $\Theta_n \in C_\lambda^{n,n}$ such that*

$$\alpha \phi_{n+1} = \delta_x \delta_y \Theta_n + \hat{\phi}_{n+1}, \quad \text{on } M^{n,n},$$

where

$$\hat{\phi}_{2m+p} = (-1)^m \frac{(p+1)!}{(p+2m)!} \sigma_x^m \sigma_y^m \phi_p,$$

for $p = k, k-1$. For $n \geq k$, there is a $\tilde{\Theta}_n \in C_\lambda^{n,n}$ such that

$$\alpha \phi_{n+1} = \delta_x \delta_y \tilde{\Theta}_n + \tilde{\phi}_{n+1}, \quad \text{on } M^{n,n},$$

where

$$\tilde{\phi}_{2m+p} = (-1)^m \frac{p!}{(p+2m)!} \sigma_x^m \sigma_y^m \alpha \phi_p$$

for $p = k+1, k$. Besides, the functions Θ_n and $\tilde{\Theta}_n$ are expressed by ψ_n, \dots, ψ_p .

5. CHERN CHARACTER OF ODD DIMENSION

As in the previous sections, assume that \mathcal{A} is an algebra over \mathbb{C} , and J is a trace ideal in \mathcal{A} with trace τ . Let X and Y be subgroups of \mathcal{A} . Assume that there is a function $q(x, y), x \in X, y \in Y$, satisfying conditions (11) and (12). Assume that there is a natural number k such that $\{x^1, y^1\} \cdots \{x^k, y^k\} \in J$. Define $\Delta = \Delta_{m,n}(x^0, \dots, x^m; y^0, \dots, y^n) = \prod_{i=0}^m \prod_{j=0}^n q(x^i, y^j)$. Let $\nu = \tau_x \tau_y$; then $(\nu f)(x^0, \dots, x^n; y^0, \dots, y^n) = f(x^n, x^0, \dots, x^{n-1}; y^n, y^0, \dots, y^{n-1})$ for $f \in C^{n,n}$.

Lemma 2. *For $n \geq k$ and $\Delta \neq 1$,*

$$(13) \quad \psi_n = (1 - \Delta)^{-1} \alpha_x \phi_n - (1 - \Delta)^{-2} \Delta \delta_x \delta_y \alpha_x \alpha_y \phi_{n-1}$$

and

$$(14) \quad (1 - \nu) \psi_n = (1 - \Delta)^{-1} [-\delta'_x (\delta_y - \delta'_y) \tau_x + (\delta_x - \delta'_x) \delta'_y] \alpha_x \phi_{n-1}.$$

Proof. It is easy to see that $\alpha_x(1 - \tau_x) = 1 - \Delta$ and $\alpha_y(1 - \tau_y) = 1 - \Delta^{-1}$. Applying α_x to both sides of the right equality in (8), we have $(1 - \Delta) \psi_n = \delta_x \alpha_x \eta_n + \alpha_x \phi_n$, since $\alpha_x \delta'_x = \delta_x \alpha_x$ (see (4)). Similarly, from (6), we get $(1 - \Delta^{-1}) \eta_n = \delta_y \alpha_y \phi_{n-1}$. Thus we obtain (13).

From (2) and $(1 - \nu) = (1 - \tau_x)\tau_y + (1 - \tau_y)$ it is easy to calculate that

$$(15) \quad (1 - \nu)\delta_x\alpha_x\delta_y\alpha_y = (1 - \Delta)\delta'_x\tau_y\delta_y\alpha_y + (1 - \Delta^{-1})\delta_x\alpha_x\delta'_y.$$

On the other hand, it is easy to see that if $\nu f = f$, then $\Delta\alpha_y f = \tau_x\alpha_x f$. From (13), (15) and the fact that $(1 - \nu)\alpha_x\phi_{n-1} = 0$, we obtain

$$(16) \quad (1 - \nu)\psi_n = (1 - \Delta)^{-1}(-\delta'_x\nu\delta_y + \delta_x\delta'_y)\alpha_x\phi_{n-1}.$$

From (3), it is easy to calculate that

$$(17) \quad \delta'_x\nu\delta_y - \delta_x\delta'_y = \delta'_x(\delta_y - \delta'_y)\tau_x - (\delta_x - \delta'_x)\delta'_y + \delta'_x\delta'_y(\nu - 1).$$

From (16) and (17), (14) follows. □

Let $\mathcal{W} = \{(x, y, c) : x \in X, y \in Y, c \in Q\}$. Define the product

$$(x^0, y^0, c^0)(x^1, y^1, c^1) = (x^0x^1, y^0y^1, c^0c^1q(x^0, y^1))$$

in \mathcal{W} ; then \mathcal{W} is a group. Define a mapping from \mathcal{W} to \mathcal{A} as $p(x, y, c) = cyx$. Then the ‘‘curvature’’ of this mapping p is defined as $\omega(w^0, w^1) = p(w^0w^1) - p(w^0)p(w^1)$, $w^0, w^1 \in \mathcal{W}$. For $n \geq k$, define the Chern character of dimension $2n - 1$ (see [1] and [6]) as

$$\begin{aligned} \text{ch}_{2n-1}(w^0, \dots, w^{2n-1}) \\ = \tau(\omega(w^0, w^1) \cdots \omega(w^{2n-2}, w^{2n-1}) - \omega(w^{2n-1}, w^0) \cdots \omega(w^{2n-3}, w^{2n-2})). \end{aligned}$$

A function $f(w^0, \dots, w^n)$ is said to be homogeneous if $f((u^0, c^0), \dots, (u^n, c^n)) = \prod_{j=0}^n c^j f((u^0, 1), \dots, (u^n, 1))$. For the homogeneous function f , we always rewrite $f((u^0, 1), \dots, (u^n, 1))$ as $f(u^0, \dots, u^n)$ or $f(x^0, \dots, x^n; y^0, \dots, y^n)$ for $u^j = (x^j, y^j)$, $x^j \in X$, $y^j \in Y$. It is obvious that ch_{2n-1} is homogeneous. Thus we only have to study $\text{ch}_{2n-1}(u^0, \dots, u^n)$ for $(u^j, 1) \in \mathcal{W}$. The Hochschild boundary bf of a homogeneous function f is

$$\begin{aligned} \sum_{j=0}^n (-1)^j q(x^j, y^{j+1}) f(x^0, \dots, x^j x^{j+1}, \dots, x^n; y^0, \dots, y^j y^{j+1}, \dots, y^n) \\ + (-1)^{n+1} q(x^{n+1}, y^0) f(x^{n+1} x^0, \dots, x^n; y^{n+1} y^0, \dots, y^n). \end{aligned}$$

A function $F(f_k, \dots, f_l)(x^0, \dots, x^m; y^0, \dots, y^m)$, $f_j \in C^{j,j}$, $j = k, \dots, l$ and $x^r \in X$, $y^s \in Y$, is said to be a linear functional if it is expressed as $\sum_{s=1}^N c_s h_s$, where c_s is a function of $q(x^i, y^j)$, $i, j = 0, \dots, m$, and $h_s(x^0, \dots, x^m; y^0, \dots, y^m)$ is of the form

$$f_j(x^{l_{01}} \cdots x^{l_{0s_0}}, \dots, x^{l_{j1}} \cdots x^{l_{js_j}}; y^{r_{01}} \cdots y^{r_{0t_0}}, \dots, y^{r_{j1}} \cdots y^{r_{jt_j}})$$

for certain $j \in \{k, \dots, l\}$ where $(l_{01}, \dots, l_{0s_0}, \dots, l_{j1}, \dots, l_{js_j})$ is $(a, a + 1, \dots, a + m)$ for some a , $(r_{01}, \dots, r_{0t_0}, \dots, r_{j1}, \dots, r_{jt_j})$ is $(c, c + 1, \dots, c + m)$ for some c , and $x^n = x^{n-m-1}$, $y^n = y^{n-m-1}$ for $n > m$.

Theorem 2. *There is a linear functional $F_m(\phi_{k-1}, \dots, \phi_{2m-3})$ such that*

$$\begin{aligned} ch_{2m-1}((x^0, y^0), \dots, (x^{2m-1}, y^{2m-1})) \\ = b(1 - \Delta)^{-1}(-1)t_x\phi_{2m-2} + F_m(\phi_{k-1}, \dots, \phi_{2m-3}). \end{aligned}$$

Proof. Let $n = 2m - 1$ and $f = -t_x^{-1} ch_n$. Then

$$f(x^0, \dots, x^n; y^0, \dots, y^n) = (1 - \nu)\tau(v^0 \dots v^{m-1})$$

where $v^j = x^{2j}y^{2j} \{x^{2j+1}, y^{2j+1}\}$. From (9) and (10), through a complicated calculation, we can prove that $\tau(v^0 \dots v^{m-1}) = \psi_n + R_n(\psi_{n-1}) + G_n(\psi_k, \dots, \psi_{n-2})$, where $G_n(\psi_k, \dots, \psi_{n-2})$ is a linear functional,

$$\begin{aligned} R_n(f) = \sum_{i=2}^{n-1} \sum_{j=0}^{2[\frac{i}{2}]-1} (-1)^{i+j-1} c_{ij} f(x^0, \dots, x^i x^{i+1}, \dots, x^n; y^0, \dots, y^j y^{j+1}, \dots, y^n) \\ + \sum_{j=0}^{n-2} (-1)^j c_{nj} f(x^{n+1}x^0, x^1, \dots, x^n; y^0, \dots, y^j y^{j+1}, \dots, y^n), \end{aligned}$$

and $c_{ij} = \prod_{l=j+1}^i q(x^l, y^l)$. We can prove that if $f \in C^{n-1, n-1}$, then

$$(18) \quad (1 - \nu)R_n(f) + (-\delta'_x(\delta_y - \delta'_y)\tau_x + (\delta_x - \delta'_x)\delta'_y)f = Q_n((1 - \nu)f) + S_n(f),$$

where

(19)

$$Q_n(g) = \sum_{i=2}^{n-1} \sum_{j=0}^{i-2} (-1)^{i+j-1} c_{ij} g(x^0, \dots, x^i x^{i+1}, \dots, x^n; y^0, \dots, y^j y^{j+1}, \dots, y^n)$$

and $S_n(f) = -t_x^{-1} b t_x(f)$, if $\nu f = f$. From (13), $G_n(\psi_k, \dots, \psi_{n-2})$ can be expressed as a linear functional $H_n(\phi_{k-1}, \dots, \phi_{n-3})$. From (13), (14), (18) and (19), we have

$$\begin{aligned} (1 - \nu)\psi_n + (1 - \nu)R_n(\psi_{n-1}) \\ = -t_x^{-1} b(1 - \Delta)^{-1} t_x \alpha_x \phi_{n-1} - S_n((1 - \Delta)^{-2} \Delta \delta_x \delta_y \alpha_x \alpha_y \phi_{n-2}), \end{aligned}$$

which proves the theorem, where $F_m = t_x S_n((1 - \Delta)^{-2} \Delta \delta_x \delta_y \alpha_x \alpha_y \phi_{n-2}) - t_x H_m$. \square

Corollary. *ch_1 and ch_3 are boundaries of cyclic cochains.*

Proof. By the proof of Theorem 2, $F_1 = 0$. Thus ch_1 is the boundary of a cyclic zero-cochain. By formula (19), through calculation we may prove that $F_2 = bA(1 - \Delta)^{-2}G$, where $A = (1 + \nu + \nu^2)$,

$$\begin{aligned} G = \frac{1}{2} f_1(x^0, x^1 x^2; y^0, y^1, y^2) q(x^0, y^0)^{-1} q(x^1, y^1)^{-1} q(x^2, y^1 y^2)^{-1} \\ - \frac{1}{2} f_1(x^0, x^1 x^2; y^1 y^2, y^0) q(x^1, y^2) + f_1(x^0, x^1 x^2; y^1, y^2 y^0) q(x^2, y^2)^{-1} \\ - f_1(x^0, x^1 x^2; y^0 y^1, y^2) q(x^0, y^0)^{-1} q(x^2, y^2)^{-1} \end{aligned}$$

and $f_1 = \alpha_x^2 \tau_x \phi_1$. Therefore ch_3 is also the boundary of some cyclic 2-cochain. \square

Remark. Although the Chern character defined here is based on the mapping from the group \mathcal{W} to the algebra \mathcal{A} , it is not difficult to prove that Theorem 2 and its corollary for the Chern character defined through the mapping from a suitable group algebra of \mathcal{W} to \mathcal{A} .

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