PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 6, June 1996

STABLE SPLITTINGS OF BO($2n$) AND BU($2n$)

DUNG YUNG YAN

(Communicated by Thomas Goodwillie)

ABSTRACT. Using the Snaith-Mitchell-Priddy splittings of $\mathrm{BO}(2n)$ and $\mathrm{BU}(2n)$, we can give new stable splittings of $\mathrm{BO}(2n)$ and $\mathrm{BU}(2n)$ respectively.

In [5] and [6] Snaith, Mitchell, and Priddy showed that the natural filtrations on $\mathrm{BO}(n)$, $\mathrm{BU}(n)$, and $\mathrm{BSP}(n)$ stably split, respectively, and in [2] Henn and Mui showed the corresponding splitting for $\mathrm{BSO}(2n+1)$ and $\mathrm{BSU}(2n+1)$. The purpose of this note is to give new stable splittings of $\mathrm{BO}(2n)$ and $\mathrm{BU}(2n)$, respectively.

Let $g_{2n}\colon\ O(2n) \to \mathrm{SO}(2n+1)$ be defined by $g_{2n}(\alpha) = \det(\alpha) \oplus \alpha$. Then we have

$$Bg_{2n}\colon\ \mathrm{BO}(2n) \to \mathrm{BSO}(2n+1).$$

Let Y_{2n} be the stable fibre of Bg_{2n}, that is,

$$Y_{2n} \overset{\delta_{2n}}{\to} \mathrm{BO}(2n) \overset{Bg_{2n}}{\to} \mathrm{BSO}(2n+1)$$

is a stable fibration.

Theorem A. *Localized at the prime 2, the stable fibre Y_{2n} of Bg_{2n} is a stable summand of the stable fibre Y_{2n+2} of Bg_{2n+2}.*

Corollary B. *Localized at the prime 2, for each n there are 2-local spectra F_{2i}, $1 \le i \le n$, such that*

$$F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n} \to \mathrm{BO}(2n) \overset{Bg_{2n}}{\to} \mathrm{BSO}(2n+1)$$

is a stable filtration and

$$\mathrm{BO}(2n) \cong \mathrm{BSO}(2n+1) \vee F_2 \vee F_4 \vee F_6 \vee \cdots \vee F_{2n}.$$

Remark 1. In [3], Mitchell and Priddy proved that the homotopy type of F_2 is $\sum^{-2} \overline{SP^4} S^0 \vee P_1^\infty$ (see [3]).

Remark 2. We have analogous theorems for $\mathrm{BU}(2n)$ and $\mathrm{BSU}(2n+1)$.

From now on we will suppress details for $\mathrm{BU}(2n)$, which can be obtained easily from those for $\mathrm{BO}(2n)$, and all spaces or spectra are implicitly localized at prime 2.

Lemma 1. *Let $Y_{2n} \overset{\delta_{2n}}{\to} \mathrm{BO}(2n) \overset{Bg_{2n}}{\to} \mathrm{BSO}(2n+1)$ be a stable fibration. Then $\mathrm{BO}(2n) \cong \mathrm{BSO}(2n+1) \vee Y_{2n}$.*

Received by the editors May 26, 1994 and, in revised form, October 20, 1994.
1991 *Mathematics Subject Classification.* Primary 55P10.
This work was partially supported by the National Science Council of R.O.C.

Proof. Since the fibre of

$$Bg_{2n}\colon\ \mathrm{BO}(2n) \to \mathrm{BSO}(2n+1)$$

is $\mathrm{SO}(2n+1)/O(2n) \cong RP^{2n}$, the Becker-Gottlieb transfer [1] associated to this fibration is a stable map

$$q_{2n}\colon\ \mathrm{BSO}(2n+1) \to \mathrm{BO}(2n)$$

such that $(q_{2n})^* \circ (Bg_{2n})^*$ is multiplication by the Euler charactertistic $\chi(RP^{2n}) = 1(\bmod 2)$. Thus q_{2n} provides a stable splitting of Bg_{2n}, and

$$\mathrm{BO}(2n) \cong \mathrm{BSO}(2n+1) \vee Y_{2n}.$$

This completes the proof.

Proof of Corollary B. This follows immediately from Theorem A and Lemma 1 by induction.

Lemma 2. *There is a stable map* $\lambda_{2n}\colon\ \mathrm{BO}(2n+2) \to \mathrm{BO}(2n)$ *such that the composite map*

$$\mathrm{BO}(2n) \overset{Bi_{2n}}{\to} \mathrm{BO}(2n+2) \overset{\lambda_{2n}}{\to} \mathrm{BO}(2n)$$

is homotopic to the identity map, where the first map is the natural inclusion map.

Proof. This lemma follows immediately from [5] and [6].

Now we can prove Theorem A.

Proof of Theorem A. Let g_{2n} be the map $g_{2n}(\alpha) = (\det\alpha) \oplus \alpha$. Then the diagram of groups

$$\begin{array}{ccc}
O(2n) & \xrightarrow{g_{2n}} & \mathrm{SO}(2n+1) \\
{\scriptstyle i_{2n}}\downarrow & & \downarrow{\scriptstyle j_{2n+1}} \\
O(2n+2) & \xrightarrow{g_{2n+2}} & \mathrm{SO}(2n+3)
\end{array}$$

is strictly commutative, where i_{2n} and j_{2n+1} are the inclusions

$$(\det(\alpha) \oplus \alpha) \oplus 1 = \det(\alpha \oplus 1) \oplus (\alpha \oplus 1).$$

This yields a strictly commutative diagram of classifying spaces, and if Y_{2n} denotes the stable fibre of Bg_{2n}, then we have a commutative diagram of stable maps:

$$\begin{array}{ccccc}
Y_{2n} & \xrightarrow{\delta_{2n}} & \mathrm{BO}(2n) & \xrightarrow{Bg_{2n}} & \mathrm{BSO}(2n+1) \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle Bi_{2n}} & & \downarrow{\scriptstyle Bj_{2n+1}} \\
Y_{2n+2} & \xrightarrow{\delta_{2n+2}} & \mathrm{BO}(2n+2) & \xrightarrow{Bg_{2n+2}} & \mathrm{BSO}(2n+3)
\end{array}$$

Now by Lemma 1, Bg_{2n} has a right inverse (in the stable category), that is, the fibration sequence splits, that is, δ_{2n} has a left inverse, say η_{2n}. By Lemma 2, Bi_{2n} has a left inverse, say λ_{2n}. It follows that k has a left inverse

$$(\eta_{2n}\lambda_{2n}\delta_{2n+2})k = \eta_{2n}\lambda_{2n}Bi_{2n}\delta_{2n} = \eta_{2n}\delta_{2n} = 1.$$

This completes the proof of Theorem A.

References

1. J. Becker and D. Gottlieb, *Transfer maps for fibrations and duality*, Compositio Math. **33** (1976), 107–133. MR **55:**9087
2. H. W. Henn and H. Mui, *Stable splittings for classifying spaces of alternating, special orthogonal and special unitary groups*, Contemp. Math., Amer. Math. Soc., Providence, RI, vol. 146, 1993, pp. 143–158. MR **94g:**55021
3. S. Mitchell and S. Priddy, *Symmetric product spectra and splittings of classifying spaces*, Amer. J. Math. **106**(1984), 219–232. MR **86g:**55009
4. S. Mitchell and S. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), 285–298. MR **85f:**55005
5. S. Mitchell and S. Priddy, *A double coset formula for Levi subgroups and splitting BGL_n*, Lecture Notes in Math., vol. 1730, Springer-Verlag, Berlin and New York, 1989, pp. 325–334. MR **90f:**55015
6. V. Snaith, *Algebraic cobordism and K-theory*, Mem. Amer. Math. Soc. **21** (1979), no. 221. MR **80k:**57060

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 30043