

POLYNOMIALS WITH ROOTS MODULO EVERY INTEGER

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ABSTRACT. Given a polynomial with integer coefficients, we calculate the density of the set of primes modulo which the polynomial has a root. We also give a simple criterion to decide whether or not the polynomial has a root modulo every non-zero integer.

1. INTRODUCTION

In [BO] and [BH] the diophantine equation

$$P(x) = n!,$$

where P is a polynomial with integer coefficients, was studied (we refer to [EO] and [Gu, Sec.D25] for related equations and more information). On probabilistic grounds, one expects that, if $\deg P \geq 2$, then the equation has only finitely many solutions. One case in which this is trivial is when the congruence

$$(1) \quad P(x) \equiv 0 \pmod{m}$$

happens to have no root for some integer m . This raises the following

Question. Given a polynomial $P(x) \in \mathbf{Z}[x]$, decide whether or not (1) has a solution for every m .

The same question is also motivated by a more general result. A *measure-preserving system* is a quadruple (X, \mathcal{B}, μ, T) , in which (X, \mathcal{B}, μ) is a probability space, and T is a measure-preserving transformation thereof. A set $R \subseteq \mathbf{N}$ is a *Poincaré set* if for any measure-preserving system (X, \mathcal{B}, μ, T) and $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists some $n \in R$ with $\mu(T^{-n}A \cap A) > 0$ [Fu, Def.3.6]. An interesting question is which “natural” sets of integers are Poincaré sets. It turns out that, for $P \in \mathbf{Z}[x]$, the set $\{P(n) : n \in \mathbf{N}\}$ is a Poincaré set if and only if (1) has a root for each m . A consequence is that, if P is such and S is a set of integers of positive (upper Banach) density, then there exist $s_1, s_2 \in S$, $s_1 \neq s_2$, and $n \in \mathbf{N}$ such that $s_2 - s_1 = P(n)$. This result was first proved by Sárközy for the polynomial $P(x) = x^2$ [Sá1] (see also [Sá2] and [Sá3], where other polynomials

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are dealt with). It does not seem to have been explicitly stated in the form above, but certainly follows from the results and the discussion in [Fu, Ch.3]. For more on this direction, see [Bo] (in particular, Theorem 6.6 there).

Another result involving polynomials that satisfy the property in question is due to Kamae and Mendes France [KM]. The well-known difference theorem of van der Corput states that, if $(x_n)_{n=1}^\infty$ is a sequence in \mathbf{R} such that for every positive integer h the sequence $(x_{n+h} - x_n)_{n=1}^\infty$ is uniformly distributed modulo 1, then $(x_n)_{n=1}^\infty$ is also uniformly distributed modulo 1 (see, for example, [KN]). Kamae and Mendes France noted that there exist sets $H \subseteq \mathbf{N}$ such that it suffices to check the difference condition for each $h \in H$ to obtain the same conclusion. One of their examples of such a set H is the set of all values assumed by some integer polynomial satisfying our condition.

Obviously, (1) is solvable for each m if P has a linear monic factor $x - a$. The interest in the question stems from the fact that there are polynomials not having a linear factor, which still enjoy this property.

Example 1. The polynomial

$$P(x) = (x^2 - 13)(x^2 - 17)(x^2 - 221)$$

has no integer (or rational) roots, but has a root modulo every integer (see [BS, p.3]).

It turns out that the question presented above is in fact decidable, and even in much more generality ([A], [FS]). In this paper we present a relatively simple answer to this question. We also find, given a polynomial $P(x) \in \mathbf{Z}[x]$, the density of the set of primes p for which (1) has a solution for $m = p$. (The fact that this set of primes has some Dirichlet density which is, moreover, a rational number, follows as a very special case from a result of Ax [A].)

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2. THE MAIN RESULTS

Given $P(x) \in \mathbf{Z}[x]$, factorize it as a product of polynomials in $\mathbf{Z}[x]$, irreducible over \mathbf{Q} :

$$P(x) = h_1(x) \cdot \dots \cdot h_\nu(x).$$

(Here we assumed implicitly that the greatest common divisor of the coefficients of the polynomial $P(x) = a_n x^n + \dots + a_0$ is 1 – otherwise the factorization is non-unique. Of course, this has no bearing on the paper, since for our results P may be replaced by $P/\gcd(a_0, a_1, \dots, a_n)$, which is a polynomial that possesses the desired property.) To state our theorem we need a few notation. First, let L be the splitting field of P over \mathbf{Q} and $G = \text{Gal}(L/\mathbf{Q})$ the Galois group of this extension. For $1 \leq i \leq \nu$, let θ_i be a fixed root of h_i , and put $K_i = \mathbf{Q}(\theta_i)$ and $H_i = \text{Gal}(L/K_i) \leq G$. Finally, set $U = \bigcup_{i=1}^\nu H_i \subseteq G$. We also need some constants.

Write:

$$h_i(x) = \sum_{j=0}^{n_i} a_{ij} x^j.$$

Denote:

$$\begin{aligned} \delta_i &= \text{the discriminant of } h_i; \\ \rho_i &= a_{in_i} \delta_i = R(h_i, h'_i) - \text{the resultant of } h_i \text{ and } h'_i \text{ [vW, §35]}; \\ \delta &= \rho_1 \cdots \rho_\nu; \\ \delta &= p_1^{\alpha_1} \cdots p_\mu^{\alpha_\mu} - \text{the prime power factorization of } \delta; \\ \Delta &= p_1^{2\alpha_1+1} \cdots p_\mu^{2\alpha_\mu+1}; \\ D &= \left(\prod_{i=1}^{\nu} \delta_i^{\frac{n_i-1}{n_i}} \right)^{n_1! \cdots n_\nu!}. \end{aligned}$$

Note that all these constants are integers, and are directly computable from the polynomials h_1, h_2, \dots, h_ν .

Theorem 1. *The following conditions are equivalent:*

- (a) P has a root mod m for every non-zero integer m ;
- (b) P has a root mod Δ , and

$$(2) \quad \bigcup_{\sigma \in G} \sigma^{-1} U \sigma = G;$$

- (c) P has a root mod Δ and mod p for each prime $p \leq 2D^A$. (Here A is an effective absolute constant, to be defined later.)

Remark 1. As follows from the results of Lagarias and Odlyzko [LO], under the Generalized Riemann Hypothesis (henceforward GRH), the term $2D^A$ in (c) may be replaced by $c_1 (\log D)^2$. Oesterlé [O] proved that one may take $c_1 = 70$. Bach [Ba] obtained further numerical results in this direction, but they are not general enough for the purposes of the present paper (see the discussion in [Ba], p.376).

Example 2. Condition (2) reveals that, for P to have the property under consideration without having rational roots, G has to be a union of proper subgroups thereof. The smallest group for which this occurs is the non-cyclic group of order 4. (Indeed, the condition is always fulfilled unless G is cyclic.) G is a union of three subgroups of order 2, so that P must be of degree 6 at least. This is the case in Example 1. With the group $G = S_3$ one can obtain a polynomial of degree 5 having the same properties:

$$P(x) = (x^3 - 19)(x^2 + x + 1).$$

In fact, in this case we have $L = \mathbf{Q}(\sqrt[3]{19}, i\sqrt{3})$, $G = S_3$ and:

$$\theta_1 = \sqrt[3]{19}, \quad \theta_2 = \frac{-1 + i\sqrt{3}}{2}.$$

The subgroup $H_1 = \text{Gal}(L/\mathbf{Q}(\sqrt[3]{19}))$ is of order 2, and the union of its conjugates is the set of 4 elements of S_3 of orders 1 and 2. The subgroup $H_2 = \text{Gal}(L/\mathbf{Q}(i\sqrt{3}))$ is of order 3. Thus condition (2) of Theorem 1 is satisfied. Now one calculates routinely

$$\delta_1 = 3^3 \cdot 19^2, \quad \delta_2 = 3,$$

so that

$$\delta = 3^4 \cdot 19^2, \quad \Delta = 3^9 \cdot 19^5.$$

It is easily verified that the congruence

$$x^2 + x + 1 \equiv 0 \pmod{19^5}$$

has a solution, and, with slightly more work, that the same holds for

$$x^3 - 19 \equiv 0 \pmod{3^9}.$$

Thus P is in fact an example as required.

Remark 2. It is easy to infer from Theorem 1 that there exists no polynomial of degree less than 5 without rational roots possessing the property in question. Thus the above example is minimal in this respect.

We mention in passing that, given a polynomial P satisfying the property under consideration, we can generate out of it many polynomials having the same property. In fact, solutions of (1), m being a high power of some fixed prime p , are all taken care of by one of the factors h_i of P . But then one may replace all other $h_j(x)$ by $h_j(px)$ (or $h_j(p^l x)$ with an arbitrary l).

Example 3. In Example 2, congruences modulo powers of 2 are taken care of by the first factor $x^3 - 19$, so in the second factor we may replace x by $4x$, say. Thus the polynomial $(x^3 - 19)(16x^2 + 4x + 1)$ is a non-monic polynomial possessing the property in question.

The *density* of a set T of primes is defined by

$$d(T) = \lim_{x \rightarrow \infty} \frac{\pi(x, T)}{\pi(x)},$$

where $\pi(x)$ is the number of all primes not exceeding x and $\pi(x, T) = |T \cap [1, x]|$ is the number of such primes belonging to T , provided that the limit exists. (Of course, in view of the Prime Number Theorem, one can replace the denominator on the right-hand side by $\frac{x}{\log x}$.)

We recall that there is also a weaker notion of density, namely that of the Dirichlet density. If the density of T exists, then so does the Dirichlet density, and the two densities coincide.

Theorem 2. *Given a polynomial $P \in \mathbf{Z}[x]$, the density of the set S of primes p for which the congruence $P(x) \equiv 0 \pmod{p}$ has a solution for $m = p$ is*

$$d(S) = \frac{\left| \bigcup_{\sigma \in G} \sigma^{-1} U \sigma \right|}{|G|}.$$

Remark 3. V. Schulze [Schu1, Schu2] proved that the density in Theorem 2 exists and is a rational number, and calculated it for some concrete polynomials. See also [A], [FS] and [L] for more general but less explicit results.

Remark 4. Theorems 1.3 and 1.4 of [LO] imply the following quantitative version of our Theorem 2:

$$(3) \quad |\pi(x, S) - d(S) \operatorname{Li} x| \leq d(S) \operatorname{Li} x^\beta + c_2 |U| x \exp \left(-c_3 \sqrt{\frac{\log x}{|G|}} \right),$$

where $d(S)$ is as in Theorem 2,

$$\beta = \max \left(1 - \frac{1}{4 \log D}, 1 - \frac{1}{c_4 D^{\frac{1}{|G|}}} \right),$$

and c_2, c_3, c_4 are effective absolute constants. Under GRH, the right-hand side of (3) may be replaced by

$$(3') \quad c_5 \left(d(S) \sqrt{x} \log \left(Dx^{|G|} \right) + |U| \log D \right),$$

c_5 being an effective absolute constant. This also follows from the results of [LO]. Oesterlé [O] obtained a version of (3') including only explicit constants.

3. AN UPPER BOUND FOR d_L

Lemma 1. *The absolute discriminant d_L of the field L divides D , and in particular $d_L \leq D$.*

Proof. Fix i and write $n = n_i$, $a_j = a_{ij}$, $\theta = \theta_i$, $K = K_i$.

Let $\theta = \theta^{(1)}, \dots, \theta^{(n)}$ be the conjugates of θ over \mathbf{Q} . Consider the following basis of K over \mathbf{Q} :

$$\begin{aligned} \omega_1 &= 1; \\ \omega_2 &= a_n \theta; \\ \omega_3 &= a_n \theta^2 + a_{n-1} \theta; \\ &\vdots \\ \omega_n &= a_n \theta^{n-1} + a_{n-1} \theta^{n-2} + \dots + a_2 \theta. \end{aligned}$$

Since $\omega_1, \dots, \omega_n$ are algebraic integers (see [Schm, p.183] for an explanation), we have $d_K | d(\omega_1, \dots, \omega_n)$. But

$$d(\omega_1, \dots, \omega_n) = |\det[\omega_{kj}]|^2 = \delta_i,$$

where ω_{kj} is obtained from ω_k upon replacing θ by $\theta^{(j)}$. Thus, $d_K | \delta_i$.

Note that we have $d_{K^{(j)}} = d_K$ for any j , where $K^{(j)} = \mathbf{Q}(\theta^{(j)})$. Hence the discriminant $d_{K'}$ of the field $K' := K'_i = \mathbf{Q}(\theta^{(1)}, \dots, \theta^{(n-1)})$ divides

$$\prod_{j=1}^{n-1} (d_{K^{(j)}})^{[K':K^{(j)}]} = (d_K)^{(n-1)[K':K]} |\delta_i^{(n-1)(n-1)}|$$

Finally, the field L is the composite of K'_1, \dots, K'_ν , hence

$$d_L | \prod_{i=1}^{\nu} (d_{K'_i})^{[L:K'_i]},$$

and the last product divides

$$\prod_{i=1}^{\nu} \left(\delta_i^{(n-1)(n-1)!} \right)^{n_1! \dots n_{i-1}! n_{i+1}! \dots n_\nu!} = D,$$

which proves the lemma.

4. PROOF OF THEOREMS 1 AND 2

Let K be a subfield of L , \mathfrak{P} a prime ideal of L unramified over K , \mathfrak{p} the prime ideal of K below \mathfrak{P} , and $L_{\mathfrak{P}}$ and $K_{\mathfrak{p}}$ the corresponding completions. Since there are natural embeddings

$$\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) \hookrightarrow \mathrm{Gal}(L/K) \hookrightarrow \mathrm{Gal}(L/\mathbf{Q}),$$

we may suppose further that

$$\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) \leq \mathrm{Gal}(L/K) \leq \mathrm{Gal}(L/\mathbf{Q}).$$

Let R_L be the ring of integers of the field L . Recall that the *Frobenius symbol* $(\frac{L/K}{\mathfrak{P}}) \in \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$ is defined uniquely by the property

$$\left(\frac{L/K}{\mathfrak{P}}\right)(\alpha) \equiv \alpha^{N_{\mathfrak{p}}} \pmod{\mathfrak{P}}$$

for all $\alpha \in R_L$ [Na, §7.3.1], and that *Artin's symbol*

$$\left[\frac{L/K}{\mathfrak{p}}\right] = \left\{ \sigma \left(\frac{L/K}{\mathfrak{P}} \right) \sigma^{-1} : \sigma \in \mathrm{Gal}(L/K) \right\}$$

is the conjugacy class of $(\frac{L/K}{\mathfrak{P}})$ in $\mathrm{Gal}(L/K)$.

We need the following elementary property of Frobenius symbols. Let \mathfrak{p} be unramified over \mathbf{Q} , and let $p \in \mathbf{Z}$ be the prime below \mathfrak{p} . Denote $f_{\mathfrak{p}} = [K_{\mathfrak{p}} : \mathbf{Q}_p]$. We claim that

$$(4) \quad \left(\frac{L/K}{\mathfrak{P}}\right) = \left(\frac{L/\mathbf{Q}}{\mathfrak{P}}\right)^{f_{\mathfrak{p}}}$$

and

$$(5) \quad \left(\frac{L/\mathbf{Q}}{\mathfrak{P}}\right)^m \in \mathrm{Gal}(L/K) \iff f_{\mathfrak{p}} | m.$$

In fact, (4) is well known and follows immediately from the definition. To prove (5) note that $(\frac{L/\mathbf{Q}}{\mathfrak{P}})$ generates the cyclic group $\mathrm{Gal}(L_{\mathfrak{P}}/\mathbf{Q}_p)$, and that

$$\begin{aligned} & [\mathrm{Gal}(L_{\mathfrak{P}}/\mathbf{Q}_p) : \mathrm{Gal}(L/K) \cap \mathrm{Gal}(L_{\mathfrak{P}}/\mathbf{Q}_p)] \\ &= [\mathrm{Gal}(L_{\mathfrak{P}}/\mathbf{Q}_p) : \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})] = [K_{\mathfrak{p}} : \mathbf{Q}_p] = f_{\mathfrak{p}}. \end{aligned}$$

We deduce both Theorems 1 and 2 from the following statement.

Lemma 2. *Let p be a prime not dividing δ . Then*

$$(6) \quad \left[\frac{L/\mathbf{Q}}{p} \right] \cap U \neq \emptyset$$

if and only if $P(x)$ has a root in \mathbf{Q}_p .

We mention that according to Lemma 1 it follows in particular that p is unramified in L .

Proof. Let $pR_L = \mathfrak{P}_1 \dots \mathfrak{P}_\tau$ be the decomposition of p in L . Then

$$(7) \quad \left[\frac{L/\mathbf{Q}}{p} \right] = \left\{ \left(\frac{L/\mathbf{Q}}{\mathfrak{P}_1} \right), \dots, \left(\frac{L/\mathbf{Q}}{\mathfrak{P}_\tau} \right) \right\}.$$

Hence (6) is equivalent to the following: for some i and j

$$(8) \quad \left(\frac{L/\mathbf{Q}}{\mathfrak{P}_j} \right) \in H_i.$$

Let \mathfrak{p} be the prime ideal of K_i below \mathfrak{P}_j . Then (5) yields that (8) is equivalent to

$$[(K_i)_{\mathfrak{p}} : \mathbf{Q}_p] = 1,$$

which may happen if and only if $h_i(x)$ has a root in \mathbf{Q}_p . This proves the lemma.

Proof of Theorem 1. (b) \implies (a): Instead of (a) we shall prove the following equivalent statement:

(a') $P(x)$ has a root in \mathbf{Q}_p for every prime p .

So, fix a prime p . If it does not divide δ , then $P(x)$ has a root in \mathbf{Q}_p by (2) and Lemma 2. Now let p divide δ . Let $\lambda \in \mathbf{Z}$ be the root of $P(x) \bmod \Delta$. Then

$$|P(\lambda)|_p < |\delta|_p^2.$$

Hence for some i

$$|h_i(\lambda)|_p < |\rho_i|_p^2.$$

On the other hand, there exist polynomials $a(x), b(x) \in \mathbf{Z}[x]$ such that

$$a(x)h_i(x) + b(x)h'_i(x) = \rho_i.$$

Hence $|h'_i(\lambda)|_p \geq |\rho_i|_p$, and we get

$$|h_i(\lambda)|_p < |h'_i(\lambda)|_p^2.$$

By Hensel's lemma [CF, Ch.2, App.C] $h_i(x)$ has a root in \mathbf{Q}_p . Hence $P(x)$ has a root in \mathbf{Q}_p , which completes the proof of (b) \implies (a').

(a) \implies (c): Trivial.

(c) \implies (b): We have to prove that any conjugacy class C of the group G intersects U . As proved in [LMO, Theorem 1.1], there exists an effectively computable absolute constant A with the following property. For any conjugacy class C there exists a prime $p \in \mathbf{Z}$, satisfying the following conditions:

- (i) p is unramified in L ;
- (ii) $\left[\frac{L/\mathbf{Q}}{p} \right] = C$;
- (iii) $p \leq 2d_L^A$.

Fix such p , and prove that there exists $\lambda \in \mathbf{Z}$ such that

$$(9) \quad |P(\lambda)|_p < |\delta|_p^2.$$

When $p|\delta$, we take λ as a root of $P(x) \bmod \Delta$. So let p not divide δ . By Lemma 1, $p \leq 2D^A$, whence $P(x)$ has a root $\lambda \bmod p$, and we get (9) since $|\delta|_p = 1$.

The same argument as above shows that the polynomial $P(x)$ has a root in \mathbf{Q}_p . Therefore $S \cap U \neq \emptyset$ by Lemma 2. This concludes the proof.

Proof of Theorem 2. Let C be a conjugacy class of G . Denote

$$T(C) = \left\{ p : \left[\frac{L/\mathbf{Q}}{p} \right] = C \right\}.$$

Applying Chebotarev density theorem in the form given in [LO] or [Schu3], we obtain $d(T(C)) = \frac{|C|}{|G|}$. Now by Lemma 2

$$d(S) = \sum_{C \cap U \neq \emptyset} d(T(C)) = \sum_{C \cap U \neq \emptyset} \frac{|C|}{|G|} = \frac{|\bigcup_{\sigma \in G} \sigma^{-1}U\sigma|}{|G|}.$$

The proof is complete.

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