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THE EXTENSIONS OF THE FERENC MÓRICZ THEOREMS

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ABSTRACT. We study the integrability of the r times differentiated complex trigonometric series using modified trigonometric sums and obtain a new necessary and sufficient condition for L^1-convergence of the rth derivative of the Fourier series. Some results of F. Móricz are deduced as corollaries.

1. INTRODUCTION

A complex null sequence $\{c_k\}$ satisfying $\sum_{k=1}^{\infty} |\Delta(c_k - c_{-k})| \log k < \infty$ is called weakly even and is denoted by $\{c_k\} \in W$. If a null sequence $\{c_k\}$ satisfies $\sum_{k=1}^{\infty} |\Delta(c_k - c_{-k})| k^r \log k < \infty$ for some $r = 0, 1, 2, \ldots$, then we write that $\{c_k\} \in W_r$, where $W_0 = W$. If there exists $\beta > 0$ such that $n^{-\beta} a_n \downarrow 0$, then the sequence $\{a_n\}$ is called a quasi-monotone sequence and is denoted by $a_n \curvearrowleft 0$.

The partial sums of the complex trigonometric series $\sum_{|n| \le \infty} c_n e^{\mathrm{int}}$ will be denoted by $s_n(c,t) = \sum_{|k| \le n} c_k e^{ikt}$, $t \in T = \mathbb{R}/2\pi Z$. If a trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $s_n(c,t) = s_n(f,t) = s_n(f)$.

Let $D_n(t)$ and $\widetilde{D}_n(t)$ denote the Dirichlet and the conjugate Dirichlet kernel respectively. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$ and $E_{-n}(t) = \sum_{k=1}^{n} e^{-ikt}$. Then the rth derivatives $D_n^{(r)}(t)$ and $\widetilde{D}_n^{(r)}(t)$ of $D_n(t)$ and $\widetilde{D}_n(t)$ can be written as

$$D_n^{(r)}(t) = E_n^{(r)}(t) + E_{-n}^{(r)}(t), \tag{1.1}$$

$$i\widetilde{D}_n^{(r)}(t) = E_n^{(r)}(t) - E_{-n}^{(r)}(t), \tag{1.2}$$

where $E_n^{(r)}(t)$ denotes the rth derivative of $E_n(t)$.

Č. V. Stanojević and V. B. Stanojević [5] introduced the following modified complex trigonometric sum:

$$u_n(c,t) = s_n(c,t) - (c_n E_n(t) + c_{-n} E_{-n}(t)).$$

The complex form of the rth derivative of this sum, obtained by Sheng [4], is

$$u_n^{(r)}(c,t) = s_n^{(r)}(c,t) - (c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)). \tag{1.3}$$

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S. Kumari and B. Ram [2] introduced another set of modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(a_j/j)k\cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(a_j/j)k\sin kx.$$

The complex form of the rth derivative of these modified sums is

$$g_n^{(r)}(c,t) = s_n^{(r)}(c,t) + \frac{i}{n+1}[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)]. \tag{1.4}$$

Remark 1. If $|n|^r c_n \to 0$ $(|n| \to \infty)$, then $\|g_n^{(r)} - u_n^{(r)}\|_1 \to 0$ $(n \to \infty)$. Observe that by partial summation, we have

$$E_n^{(r+1)}(t) = -i\sum_{k=0}^{n} E_k^{(r)}(t) + i(n+1)E_n^{(r)}(t)$$

and similarly for $E_{-n}^{(r+1)}(t)$. Also from (1.3) we note that

$$u_{n+1}^{(r)}(c,t) = s_n^{(r)}(c,t) - c_{n+1}E_n^{(r)}(t) - c_{-(n+1)}E_{-n}^{(r)}(t).$$

Hence,

$$\begin{aligned} u_{n+1}^{(r)}(c,t) - g_n^{(r)}(c,t) = {} & -c_{n+1}\frac{1}{n+1}\sum_{k=0}^{n} E_k^{(r)}(t) \\ & - c_{-(n+1)}\frac{1}{n+1}\sum_{k=1}^{n} E_{-k}^{(r)}(t). \end{aligned}$$

If we assume $|n|^r c_n \to 0$ $(|n| \to \infty)$, then by partial summation and the well-known properties of Fejér kernels, it follows that $\|g_n^{(r)} - u_n^{(r)}\|_1 \to 0$ $(n \to \infty)$.

Concerning the L^1-convergence of complex trigonometric series, F. Móricz [3] improved the result of Č. V. Stanojević and V. B. Stanojević [5] by assuming a weaker condition. The aim of this paper is to give sufficient conditions for the integrability of the r-times differentiated trigonometric series using complex trigonometric sums (1.3) and (1.4) and to obtain necessary and sufficient conditions for the L^1-convergence of the rth derivative of the Fourier series. The case $r = 0$ of our theorems yields the results of F. Móricz [3].

2. Results

Let $1 < p \leq 2$ be a real number. Denote by q the conjugate exponent to p, i.e., $1/p + 1/q = 1$, by I_m the dyadic interval $[2^{m-1}, 2^m)$ for $m \geq 1$, and by $\|\cdot\|_1$ the $L^1(-\pi,\pi)$-norm.

We prove the following theorems for the sum (1.3):

Theorem 1. *Let* $\{c_k\} \in W_r$ *and*

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)}\left(\sum_{k\in I_m} |\Delta c_k|^p\right)^{1/p} < \infty, \tag{2.1}$$

for some $1 < p \leq 2$ *and* $r \geq 0$. *Then*

(i) $\lim_{n\to\infty} s_n^{(r)}(c,t) = f^{(r)}(t)$ *for all* $0 < |t| \leq \pi$,

(ii) $f^{(r)}(t) \in L^1(T)$ *and* $\|u_n^{(r)}(c) - f^{(r)}\|_1 = o(1)$ *as* $n \to \infty$,

(iii) $\|s_n^{(r)}(f) - f^{(r)}\|_1 = o(1)$ *as* $n \to \infty$ *if and only if* $|n|^r \hat{f}(n) \log |n| = o(1)$ *as* $|n| \to \infty$.

Theorem 2. *Let* $\{c_k\} \in W_r$ *and*

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k\in I_m} |\Delta(c_k + c_{-k})|^p \right)^{1/p} < \infty, \tag{2.2}$$

for some $1 < p \leq 2$ *and* $r \geq 0$. *Then statements* (i)–(iii) *of Theorem* 1 *hold.*

Taking $r = 0$ in Theorems 1 and 2 we obtain Theorems 3 and 2 of F. Móricz [3] respectively.

Considering the sums (1.4) instead of (1.3) and in view of the preceding Remark in Section 1, statement (ii) in Theorems 1 and 2 can be replaced by:

(ii′) $f^{(r)}(t) \in L^1(T)$ and $\|g_n^{(r)}(c) - f^{(r)}\|_1 \to 0$ $(n \to \infty)$.

Thus we have the following results:

Theorem 3. *Under the hypothesis of Theorem* 1*, statements* (i), (ii′) *and* (iii) *hold.*

Theorem 4. *Under the hypothesis of Theorem* 2*, statements* (i), (ii′) *and* (iii) *hold.*

3. Lemmas

Lemma 1 (Sheng [4])**.** $\|D_n^{(r)}\|_1 = (4/\pi) n^r \log n + O(n^r)$, $n \to \infty$, $r \in \{0, 1, 2, \dots\}$.

Lemma 2 (Sheng [4])**.** $\|\widetilde{D}_n^{(r)}\|_1 = O(1)(n^r \log n)$, $n \to \infty$, $r \in \{0, 1, 2, \dots\}$.

Lemma 3 (Sheng [4])**.** *Let* r *be a non-negative integer and* $x \in [\pi/n, \pi]$*, where* $n \geq 1$. *Then*

$$\begin{aligned} D_n^{(r)}(x) = &\sum_{k=0}^{r-1} \frac{(n+1/2)^k \sin[(n+1/2)x + k\pi/2]}{(\sin x/2)^{r+1-k}} \Phi_k(x) \\ &+ \frac{(n+1/2)^r \sin[(n+1/2)x + r\pi/2]}{2 \sin x/2}, \end{aligned}$$

where each Φ_k *denotes an appropriate bounded function dependent on* r *but independent of* n.

Lemma 4 (Sheng [4])**.** *For each non-negative integer* n,

$$\|c_n E_n^{(r)} + c_{-n} E_{-n}^{(r)}\|_1 = o(1), \qquad n \to \infty,$$

holds if and only if $|n|^r c_n \log |n| = o(1)$, $|n| \to \infty$, *where* $\{c_n\}$ *is a complex sequence.*

Lemma 5. *Let* $\{c_k\}$ *be a sequence of complex numbers. Then for any* $1 < p \leq 2$ *and* $n \geq 1$, $r \geq 0$

$$\frac{1}{n} \int_0^{\pi} \left| \sum_{k=n}^{2n-1} c_k D_k^{(r)}(x) \right| dx \leq A_{pr} n^r \left(\frac{1}{n} \sum_{k=n}^{2n-1} |c_k|^p \right)^{1/p}, \tag{3.1}$$

where A_{pr} *is a constant depending upon* p *and* r.

This lemma is an extension of Lemma 2.3 of Bojanic and Stanojević [1].

Proof. We write

$$\begin{aligned}\frac{1}{n}\int_0^\pi \left|\sum_{k=0}^{2n-1} c_k D_k^{(r)}(x)\right| dx &= \frac{1}{n}\int_0^{\pi/n} \left|\sum_{k=n}^{2n-1} c_k D_k^{(r)}(x)\right| dx \\ &\quad + \frac{1}{n}\int_{\pi/n}^{\pi} \left|\sum_{k=n}^{2n-1} c_k D_k^{(r)}(x)\right| dx \\ &= I_1 + I_2, \quad \text{say.}\end{aligned} \tag{3.2}$$

Since $\|D_k^{(r)}\|_1 = O(k^{r+1})$, for the first integral in (3.2), we have

$$I_1 = O(1)\left\{n^{r-1}\sum_{k=n}^{2n-1}|c_k|\right\},$$

and now by Hölder's inequality, we have

$$I_1 = O(1)\left\{n^r\left(\frac{1}{n}\sum_{k-n}^{2n-1}|c_k|^p\right)^{1/p}\right\}.$$

We now estimate I_2

$$I_2 = \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k D_k^{(r)}(x)\right| dx.$$

From Lemma 3, we have

$$\begin{aligned}I_2 &= \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k\left(\sum_{\lambda=0}^{r-1}\frac{(k+1/2)^\lambda \sin[(k+1/2)x+\lambda\pi/2]}{(\sin x/2)^{r+1-\lambda}}\Phi_\lambda\right)\right| dx \\ &\quad + \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k\frac{(k+1/2)^r\sin[(k+1/2)x+r\pi/2]}{2\sin x/2}\right| dx \\ &\le \Phi_{2n-1}^{(1)}(x) + \Phi_{2n-1}^{(2)}(x),\end{aligned}$$

where

$$\Phi_{2n-1}^{(1)}(x) = \sum_{\lambda=1}^{r}\Phi_{2n-1,\lambda}^{(1)}(x),$$

$$\Phi_{2n-1,\lambda}^{(1)}(x) = \frac{1}{n}\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k\frac{(k+1/2)^\lambda\sin[(k+1/2)x+\lambda\pi/2]}{(\sin x/2)^{r+1-\lambda}}\Phi_\lambda(x)\right| dx.$$

Since Φ_λ are bounded, it can be shown by Hölder's inequality that

$$\begin{aligned}\Phi_{2n-1,\lambda}^{(1)}(x) = O(1)\Bigg\{&\frac{1}{n}\left(\int_{\pi/n}^{\pi}\frac{dx}{(\sin x/2)^{(r+1-\lambda)p}}\right)^{1/p} \\ &\cdot\left(\int_{\pi/n}^{\pi}\left|\sum_{k=n}^{2n-1} c_k(k+1/2)^\lambda[\sin(k+1/2)x+\lambda\pi/2]\right|^q dx\right)^{1/q}\Bigg\}.\end{aligned}$$

Since

$$\int_{\pi/n}^{\pi} \frac{dx}{(\sin x/2)^{(r+1-\lambda)p}} \leq \pi^{(r+1-\lambda)p} \int_{\pi/n}^{\pi} \frac{dx}{x^{(r+1-\lambda)p}} \leq \frac{\pi}{(r+1-\lambda)p-1} n^{(r+1-\lambda)p-1},$$

it follows that

$$\Phi_{2n-1,\lambda}^{(1)}(x) \leq \left(\frac{\pi}{(r+1-\lambda)p-1}\right)^{1/p} n^{r-\lambda-1/p} \cdot \left(\int_0^{\pi} \left|\sum_{k=n}^{2n-1} c_k \left(k+\frac{1}{2}\right)^{\lambda} \sin[(k+1/2)x+\lambda\pi/2]\right|^q dx\right)^{1/q}.$$

Now, by using Hausdorff-Young inequality, we get

$$\left(\frac{1}{\pi}\int_0^{\pi} \left|\sum_{k=n}^{2n-1} c_k (k+1/2)^{\lambda} \sin[(k+1/2)x+\lambda\pi/2]\right|^q dx\right)^{1/q} \leq 2\left(\sum_{k=n}^{2n-1} |c_k(k+1/2)^{\lambda}|^p\right)^{1/p} \leq 2^{1+\lambda} n^{\lambda} \left(\sum_{k=n}^{2n-1} |c_k|^p\right)^{1/p}.$$

Thus,

$$\Phi_{2n-1,\lambda}^{(1)}(x) \leq A_{pr}^{(0)} n^r \left(\frac{1}{n}\sum_{k=n}^{2n-1} |c_k|^p\right)^{1/p}.$$

Therefore

$$\Phi_{2n-1}^{(1)}(x) \leq A_{pr}^{(1)} n^r \left(\frac{1}{n}\sum_{k=n}^{2n-1} |c_k|^p\right)^{1/p}.$$

Similarly,

$$\Phi_{2n-1}^{(2)}(x) \leq A_{pr}^{(2)} n^r \left(\frac{1}{n}\sum_{k=n}^{2n-1} |c_k|^p\right)^{1/p}.$$

Combining the above estimates, we get

$$I_1 + I_2 \leq A_{pr} n^r \left(\frac{1}{n}\sum_{k=n}^{2n-1} |c_k|^p\right)^{1/p}.$$

This completes the proof of Lemma 5.

Lemma 6. *Let r be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $A_{r\varepsilon} > 0$ such that for all $\varepsilon \leq |t| \leq \pi$ and all $n \geq 1$,*

$$|E_n^{(r)}(t)|, |E_{-n}^{(r)}(t)| \leq A_{r\varepsilon} n^r/|t|$$

and

$$|D_n^{(r)}(t)|, |\widetilde{D}_n^{(r)}(t)| \leq 2A_{r\varepsilon} n^r/|t|.$$

Proof. The case $r=0$ is trivial. For $r\geq 1$, we have

$$-i^r E_n^{(r)}(t)=\sum_{k=0}^{n} k^r e^{ikt}=\sum_{k=0}^{n}(\Delta k^r)E_k(t)+(n+1)^r E_n(t),$$

and so

$$|E_n^r(t)|\leq (A_{0\varepsilon}/|t|)\left\{\left(\sum_{k=0}^{n}|\Delta k^r|\right)+(n+1)^r\right\}\leq A_{r\varepsilon}n^r/|t|$$

for some constant $A_{r\varepsilon}$. Since

$$E_{-n}^{(r)}(t)|=(-1)^r E_n^{(r)}(-t),$$

we obtain $|E_{-n}^{(r)}(t)|\leq A_{r\varepsilon}n^r/|t|$. The other two inequalities follow from the equations (1.1) and (1.2).

4. Proofs of the theorems

Proof of Theorem 1. Performing Abel's transformation on the rth derivative of the partial sums of a general trigonometric series, it is easy to see that

$$\begin{aligned} s_n^{(r)}(c,t)=&\sum_{k=0}^{n-1}\Delta c_k D_k^{(r)}(t)+\sum_{k=0}^{n-1}\Delta(c_{-k}-c_k)E_{-k}^{(r)}(t)\\ &+c_n E_n^{(r)}(t)+c_{-n}E_{-n}^{(r)}(t). \end{aligned}$$

By the use of Lemma 6 and Hölder's inequality, we get

$$\begin{aligned} \sum_{k=1}^{\infty}|\Delta c_k D_k^{(r)}(t)|&\leq \lim_{n\to\infty}\frac{A_{r+1}}{|t|}\left(\sum_{k=1}^{n}k^r|\Delta c_k|\right)\\ &=\lim_{m\to\infty}\frac{A_{r+1}}{|t|}\sum_{j=1}^{m}\left(\sum_{k=2^{j-1}}^{2^j-1}k^r|\Delta c_k|\right)\quad \text{for } n=2^m-1\\ &=\lim_{m\to\infty}\frac{A_{r+1}}{|t|}\sum_{j=1}^{m}2^{j(1/q+r)}\left(\sum_{k\in I_j}|\Delta c_k|^p\right)^{1/p}<\infty \end{aligned}$$

and

$$\begin{aligned} \sum_{k=3}^{\infty}|\Delta(c_{-k}-c_k)||E_{-k}^{(r)}(t)|&\leq\frac{A_{r+1}}{|t|}\left\{\sum_{k=3}^{\infty}k^r|\Delta(c_{-k}-c_k)|\right\}\\ &\leq\frac{A_{r+1}}{|t|}\left\{\sum_{k=3}^{\infty}k^r\log k|\Delta(c_{-k}-c_k)|\right\}<\infty, \end{aligned}$$

where A_{r+1} is a suitable constant. Also $\lim_{n\to\infty}\{c_n E_n^{(r)}(t)+c_{-n}E_{-n}^{(r)}(t)\}=0$ for $0<|t|\leq\pi$, as $\{c_k\}$ is a null sequence. These imply that $f^{(r)}(t)=\sum_{k=1}^{\infty}\Delta c_k D_k^{(r)}(t)+\sum_{k=1}^{\infty}\Delta(c_{-k}-c_k)E_{-k}^{(r)}(t)$ exists for $0<|t|\leq\pi$, and thus the proof of (i) is completed.

Furthermore, from the above and (1.3), for $t \neq 0$, we have

$$f^{(r)}(t) - u_n^{(r)}(c,t) = \sum_{k=n}^{\infty} \Delta c_k D_k^{(r)}(t) + \sum_{k=n}^{\infty} \Delta(c_{-k} - c_k) E_{-k}^{(r)}(t).$$

This implies that

$$\|f^{(r)} - u_n^{(r)}(c)\|_1 \leq \left\| \sum_{k=n}^{\infty} \Delta c_k D_k^{(r)} \right\|_1 + \sum_{k=n}^{\infty} |\Delta(c_{-k} - c_k)| \, \|E_{-k}^{(r)}\|_1.$$

Lemma 1, Lemma 2 and Lemma 5 imply that

$$\begin{aligned}\|f^{(r)} - u_n^{(r)}(c)\|_1 \leq A_{pr} \sum_{m=j}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta c_k|^p \right)^{1/p} \\ + O\left(\sum_{k=n}^{\infty} |\Delta(c_{-k} - c_k)| k^r \log k \right) = o(1), \qquad n \to \infty,\end{aligned}$$

by the hypothesis of the theorem; here $j = j(n)$ denotes the integer for which $2^{j-1} \leq n < 2^j$. Since $u_n^{(r)}(c,t)$ is a polynomial, it follows that $f^{(r)} \in L^1(T)$, which proves assertion (ii).

We further notice that

$$\begin{aligned}\|f^{(r)} - s_n^{(r)}(f)\|_1 &= \|f^{(r)} - u_n^{(r)}(c) + u_n^{(r)}(c) - s_n^{(r)}(f)\|_1 \\ &\leq \|f^{(r)} - u_n^{(r)}\|_1 + \|u_n^{(r)}(c) - s_n^{(r)}(f)\|_1 \\ &= \|f^{(r)} - u_n^{(r)}(c)\|_1 + \|\hat{f}(n) E_n^{(r)} + \hat{f}(-n) E_{-n}^{(r)}\|_1\end{aligned}$$

and

$$\begin{aligned}\|\hat{f}(n) E_n^{(r)} + \hat{f}(-n) E_{-n}^{(r)}\|_1 &= \|u_n^{(r)}(c) - s_n^{(r)}(f)\|_1 \\ &\leq \|f^{(r)} - s_n^{(r)}(f)\|_1 + \|f^{(r)} - u_n^{(r)}(c)\|_1.\end{aligned}$$

Since $\|f^{(r)} - u_n^{(r)}(c)\|_1 = o(1)$, $n \to \infty$, by (ii), and $\|\hat{f}(n) E_n^{(r)} + \hat{f}(-n) E_{-n}^{(r)})\|_1 = o(1)$, $n \to \infty$, if and only if $n^r \hat{f}(n) \log n = o(1)$, $|n| \to \infty$, by Lemma 4, then assertion (iii) follows.

Proof of Theorem 2. As before, an application of Abel's transformation yields

$$\begin{aligned}s_n^{(r)}(c,t) = \sum_{k=0}^{n-1} \Delta c_k D_k^{(r)}(t) + \sum_{k=0}^{n-1} \Delta(c_{-k} - c_k) E_{-k}^{(r)}(t) \\ + c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t).\end{aligned}$$

Then making use of (1.1) and (1.2), we get

$$\begin{aligned}s_n^{(r)}(c,t) = \frac{1}{2} \sum_{k=0}^{n-1} \Delta(c_k + c_{-k}) D_k^{(r)}(t) + i \sum_{k=0}^{n-1} \Delta(c_{-k} - c_k) \widetilde{D}_k^{(r)}(t) \\ + c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t).\end{aligned}$$

The rest of the arguments are similar to the proof of Theorem 1, and therefore we omit them.

Proofs of Theorems 3 *and* 4. We observe that under the assumptions of Theorems 1 and 2, $\{c_k\} \in W_r$ and (2.1), respectively (2.2). Thus $\sum_{|k|=n}^{\infty} |k|^r |\Delta c_k| \to 0$ ($n \to \infty$), which together with $c_n \to 0$ ($|n| \to \infty$) implies that $|n|^r c_n \to 0$ ($|n| \to \infty$). Hence, by Remark of Section 1, Theorems 3 and 4 follow.

5. Conclusions

1. A sequence $\{c_k\}$ of complex numbers is said to belong to the class $S^*_{p\alpha r}$ of Sheng [4] if

$$\{c_k\} \in W_r, \tag{5.1}$$

(5.2) there exists a sequence $\{A_k\}$ of positive numbers such that

$$A_k \curvearrowleft 0 \quad \text{and} \quad \sum_{k=1}^{\infty} k^\alpha A_k < \infty \quad \text{for some } \alpha \geq 0$$

and

$$\frac{1}{n^{p(\alpha-r)+1}} \sum_{k=1}^{n} \frac{|\Delta c_k|^p}{A_k^p} = O(1) \quad \text{for some } \alpha \geq 0,\ 1 < p \leq 2. \tag{5.3}$$

Conditions (5.2) and (5.3) imply (2.1). In fact, by (5.3) and quasi-monotonicity of $\{A_k\}$,

$$\left(\sum_{k \in I_m} |\Delta c_k|^p\right)^{1/p} \leq K 2^{m/p} 2^{m(\alpha-r)} A_{2^{m-1}}$$

with an absolute constant $K > 0$. Hence,

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta c_k|^p\right)^{1/p} &\leq K \sum_{m=1}^{\infty} 2^{m/p+m/q} 2^{m\alpha} A_{2^{m-1}} \\ &= 2^{1+\alpha} K \sum_{m=8}^{\infty} 2^m 2^{m\alpha} A_{2^m} < \infty \quad \text{due to (5.2).}\end{aligned}$$

Therefore Theorem 1 implies the following:

Theorem A (Sheng [4]). *Let* $\{c_k\} \in S^*_{p\alpha r}$, $\alpha \geq 0$ *and* $r \in \{0, 1, 2, \ldots, [\alpha]\}$. *Then, for* $t \neq 0$

(i) $\lim_{n\to\infty} s_n^{(r)}(c, t) = f^{(r)}(t)$,

(ii) $f^{(r)}(t) \in L^1(T)$

(iii) $\|s_n^{(r)}(f) - f^{(r)}\|_1 = o(1)$ *as* $n \to \infty$ *if and only if* $|n|^r \hat{f}(n) \log |n| = o(1)$ *as* $|n| \to \infty$.

2. The following examples show that Theorems 1 and 2 are not comparable to one another.

Example 1. Let a null sequence $\{c_k\}$ be defined by $\Delta c_k = 1/2^{mr} m^3$ for $k = 2^m$, $\Delta c_k = -1/2^{mr} m^3$ for $k = -2^m$ with $m \geq 0$, and let $\Delta c_k = 0$ otherwise. Then, $\sum_{k=1}^{\infty} |\Delta(c_k - c_{-k})| k^r \log k < \infty$ and (2.2) is satisfied, but condition (2.1) is not satisfied. Thus, only Theorem 2 applies in this case.

Example 2. Let a null sequence $\{c_k\}$ be defined by $\Delta c_k = 1/k^{r+2}$ for $k \geq 1$, $\Delta c_k = 1/2^{mr}m^3$ for $k = -2^m$ with $m \geq 0$ and $\Delta c_k = 0$ otherwise. Then, $\sum_{k=1}^{\infty} |\Delta(c_k - c_{-k})|k^r \log k < \infty$ and (2.1) is satisfied, but condition (2.2) is not satisfied. Thus, only Theorem 1 applies in this case.

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