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MINIMAL PRIME IDEALS IN ENVELOPING ALGEBRAS OF LIE SUPERALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over a field of characteristic zero. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. We show that when $\mathfrak{g} = b(n)$, then $U(\mathfrak{g})$ is not semiprime, but it has a unique minimal prime ideal; it follows then that when $\mathfrak{g}$ is classically simple, $U(\mathfrak{g})$ has a unique minimal prime ideal. We further show that when $\mathfrak{g}$ is a finite dimensional nilpotent Lie superalgebra, then $U(\mathfrak{g})$ has a unique minimal prime ideal.

Throughout let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over a field k of characteristic zero (see [S] or [K] for general definitions). Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. When $\mathfrak{g}$ is a Lie algebra $U(\mathfrak{g})$ is a domain, but when $\mathfrak{g}$ is a Lie superalgebra $U(\mathfrak{g})$ need not be semiprime. In this paper we will present some classes of Lie superalgebras for which we can prove that $U(\mathfrak{g})$ has a unique minimal prime ideal; we begin by noting that we know of no example of a finite dimensional Lie superalgebra $\mathfrak{g}$ for which $U(\mathfrak{g})$ does *not* have a unique minimal prime ideal.

A. Bell gave a sufficient condition [Be, Theorem 1.5] for U to be a prime ring (so that 0 is the unique minimal prime ideal). Let $([y_i, y_j])$ be the $m \times m$ matrix of brackets of basis elements of $\mathfrak{g}_1$, and let $d(\mathfrak{g})$ be the determinant of that matrix, considered as a matrix with elements in the symmetric algebra $S(\mathfrak{g}_1)$. If $d(\mathfrak{g}) \neq 0$, then U is prime. Using this criterion, Bell showed [Be, Corollary 3.6] that all the classically simple, finite dimensional Lie superalgebras $\mathfrak{g}$ (except for one class $b(n)$) (called $P(n-1)$ in the notation of [K]) have $d(\mathfrak{g}) \neq 0$, and hence U is prime. Since $d(b(n)) = 0$, Bell's criterion could not determine whether $U(b(n))$ was prime. Behr [B] had shown $U(b(2))$ was not semiprime, but $b(n)$ is classically simple only for $n \geq 3$. Bell [Be, Section 1.6] had further shown that when $\mathfrak{g}_0$ is supercentral in $\mathfrak{g}$, then U has a unique minimal prime ideal, and he raised the question of whether U *always* has a unique minimal prime ideal. Here, in Section 2, we will show that $U(b(n))$ is not semiprime for all n, but that $U(b(n))$ always has a unique minimal prime ideal. Hence the enveloping algebra $U(\mathfrak{g})$ of any classically simple Lie superalgebra $\mathfrak{g}$ has a unique minimal prime ideal. This result has been used by Letzter and Musson in [LM]. Wilson [W] has shown that $W(n)$, a class of simple Lie superalgebras of Cartan type, has a prime enveloping algebra when n is even and $n \geq 4$.

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In Section 3 we consider a second class of Lie superalgebras, the nilpotent Lie superalgebras. We show that the enveloping algebra of a nilpotent Lie superalgebra always has a unique minimal prime ideal.

Both proofs make use of the fact, noted by Behr [B, Corollary, p. 21] that U always has an Artinian quotient ring Q, obtained by inverting the nonzero elements of $U_0 = U(\mathfrak{g}_0)$. In Section 1 we note the fact that this quotient ring will always be a quasi-Frobenius ring; we will use this fact in Section 2 in the proof that $U(b(n))$ has a unique minimal prime ideal. This result is an easy consequence of results of Stafford and Zhang [SZ] on Auslander-Gorenstein rings. When $\mathfrak{g}$ is a Lie algebra, its enveloping algebra has many nice homological properties, properties not holding for general Noetherian rings; these properties include its finite global dimension, and the even stronger Auslander-regular property (which makes its homological properties more like those of a commutative regular ring). As was noted by Behr [B, Proposition 5], the enveloping algebras of Lie superalgebras need not have finite global dimension (e.g. only one class of classically simple Lie superalgebras $osp(1,n)$ has an enveloping algebra with finite global dimension). However, as noted in [KKS], the enveloping algebra of a Lie superalgebra always has finite injective dimension, and here we note the fact that it has the stronger property of being an Auslander-Gorenstein ring.

1. Homological properties of $U(\mathfrak{g})$

In this section we will note some homological facts about U, which we will use to show that Q is a quasi-Frobenius ring. This result will be used in Section 2 to show that $U(b(n))$ has a unique minimal prime ideal.

1.1. Let $\{x_1, \dots, x_n\}$ be a basis for $\mathfrak{g}_0$, and let $\{y_1, \dots, y_m\}$ be a basis for $\mathfrak{g}_1$. We shall make frequent use of the fact that U has a PBW basis (see e.g. [B, Section 2]); namely, any element of U can be written uniquely as a linear combination of the elements of the form $x_1^{e_1} \cdots x_n^{e_n} y_1^{f_1} \cdots y_m^{f_m}$, for nonnegative integers e_i, and $f_j = 0$ or 1. As noted in Behr [B], when U is filtered by taking the x's and y's to have degree one, the associated graded ring $Gr(U) = \Lambda(y_1, \dots, y_m)[x_1, \dots, x_n]$ is a polynomial ring in n indeterminates over the exterior algebra $\Lambda(y_1, \dots, y_m)$. It then follows [KKS, Proposition 2.3] that U has finite (right and left) injective dimension, $\operatorname{injdim}(U) = \dim(\mathfrak{g}_0)$, the vector space dimension of $\mathfrak{g}_0$. We will use the homological properties of $Gr(U)$ to deduce further homological properties of U.

Recall that a ring R is called *Auslander-Gorenstein* if R is a Noetherian ring of finite (right and left) injective dimension with the additional property that for every finitely generated R-module M and every submodule $N \subseteq \operatorname{Ext}^j_R(M,R)$, one has $\operatorname{Ext}^i_R(N,R) = 0$ for all $i < j$. Let $j(M) = \min\{j : \operatorname{Ext}^j_R(M,R) \neq 0\}$; then R is called *(GKdim) Cohen-Macaulay* provided that $\operatorname{GKdim}(R) < \infty$, and $j(M) + \operatorname{GKdim}(M) = \operatorname{GKdim}(R)$ holds for every finitely generated R-module M.

Using a result of Stafford-Zhang, it follows easily that U is an Auslander-Gorenstein, Cohen-Macaulay ring.

Theorem 1.2. *The enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra $\mathfrak{g}$ is an Auslander-Gorenstein, (GKdim) Cohen-Macaulay ring.*

Proof. Filtering U as in 1.1, the associated graded ring $Gr(U)$ is a Noetherian PI ring of finite injective dimension, and the result follows from [SZ, Corollary 4.5]. ☐

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The result below is a consequence of properties that Levasseur [L, Proposition 5.9] has shown hold for any Auslander-Gorenstein, Cohen-Macaulay k-algebra (see also [MR, Proposition 8.3.11]).

Proposition 1.3. *Let U be the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra $\mathfrak{g}$. Then:*

(1) *GKdim is exact and finitely partitive for finitely generated U-modules.*
(2) *U is homogeneous for GKdim.*

It now follows by an argument outlined in Stafford-Zhang [SZ, remarks following proof of Proposition 5.2], and the results of Levasseur [L, Proposition 5.9] that any Auslander-Gorenstein, Cohen-Macaulay k-algebra has a quotient ring that is a quasi-Frobenius ring. We include the argument for completeness.

Theorem 1.4. *An Auslander-Gorenstein, Cohen-Macaulay k-algebra A has a (right and left) quotient ring Q that is a quasi-Frobenius ring.*

Proof. Levasseur [L, Proposition 5.9(ii)] has shown that A has an Artinian quotient ring Q. To show that Q is self-injective it suffices to show that $\operatorname{Ext}^i_Q(S,Q)=0$ for all simple right Q-modules S and all $i\geq 1$. Since Q is Artinian, any simple Q-module will occur as a summand of Q/M, where M is a maximal ideal of Q. Expansion and contraction are 1-1 correspondences between the prime ideals of Q and the prime ideals of A that do not intersect the set of regular elements of A (see e.g. [MR, Proposition 2.1.16, page 47]), so we may assume that $M=PA$, for P a prime ideal of A. By [SZ, Lemma 3.3] $\operatorname{Ext}^i_Q(Q/PQ,Q)\cong Q\otimes_A \operatorname{Ext}^i_A(A/P,A)$. Let $i\geq 1$ and $q\otimes a\in Q\otimes_A \operatorname{Ext}^i_A(A/P,A)$. Let $C=Aa$ be the cyclic left A-submodule of $\operatorname{Ext}^i_A(A/P,A)$ generated by a. Since A is Auslander-Gorenstein, we have $\operatorname{Ext}^j_A(C,A)=0$ for all $j<i$. It follows that $j(C)\geq 1$, and since A is Cohen Macaulay, we have $\operatorname{GKdim}(C)<\operatorname{GKdim}(A)$. Let $C=A/I$ for I a left ideal of A. Since GKdim is an exact dimension function on finitely generated A-modules, A is homogeneous for GKdim, and $N(A)$ is (weakly) ideal invariant under GKdim (see e.g. [MR, Corollary 8.3.16]), then by [MR, Proposition 6.8.14(ii) and Theorem 6.8.15] there is a regular element $c\in I$, and hence $q\otimes a=qc^{-1}\otimes ca=0$. It follows that $Q\otimes_A \operatorname{Ext}^i_A(A/P,A)=0$, and hence $\operatorname{Ext}^i_Q(S,Q)=0$. □

Corollary 1.5. *The quotient ring Q of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra $\mathfrak{g}$ is a quasi-Frobenius ring.*

2. Minimal prime ideals in $U(b(n))$

In this section we show that $U(b(n))$ is never semiprime, but has a unique minimal prime ideal for all n.

2.1. Throughout this section let $\mathfrak{g}=b(n)=\left\{\begin{bmatrix} A & 0\\ 0 & -A^t\end{bmatrix}\right\}\oplus\left\{\begin{bmatrix} 0 & B\\ C & 0\end{bmatrix}\right\}=\mathfrak{g}_0\oplus\mathfrak{g}_1$, where A is an $n\times n$ matrix of trace zero, B is an $n\times n$ symmetric matrix, and C is an $n\times n$ skew-symmetric matrix, with bracket defined on homogeneous elements by $[x,y]=xy+(-1)^{ij+1}yx$ when $x\in\mathfrak{g}_i$ and $y\in\mathfrak{g}_j$. For $n\geq 3$ it has been shown that $b(n)$ is classically simple. Basis elements of $\mathfrak{g}_1$ are of two types:

(Type I): Let $\{y_1,y_2,\ldots,y_r\}$ be a basis for $\left\{\begin{bmatrix} 0 & B\\ 0 & 0\end{bmatrix}\right\}$. Note that $r=\frac{n(n+1)}{2}$. The k-space spanned by the type I basis elements is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$.

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(Type II): Let $\{z_1, z_2, \dots, z_s\}$ be a basis for $\left\{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right\}$. Note that $s = \frac{n(n-1)}{2}$. The k-space spanned by the type II basis elements is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$

The fact that there are more basis elements of type I than of type II will be used in our argument. In choosing the PBW basis for $U = U(\mathfrak{g})$ we will regard the x's $< y$'s, and y's $< z$'s, where the x's are basis elements of $\mathfrak{g}_0$.

2.2. We next collect some facts about products in U of type I and type II elements:

(1) Note that $2y_i^2 = y_i y_i + y_i y_i = [y_i, y_i] = 0$. Similarly $z_i^2 = 0$.
(2) Also $y_i y_j + y_j y_i = [y_i, y_j] = 0$. Hence the y's anti-commute (and similarly the z's anti-commute).
(3) If $Y = y_{i_1} y_{i_2} \cdots y_{i_q}$ is a monomial in U of elements y_{i_n} of type I, we call q the length of Y, and we write $\ell(Y) = q$. We similarly talk about the length of a monomial of x's or of z's.
(4) Thus any product of y's of length greater than r is zero (and any product of z's of length greater than s is zero).

Lemma 2.3. *In $U(\mathfrak{g})$:*

(a) *If $Y = y_{i_1} y_{i_2} \cdots y_{i_q}$ is a product of type I elements of length q, and if $u \in U_0 = U(\mathfrak{g}_0)$, then $Yu = \Sigma u_\alpha Y_\alpha$, where $u_\alpha \in U_0$, $\ell(u_\alpha) \leq \ell(u)$, and Y_α is a product of y's with $\ell(Y_\alpha) = \ell(Y) = q$ for all α. If $Y = y_1 \cdots y_r$ is the product of all the type I basis elements, then $Yu = u'Y$ for some $u' \in U_0$. Similarly for $Z = z_{j_1} z_{j_2} \cdots z_{j_p}$.*

(b) *If Y is a product of type I basis elements and Z is a product of type II basis elements, then $ZY = \Sigma u_\alpha Y_\alpha Z_\alpha$ where $u_\alpha \in U_0$, Z_α is a product of z's, and $\ell(Y_\alpha) - \ell(Z_\alpha) = \ell(Y) - \ell(Z)$ for all α.*

Proof. For (a) note that in $U(\mathfrak{g})$ if $x \in \mathfrak{g}_0$, and if $y \in \mathfrak{g}_1$ is of type I, then $xy - yx = [x, y] = \Sigma a_i y_i \in \mathfrak{g}_1$. Then part (a) follows by induction. (The result for the z's is similar.)

For (b) note that if y, z are of types I and II (respectively) in $\mathfrak{g}_1$, then in $U(\mathfrak{g})$ we have $yz + zy = [y, z] \in \mathfrak{g}_0$. Again induction gives the result. □

Behr [B] showed that $U(b(2))$ is not semiprime ($b(2)$ is not simple). We next show that all $U(b(n))$ are not semiprime. In fact it follows from the proof of the following proposition that if $Y = y_1 \cdots y_r$ is the product of all y's of type I, then $(YU)^2 = 0$, and hence $U(b(n))$ always has the nilpotent ideal YU.

Proposition 2.4. *The ideal generated by all products of at least $s+1$ type I elements is in the nil radical, $\langle\{y_{i_1} y_{i_2} \cdots y_{i_{s+1}}\}\rangle \subseteq N(U)$.*

Proof. Let Y be such a product of at least $s+1$ type I elements. In YUY elements are sums of terms of the form YuY_1Z_1Y where $u \in U_0$, Y_1 is a product of type I elements, and Z_1 is a product of type II elements. Note that by Lemma 2.3(a), YuY_1Z_1Y is a sum of terms of the form $u'Y'Y_1Z_1Y$ for $u' \in U_0$ and Y' a product of type I elements with $\ell(Y') = \ell(Y) \geq s+1$. Next note that by Lemma 2.3(b) Z_1Y is a sum of terms of the form $u_\alpha Y_\alpha Z_\alpha$, where $u_\alpha \in U_0$ and $\ell(Y_\alpha) - \ell(Z_\alpha) = \ell(Y) - \ell(Z_1) \geq s + 1 - \ell(Z_1) \geq 1$, and hence $\ell(Y_\alpha) \geq 1$. Thus elements of the form $u'Y'Y_1Z_1Y$ are sums of elements of the form $u'Y'Y_1u_\alpha Y_\alpha Z_\alpha$, and by Lemma 2.3(a) these elements are sums of elements of the form $u''Y''Y_1'Y_\alpha Z_\alpha$, where $\ell(Y'') = \ell(Y') \geq s+1$. Hence elements in YUY are of the form $\Sigma u_\beta Y_\beta Z_\beta$, where $\ell(Y_\beta) \geq$

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$s+2$. It follows by induction that elements in $Y(UY)^k$ are of the form $\Sigma u_\beta Y_\beta Z_\beta$, where $\ell(Y_\beta) \geq s+1+k$. Consequently, $Y(UY)^{r-s}=0$, and $Y \in N(U)$. □

We can now prove the main result of this section. Throughout the following we will use Y_i to denote a monomial in the y's and Z_j to denote a monomial in the z's.

Theorem 2.5. *The enveloping algebra $U(b(n))$ has a unique minimal prime ideal.*

Proof. By Behr [B, Corollary, page 21] the elements of U_0^* form a two-sided Ore set of regular elements of U, and localizing at U_0^* yields an Artinian two-sided classical quotient ring Q. Let Q_0 denote the division ring of quotients of U_0; then the set of monomials in the basis elements of $\mathfrak{g}_1$ form a basis for Q over Q_0. As noted in Corollary 1.5, Q is quasi-Frobenius. By [MR, Proposition 2.1.16, page 47] expansion and contraction are 1-1 correspondences between the prime ideals of Q and the prime ideals of U that do not intersect U_0^*; furthermore, $N(Q)=N(U)Q=QN(U)$ and $N(U)=N(Q)\cap U$. It is sufficient to show that $N(U)$ is prime, for since $N(U)$ is nilpotent, it would then be the unique minimal prime ideal of U; hence it is sufficient to show that $N(Q)$ is prime. Since Q is Artinian, this is the case exactly when $Q/N(Q)$ is simple Artinian, or equivalently, when Q has exactly one simple module up to isomorphism. Since Q is quasi-Frobenius, every simple right Q-module is isomorphic to a minimal right ideal of Q (see, for example, [AF, Corollary 31.4, page 340]); hence it will be sufficient to show that all minimal right ideals of Q are isomorphic.

Let Y be the product in U of all the type I basis elements of $\mathfrak{g}_1$ and let Z be the product of all the type II basis elements of $\mathfrak{g}_1$. By Lemma 2.3(a) if $u \in U_0$, then $Yu = u'Y$ for some $u' \in U_0$, and thus $Yu^{-1} = (u')^{-1}Y$; it follows that if $q \in Q_0$, then $Yq = q'Y$ for some $q' \in Q_0$. The same property holds for Z. Consider the right ideal YZQ of Q; by 2.2.2 the right ideal YZQ does not depend on the order of the factors of Y and Z.

Claim 1. The right ideal YZQ is a minimal right ideal of Q.

Proof of Claim. Take an arbitrary nonzero element YZq of YZQ. We can write $Zq = \sum_{i,j} t_{i,j} Y_i Z_j$ where each $t_{i,j}$ is a nonzero element of Q_0, and thus have $YZq = \sum_{i,j} Y t_{i,j} Y_i Z_j = \sum_{i,j} t'_{i,j} Y Y_i Z_j$ where $t'_{i,j} \in Q_0$ for each i,j. Since $YY_i = 0$ if $\ell(Y_i) > 0$, we can write YZq in the form $YZq = Y\sum_j s_j Z_j$ where $s_j \in Q_0^*$ for all j. Let j_0 be such that Z_{j_0} has no proper subproduct that occurs in the sum $\sum_j s_j Z_j$, and let Z'_{j_0} be the product of all z's not occurring as a factor of Z_{j_0}. Since the z's anti-commute, and since the square of any z is zero, we have $Z_j Z'_{j_0} = 0$ for $j \neq j_0$. Consequently, $YZqZ'_{j_0} = Y\sum_j s_j Z_j Z'_{j_0} = Ys_{j_0} Z_{j_0} Z'_{j_0} = \pm Y s_{j_0} Z = YZs'_{j_0}$ for some $s'_{j_0} \in Q_0^*$. Thus $YZ \in YZqQ$ for every q, and the right ideal YZQ is minimal.

Claim 2. If I is a minimal right ideal of Q, then I is isomorphic to YZQ.

Proof of Claim. Take $I = qQ$. We can write $q = \sum_{i,j} t_{i,j} Y_{i,j} Z_j$ where for each i,j we have that $t_{i,j}$ is a nonzero element of Q_0. As in the proof of Claim 1, take j_0 so that Z_{j_0} has no proper subproduct that occurs in the sum, and let Z'_{j_0} be the product of the z's not occurring as a factor of Z_{j_0}. Multiplying by Z'_{j_0} yields $q' = qZ'_{j_0} = \sum_i t_{i,j_0} Y_i Z = \sum_i \pm s_i Y_{i,j_0} Z$ where $s_i = \pm t_{i,j_0}$ for each i. Observe that $q' \neq 0$ since the terms in the sum are distinct basis elements for Q over Q_0; since I is simple, we have $I = q'Q$. Take a common denominator s for the s_i's, and write $s_i = s^{-1}u_i$ for $s, u_i \in U_0$. Then $I \cong sI = q''Q$ where $q'' = sq' = \sum_i u_i Y_i Z$. By Lemma 2.3(a) we can write $q'' = \sum_j Y_j v_j Z = (\sum_j Y_j v_j)Z$ where each v_j is a

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nonzero element of U_0. Again take j_0 so that no proper subproduct of Y_{j_0} occurs in the sum, and let Y'_{j_0} be the product of all the y's not occurring as a factor of Y_{j_0}. Multiplying on the left by Y'_{j_0} yields $Y'_{j_0}q'' = Y'_{j_0}Y_{j_0}v_{j_0}Z = \pm Yv_{j_0}Z = YZv'_{j_0}$ for some $v'_{j_0} \in U_0^*$. Observing that $Y'_{j_0}q'' \neq 0$, we have that $Y'_{j_0}q''Q = YZQ$. Hence $I \cong sI \cong Y'_{j_0}sI = YZQ$ as desired.

By the remarks at the beginning of the proof, the result now follows from Claims 1 and 2. □

3. Minimal prime ideals in enveloping algebras of nilpotent Lie superalgebras

In this section we show that if $\mathfrak{g}$ is nilpotent, then $U(\mathfrak{g})$ has a unique minimal prime ideal.

3.1. The *center* of $\mathfrak{g}$, $Z(\mathfrak{g})$, is defined by $Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x,y] = 0 \text{ for all } y \in \mathfrak{g}\}$; the higher centers are defined by $Z_i(\mathfrak{g})/Z_{i-1}(\mathfrak{g}) = Z(\mathfrak{g}/Z_{i-1}(\mathfrak{g}))$. Recall that $\mathfrak{g}$ is called *nilpotent* if $Z_\ell(\mathfrak{g}) = \mathfrak{g}$ for some ℓ. Let $\{x_1, x_2, \dots, x_n\}$ be a basis for $\mathfrak{g}_0$, and let $\{y_1, y_2, \dots, y_m\}$ be a basis for $\mathfrak{g}_1$. In the enveloping algebra $U = U(\mathfrak{g})$ call a monomial $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}y_1^{f_1}y_2^{f_2}\cdots y_m^{f_m}$ in the PBW basis *even* if $f_1 + f_2 + \cdots + f_m$ is even; recall that each f_i is either 0 or 1 so that a monomial is even if the number of 'y-factors' is even. Call all other monomials *odd*. If $a \in U$, group even and odd monomials together and write $a = a_0 + a_1$; this decomposition of elements yields a Z_2-grading of U. Let $\sigma : U \longrightarrow U$ be defined by $\sigma(a) = \sigma(a_0 + a_1) = a_0 - a_1$; the map σ is an automorphism of U. Furthermore, an ideal I is graded if and only if $\sigma(I) = I$.

Letzter has noted that the following theorem follows from [Le, Remark 3.11, page 254].

Theorem 3.2. *Let $\mathfrak{g}$ be a solvable Lie superalgebra over an algebraically closed field of characteristic* 0. *If $U(\mathfrak{g})$ does not have a unique minimal prime, then it has exactly two minimal primes P and $\sigma(P)$. Then $P \cap \sigma(P)$ is the unique minimal graded prime ideal and is $N(U)$, the nil radical of U.*

Since U has an Artinian two-sided classical quotient ring Q that is obtained by inverting the nonzero elements of $U_0 = U(\mathfrak{g}_0)$, the monomials in the y's form a basis for Q over $Q_0 = Q(U_0)$. We can extend the grading to Q; the automorphism σ extends to Q by $\sigma(s^{-1}a) = s^{-1}\sigma(a)$ where $s \in U_0^*$ and $a \in U$.

While it is possible for Q to have proper idempotents, this is not the case for U.

Proposition 3.3. *Let $\mathfrak{g}$ be a nilpotent Lie superalgebra with enveloping algebra $U = U(\mathfrak{g})$. Then* 0 *and* 1 *are the only idempotents of U.*

Proof. Induct on the dimension of $\mathfrak{g}$. The result is true if $\mathfrak{g}$ has dimension 1. Suppose the result is true for all nilpotent Lie superalgebras with dimension less than that of $\mathfrak{g}$. Consider $Z(\mathfrak{g})$, which is nonzero since $\mathfrak{g}$ is nilpotent. Since $Z(\mathfrak{g})$ is a graded Lie ideal, it must contain a homogeneous element.

Case 1. Suppose that $Z(\mathfrak{g})$ contains an odd homogeneous element y. Let e be a nonzero idempotent of U. Then $e + yU$ is a nonzero element of U/yU since yU is nilpotent. Thus if e is a proper nonzero idempotent of U, both e and $1 - e$ are nonzero idempotents of U/yU. This cannot be for $U/yU \cong U(\mathfrak{g}/ky)$ has no proper idempotents by induction.

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Case 2. Suppose that $Z(\mathfrak{g})$ contains an even homogeneous element z and let e be a proper idempotent of U. We may assume that z is a basis element of $\mathfrak{g}$. Suppose that $e \in zU$. From the PBW basis of U we see that we can write $e = z^i u$ where u is not in zU. Then $zu = e = e^2 = z^2u^2$ and we have $u = zu^2 \in zU$, which is a contradiction. Hence $e + zU$ must be a nonzero idempotent of U/zU. Thus $e + zU$ and $(1 - e) + zU$ are proper idempotents of $U/zU \cong U(\mathfrak{g}/kz)$; this is a contradition by induction. □

The following proposition gives consequences of U having two minimal prime ideals. Let $\bar{Q} = Q/N(Q)$ and let $\bar{a} = a + N(Q)$. Since $J(Q) = N(Q) = N(U)Q = QN(U)$ (for example, see [CH, Lemma 9.2, page 125]), we have $\sigma(N(Q)) = \sigma(N(U)Q) = N(U)Q = N(Q)$ and hence $N(Q)$ is σ-stable. Thus σ induces an automorphism $\bar{\sigma}$ on $\bar{Q}$.

Proposition 3.4. *Let $U = U(\mathfrak{g})$ for a nilpotent finite dimensional Lie superalgebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0. If U has two minimal prime ideals, then there exist elements c and s in U such that*

(1) *c is an odd regular element of U,*
(2) *$s \in U_0$,*
(3) *$\bar{c}^2 = \dfrac{\bar{s}^2}{4}$, and*
(4) *$\bar{c}\bar{u}\bar{s} = \bar{s}\bar{u}\bar{c}$ for all $u \in U$. Note that s and c commute modulo $N(Q)$.*

Conversely, given such elements c and s, $\frac{1}{2} + cs^{-1} + N(Q)$ is a central idempotent of $Q/N(Q)$.

Proof. Suppose that U has two minimal prime ideals, P and $\sigma(P)$. Since Q is obtained by classically localizing at the two-sided Ore set $S = U_0^*$ all of whose elements are regular in U, there is a correspondence between the prime ideals of Q and the prime ideals of U that do not intersect S [MR, Proposition 2.1.16, page 47]. Since Q is Artinian, Q has Krull dimension 0 and every prime ideal of Q is the expansion of one of the two minimal prime ideals of U. Since U is Noetherian, neither of the minimal prime ideals P nor $\sigma(P)$ can contain a regular element. Consequently Q also has exactly two minimal prime ideals (which are the only prime ideals of Q), namely PQ and $\sigma(P)Q$. The above correspondence shows that if Q does not have a unique minimal prime, then neither can U.

Consider $\bar{Q} = Q/N(Q) = Q/J(Q)$. Then Q has more than one prime ideal if and only if $\bar{Q}$ does, and since $\bar{Q}$ is semisimple Artinian, this is the case if and only if $\bar{Q}$ has a proper central idempotent. In this case $\bar{Q}$ will have exactly two proper central idempotents corresponding to the prime ideals of Q; call them $\bar{e}$ and $1 - \bar{e}$. Furthermore $\bar{\sigma}(\bar{e}) = 1 - \bar{e}$. Write $\bar{e} = e + J$ where e is an idempotent of Q and write $e = a + b$ where a is even and b is odd. Then $\bar{\sigma}(\bar{e}) = 1 - \bar{e} = 1 - \bar{a} - \bar{b} = \bar{a} - \bar{b}$ and hence $\bar{a} = \frac{1}{2}$. Since $\bar{e}$ is central, so are $\bar{a}$ and $\bar{b}$. Then $\bar{a}^2 + 2\bar{a}\bar{b} + \bar{b}^2 = \bar{a} + \bar{b}$ which implies that $\frac{1}{4} + \bar{b} + \bar{b}^2 = \frac{1}{2} + \bar{b}$ and consequently that $\bar{b}^2 = \frac{1}{4}$. Write $\bar{b} = \bar{c}\bar{s}^{-1}$ where $s \in U_0$ and c is an odd element of U. Then $\bar{c}^2 = \frac{1}{4}\bar{s}^2$. Note that c must be a regular element of U. For any $u \in U$ we have that $\bar{c}\bar{s}^{-1}\bar{u} = \bar{u}\bar{c}\bar{s}^{-1}$ and in particular $\bar{c}$ and $\bar{s}$ commute. Thus we have $\bar{c}\bar{u}\bar{s} = \bar{s}\bar{u}\bar{c}$ for all $u \in U$.

Calculation shows that the converse follows. □

The following lemma appeared in Bell-Musson [BM, Lemma 1.10, page 406].

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Lemma 3.5. *Let $\mathfrak{g}$ be a nilpotent Lie superalgebra with center $Z(\mathfrak{g}) = kz$ where z is even. Let y be a homogeneous element of $Z_2(\mathfrak{g}) - Z(\mathfrak{g})$ where $Z_2(\mathfrak{g})/Z(\mathfrak{g}) = Z(\mathfrak{g}/Z(\mathfrak{g}))$. Then there exists a homogeneous element x of the same parity as y and a graded ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ such that*

(1) $[y, x] = z$,
(2) $\mathfrak{h}$ *is the centralizer of y in $\mathfrak{g}$, and*
(3) $\mathfrak{g} = \mathfrak{h} \oplus kx$.

We need the following simple lemma.

Lemma 3.6. *Let $\mathfrak{g}$ be a Lie superalgebra with $\dim(Z(\mathfrak{g}) \cap \mathfrak{g}_0) \geq 2$; say z and z_1 are linearly independent such elements. If u is a nonzero element of U, then there are only finitely many λ's in k such that $u \in (z + \lambda z_1)U$.*

Proof. Let z_λ denote $z + \lambda z_1$. If $\mu \neq \lambda$, then z_λ is a regular element of $U/z_\mu U \cong U(\mathfrak{g}/kz_\mu)$. Hence if $u = z_\lambda u_1 = z_\mu u_2$ for $u_1, u_2 \in U$, then $u_1 = z_\mu u_3$ for some $u_3 \in U$. Consequently, $u_1 \in z_\mu U$ for every z_μ such that $u \in z_\mu U$ with $\mu \neq \lambda$. Since the high term of u_1 is less than the high term of u in the length lexicographic ordering of monomials, an induction argument yields the result. □

We need the following lemma of Bell and Musson [BM, Lemma 1.5, page 404].

Lemma 3.7. *Let $A = A_0 \oplus A_1$ be a Z_2-graded k-algebra with $\sigma : A \longrightarrow A$ defined by $\sigma(a_0 + a_1) = a_0 - a_1$. Let δ be a σ-derivation on A, and suppose that there is an even $h \in A$ with $\delta(h) = 0$. Let y' be an odd supercentral element of A with $\delta(y') = 1$, and let S be the Ore extension $A[t; \sigma, \delta]$. Set $B = \delta(y'A)$. Then B is a subalgebra of A such that $A = B \oplus y'A$, and so $A/y'A \cong B$. Moreover, $t^2 - h$ is central in S and $S/(t^2 - h)S \cong A/y'A \otimes M_2(k)$.*

We can now prove the main result of this section.

Theorem 3.8. *Let $\mathfrak{g}$ be a nilpotent Lie superalgebra. Then $U(\mathfrak{g})$ has a unique minimal prime ideal.*

Proof. First assume that k is algebraically closed. The proof will be by induction on the dimension of $\mathfrak{g}$. The result is clearly true if $\mathfrak{g}$ has dimension 1. Suppose that the result is true for all nilpotent Lie superalgebras with dimension less than that of $\mathfrak{g}$. Furthermore, assume that U has two minimal prime ideals.

Case 1. There is an odd element $y \in Z(\mathfrak{g})$. Since yU is a nilpotent ideal of U, there is a correspondence between the minimal prime ideals of U and those of U/yU. This contradicts the induction hypothesis, for $U/yU \cong U(\mathfrak{g}/(ky))$.

Case 2. We have $Z(\mathfrak{g}) \subset \mathfrak{g}_0$ and $\dim Z(\mathfrak{g}) \geq 2$. Let c and s be the elements of U given by Proposition 3.4. Take a nonzero $z \in Z(\mathfrak{g})$. It follows from Proposition 3.4 that $\bar{e} = \overline{\frac{1}{2} + cs^{-1}}$ is a central idempotent of $Q(U/zU)/N(Q(U/zU))$ provided s is not an element of zU, for $U/zU \cong U(\mathfrak{g}/kz)$ would contain $s + zU$ as an element of the Ore set $U_0^*(\mathfrak{g}/kz)$. By Lemma 3.6, z can be chosen so that s is not an element of U. It is conceivable that this could be the zero idempotent; that is, $\frac{1}{2} + cs^{-1} + zU \in N(Q(U/zU))$. In this case $s + 2c + zU$ is an element of the nilpotent ideal $N(U/zU)$. Recall that $Q(U(\mathfrak{g}))$ is spanned by all the monomials in the basis elements of $\mathfrak{g}_1$ over the division ring $Q(U_0)$, and that $N(Q)^i = N(U)^i Q$; hence $N(U/zU)$ must be nilpotent of index less than or equal to $p = 2^n$ ($n =$ dim $\mathfrak{g}_1$). Note that $(s + 2c)^p$ and $(s - 2c)^p$ are nonzero since $\bar{e}$ and $1 - \bar{e}$ are proper idempotents of $\bar{Q}$. Again by Lemma 3.6 and the fact that k is infinite, we can

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choose z so that s is not in zU, $(s+2c)^p$ is not in zU, and $(s-2c)^p$ is not in zU. Thus $\bar{e}$ is a proper nonzero central idempotent of $Q(U/zU)/N(Q(U/zU))$. Since $U/zU \cong U(\mathfrak{g}/kz)$, this contradicts the induction hypothesis.

Case 3. We have $Z(\mathfrak{g}) \subset \mathfrak{g}_0$ and $\dim Z(\mathfrak{g}) = 1$. Let x, y be as in Lemma 3.5 and write $\mathfrak{g} = \mathfrak{h} \oplus kx$ where $\mathfrak{h}$ is the centralizer of y. We have three subcases.

Case 3a. Suppose that x and y are both even. Then we have $U(\mathfrak{g}) \cong U(\mathfrak{h})[x;\delta]$. Since $\mathfrak{h}$ is again nilpotent, the induction hypothesis implies that $U(\mathfrak{h})$ has a unique minimal prime ideal. The theorem now follows by [MR, Theorem 1.2.9, page 16 and Proposition 14.2.3, page 495].

Case 3b. Suppose that x and y are both odd and $[y,y]=0$. Let $h=[x,x]/2$. Then we have $U(\mathfrak{g}) = U(\mathfrak{h})[x;\sigma,\delta]/\langle x^2-h\rangle$; localizing at the powers of z yields $U(\mathfrak{g})_{(z^i)} = U(\mathfrak{h})_{(z^i)}[x;\sigma,\delta]/\langle x^2-h\rangle$ where σ and δ have been extended to all of $U(\mathfrak{h})_{(z^i)}$. Hence letting $A = U(\mathfrak{h})_{(z^i)}$ and applying Lemma 3.7 we have $U(\mathfrak{g})_{(z^i)} \cong A/y'A \otimes M_2(k) \cong M_2(A/y'A)$. By induction $U(\mathfrak{h})$ has a unique minimal prime ideal and hence so does $U(\mathfrak{h})_{(z^i)}$. Since y is a supercentral odd element of $\mathfrak{h}$, the ideal $y'A = yz^{-1}A$ is a nilpotent ideal of A, and thus $A/y'A$ has a unique minimal prime. Hence so does $U(\mathfrak{g})_{(z^i)}$. Since z cannot be contained in any minimal prime of $U(\mathfrak{g})$ (see the proof of Proposition 3.4), U has a unique minimal prime ideal.

Case 3c. Suppose that $[y,y] \neq 0$ for all $y \in Z_2(\mathfrak{g}) - Z(\mathfrak{g})$. Let x be as in Lemma 3.5. Since $y \in Z_2(\mathfrak{g}), [y,y] = \lambda z$ for some scalar λ. Suppose that y and y' represent linearly independent cosets in $Z_2(\mathfrak{g})/Z(\mathfrak{g}) = Z_2(\mathfrak{g})/kz$; by Case 3a we may assume that y' is also odd. Let $[y',y'] = \mu z$ and $[y,y'] = \rho z$. Consider $[y+ay', y+ay'] = (\lambda + 2\rho a + \mu a^2)z$. Since k is algebraically closed, there exists $a \in k$ such that $\lambda + 2\rho a + \mu a^2 = 0$. In this case $[y+ay', y+ay'] = 0$, which is a contradiction. Thus we may assume that $\dim Z_2(\mathfrak{g})/Z(\mathfrak{g}) = 1$. Then in Lemma 3.5 we may take $x = y$ and write $\mathfrak{g} = \mathfrak{h} \oplus ky$; recall that $\mathfrak{h}$ is the centralizer of y. If $w \in Z_2(\mathfrak{h})$, then $[w,y]=0$ and $w \in Z_2(\mathfrak{g})$. Since $\dim Z_2(\mathfrak{g})/Z(\mathfrak{g}) = 1$, we have that $w = \alpha z$ for some scalar α. The nilpotence of $\mathfrak{h}$ implies that $\mathfrak{h} = kz$ and $\mathfrak{g} = kz \oplus ky$. Hence $d(\mathfrak{g}) \neq 0$ and it follows from Bell [Be, Theorem 1.5] that $U(\mathfrak{g})$ is prime.

If k is not algebraically closed, let K be the algebraic closure of k and let $\mathcal{B} = \{b_\alpha : \alpha \in \mathcal{A}\}$ be a basis for K over k. Then $\mathcal{B}$ is a central free basis for $U_K(\mathfrak{g})$ over $U_k(\mathfrak{g})$. Since K is algebraically closed, the preceding part of the proof implies that $U_K(\mathfrak{g})$ has a unique minimal prime ideal P with $P = N(U_K(\mathfrak{g}))$. Let $p = P \cap U_k(\mathfrak{g})$. Suppose that $uU_k(\mathfrak{g})v \subseteq p$ for $u, v \in U_k(\mathfrak{g})$ with u not an element of p. Since $ub_\alpha U_k(\mathfrak{g})v \subseteq pb_\alpha \subseteq P$ for all $\alpha \in \mathcal{A}$, we have $uU_K(\mathfrak{g})v \subseteq P$; consequently, $v \in P \cap U_k(\mathfrak{g}) = p$, and p is a prime ideal. Since p is a nilpotent prime ideal, p is the unique minimal prime ideal of $U_k(\mathfrak{g})$. ☐

The following example shows that, while U has no nontrivial idempotents, it is possible that Q contains nontrivial idempotents.

3.9. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the subalgebra of 4×4 matrices, where $\mathfrak{g}_0$ is the vector space spanned by the matrix units $\{x_1 = e_{1,2}, x_2 = e_{1,3}, x_3 = e_{2,3}\}$, $\mathfrak{g}_1$ is the vector space spanned by the matrix units $\{y_1 = e_{1,4}, y_2 = e_{4,3}, y_3 = e_{4,2}\}$, and the bracket is defined on basis elements as $[g_i, g_j] = g_i g_j + (-1)^{ij+1} g_j g_i$. In the notation of [S], $\mathfrak{g}$ is a subalgebra of $pl(3,1)$. The only nonzero brackets involving basis elements are: $[x_1, x_3] = x_2, [x_3, y_3] = -y_2, [y_1, y_2] = x_2$, and $[y_1, y_3] = x_1$. It is easy to check that:

(1) $Z_4 = \mathfrak{g}$, *and so* $\mathfrak{g}$ *is a nilpotent Lie superalgebra.*

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(2) $y_2y_3Uy_2y_3 = 0$, so U *is not semiprime.*
(3) $x_2 \in Z(\mathfrak{g})$.
(4) $e = y_1y_2x_2^{-1}$ *is a nontrivial idempotent of* Q.

Let $\mathfrak{h}$ be the k-span of $\{x_2, x_3, y_1, y_2\}$. Then it can be shown that $\mathfrak{h}$ is a nilpotent Lie superalgebra, $U(\mathfrak{h})$ is prime, and $e = y_1y_2x_2^{-1}$ is a proper idempotent of $Q(U(\mathfrak{h}))$.

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