

MINIMAL PRIME IDEALS IN ENVELOPING ALGEBRAS OF LIE SUPERALGEBRAS

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ABSTRACT. Let \mathfrak{g} be a finite dimensional Lie superalgebra over a field of characteristic zero. Let $U(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} . We show that when $\mathfrak{g} = b(n)$, then $U(\mathfrak{g})$ is not semiprime, but it has a unique minimal prime ideal; it follows then that when \mathfrak{g} is classically simple, $U(\mathfrak{g})$ has a unique minimal prime ideal. We further show that when \mathfrak{g} is a finite dimensional nilpotent Lie superalgebra, then $U(\mathfrak{g})$ has a unique minimal prime ideal.

Throughout let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over a field k of characteristic zero (see [S] or [K] for general definitions). Let $U = U(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} . When \mathfrak{g} is a Lie algebra $U(\mathfrak{g})$ is a domain, but when \mathfrak{g} is a Lie superalgebra $U(\mathfrak{g})$ need not be semiprime. In this paper we will present some classes of Lie superalgebras for which we can prove that $U(\mathfrak{g})$ has a unique minimal prime ideal; we begin by noting that we know of no example of a finite dimensional Lie superalgebra \mathfrak{g} for which $U(\mathfrak{g})$ does *not* have a unique minimal prime ideal.

A. Bell gave a sufficient condition [Be, Theorem 1.5] for U to be a prime ring (so that 0 is the unique minimal prime ideal). Let $([y_i, y_j])$ be the $m \times m$ matrix of brackets of basis elements of \mathfrak{g}_1 , and let $d(\mathfrak{g})$ be the determinant of that matrix, considered as a matrix with elements in the symmetric algebra $S(\mathfrak{g}_1)$. If $d(\mathfrak{g}) \neq 0$, then U is prime. Using this criterion, Bell showed [Be, Corollary 3.6] that all the classically simple, finite dimensional Lie superalgebras \mathfrak{g} (except for one class $b(n)$) (called $P(n-1)$ in the notation of [K]) have $d(\mathfrak{g}) \neq 0$, and hence U is prime. Since $d(b(n)) = 0$, Bell's criterion could not determine whether $U(b(n))$ was prime. Behr [B] had shown $U(b(2))$ was not semiprime, but $b(n)$ is classically simple only for $n \geq 3$. Bell [Be, Section 1.6] had further shown that when \mathfrak{g}_0 is supercentral in \mathfrak{g} , then U has a unique minimal prime ideal, and he raised the question of whether U *always* has a unique minimal prime ideal. Here, in Section 2, we will show that $U(b(n))$ is not semiprime for all n , but that $U(b(n))$ always has a unique minimal prime ideal. Hence the enveloping algebra $U(\mathfrak{g})$ of any classically simple Lie superalgebra \mathfrak{g} has a unique minimal prime ideal. This result has been used by Letzter and Musson in [LM]. Wilson [W] has shown that $W(n)$, a class of simple Lie superalgebras of Cartan type, has a prime enveloping algebra when n is even and $n \geq 4$.

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In Section 3 we consider a second class of Lie superalgebras, the nilpotent Lie superalgebras. We show that the enveloping algebra of a nilpotent Lie superalgebra always has a unique minimal prime ideal.

Both proofs make use of the fact, noted by Behr [B, Corollary, p. 21] that U always has an Artinian quotient ring Q , obtained by inverting the nonzero elements of $U_0 = U(\mathfrak{g}_0)$. In Section 1 we note the fact that this quotient ring will always be a quasi-Frobenius ring; we will use this fact in Section 2 in the proof that $U(b(n))$ has a unique minimal prime ideal. This result is an easy consequence of results of Stafford and Zhang [SZ] on Auslander-Gorenstein rings. When \mathfrak{g} is a Lie algebra, its enveloping algebra has many nice homological properties, properties not holding for general Noetherian rings; these properties include its finite global dimension, and the even stronger Auslander-regular property (which makes its homological properties more like those of a commutative regular ring). As was noted by Behr [B, Proposition 5], the enveloping algebras of Lie superalgebras need not have finite global dimension (e.g. only one class of classically simple Lie superalgebras $osp(1, n)$ has an enveloping algebra with finite global dimension). However, as noted in [KKS], the enveloping algebra of a Lie superalgebra always has finite injective dimension, and here we note the fact that it has the stronger property of being an Auslander-Gorenstein ring.

1. HOMOLOGICAL PROPERTIES OF $U(\mathfrak{g})$

In this section we will note some homological facts about U , which we will use to show that Q is a quasi-Frobenius ring. This result will be used in Section 2 to show that $U(b(n))$ has a unique minimal prime ideal.

1.1. Let $\{x_1, \dots, x_n\}$ be a basis for \mathfrak{g}_0 , and let $\{y_1, \dots, y_m\}$ be a basis for \mathfrak{g}_1 . We shall make frequent use of the fact that U has a PBW basis (see e.g. [B, Section 2]); namely, any element of U can be written uniquely as a linear combination of the elements of the form $x_1^{e_1} \cdots x_n^{e_n} y_1^{f_1} \cdots y_m^{f_m}$, for nonnegative integers e_i , and $f_j = 0$ or 1. As noted in Behr [B], when U is filtered by taking the x 's and y 's to have degree one, the associated graded ring $Gr(U) = \Lambda(y_1, \dots, y_m)[x_1, \dots, x_n]$ is a polynomial ring in n indeterminates over the exterior algebra $\Lambda(y_1, \dots, y_m)$. It then follows [KKS, Proposition 2.3] that U has finite (right and left) injective dimension, $\text{injdim}(U) = \dim(\mathfrak{g}_0)$, the vector space dimension of \mathfrak{g}_0 . We will use the homological properties of $Gr(U)$ to deduce further homological properties of U .

Recall that a ring R is called *Auslander-Gorenstein* if R is a Noetherian ring of finite (right and left) injective dimension with the additional property that for every finitely generated R -module M and every submodule $N \subseteq \text{Ext}_R^j(M, R)$, one has $\text{Ext}_R^i(N, R) = 0$ for all $i < j$. Let $j(M) = \min\{j : \text{Ext}_R^j(M, R) \neq 0\}$; then R is called *(GKdim) Cohen-Macaulay* provided that $\text{GKdim}(R) < \infty$, and $j(M) + \text{GKdim}(M) = \text{GKdim}(R)$ holds for every finitely generated R -module M .

Using a result of Stafford-Zhang, it follows easily that U is an Auslander-Gorenstein, Cohen-Macaulay ring.

Theorem 1.2. *The enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra \mathfrak{g} is an Auslander-Gorenstein, (GKdim) Cohen-Macaulay ring.*

Proof. Filtering U as in 1.1, the associated graded ring $Gr(U)$ is a Noetherian PI ring of finite injective dimension, and the result follows from [SZ, Corollary 4.5]. \square

The result below is a consequence of properties that Levasseur [L, Proposition 5.9] has shown hold for any Auslander-Gorenstein, Cohen-Macaulay k -algebra (see also [MR, Proposition 8.3.11]).

Proposition 1.3. *Let U be the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra \mathfrak{g} . Then:*

- (1) *GKdim is exact and finitely partitive for finitely generated U -modules.*
- (2) *U is homogeneous for GKdim.*

It now follows by an argument outlined in Stafford-Zhang [SZ, remarks following proof of Proposition 5.2], and the results of Levasseur [L, Proposition 5.9] that any Auslander-Gorenstein, Cohen-Macaulay k -algebra has a quotient ring that is a quasi-Frobenius ring. We include the argument for completeness.

Theorem 1.4. *An Auslander-Gorenstein, Cohen-Macaulay k -algebra A has a (right and left) quotient ring Q that is a quasi-Frobenius ring.*

Proof. Levasseur [L, Proposition 5.9(ii)] has shown that A has an Artinian quotient ring Q . To show that Q is self-injective it suffices to show that $\text{Ext}_Q^i(S, Q) = 0$ for all simple right Q -modules S and all $i \geq 1$. Since Q is Artinian, any simple Q -module will occur as a summand of Q/M , where M is a maximal ideal of Q . Expansion and contraction are 1-1 correspondences between the prime ideals of Q and the prime ideals of A that do not intersect the set of regular elements of A (see e.g. [MR, Proposition 2.1.16, page 47]), so we may assume that $M = PA$, for P a prime ideal of A . By [SZ, Lemma 3.3] $\text{Ext}_Q^i(Q/PQ, Q) \cong Q \otimes_A \text{Ext}_A^i(A/P, A)$. Let $i \geq 1$ and $q \otimes a \in Q \otimes_A \text{Ext}_A^i(A/P, A)$. Let $C = Aa$ be the cyclic left A -submodule of $\text{Ext}_A^i(A/P, A)$ generated by a . Since A is Auslander-Gorenstein, we have $\text{Ext}_A^j(C, A) = 0$ for all $j < i$. It follows that $j(C) \geq 1$, and since A is Cohen Macaulay, we have $\text{GKdim}(C) < \text{GKdim}(A)$. Let $C = A/I$ for I a left ideal of A . Since GKdim is an exact dimension function on finitely generated A -modules, A is homogeneous for GKdim , and $N(A)$ is (weakly) ideal invariant under GKdim (see e.g. [MR, Corollary 8.3.16]), then by [MR, Proposition 6.8.14(ii) and Theorem 6.8.15] there is a regular element $c \in I$, and hence $q \otimes a = qc^{-1} \otimes ca = 0$. It follows that $Q \otimes_A \text{Ext}_A^i(A/P, A) = 0$, and hence $\text{Ext}_Q^i(S, Q) = 0$. \square

Corollary 1.5. *The quotient ring Q of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie superalgebra \mathfrak{g} is a quasi-Frobenius ring.*

2. MINIMAL PRIME IDEALS IN $U(b(n))$

In this section we show that $U(b(n))$ is never semiprime, but has a unique minimal prime ideal for all n .

2.1. Throughout this section let $\mathfrak{g} = b(n) = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where A is an $n \times n$ matrix of trace zero, B is an $n \times n$ symmetric matrix, and C is an $n \times n$ skew-symmetric matrix, with bracket defined on homogeneous elements by $[x, y] = xy + (-1)^{ij+1}yx$ when $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$. For $n \geq 3$ it has been shown that $b(n)$ is classically simple. Basis elements of \mathfrak{g}_1 are of two types:

(Type I): Let $\{y_1, y_2, \dots, y_r\}$ be a basis for $\left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right\}$. Note that $r = \frac{n(n+1)}{2}$.

The k -space spanned by the type I basis elements is a \mathfrak{g}_0 -submodule of \mathfrak{g}_1 .

(Type II): Let $\{z_1, z_2, \dots, z_s\}$ be a basis for $\left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \right\}$. Note that $s = \frac{n(n-1)}{2}$.

The k -space spanned by the type II basis elements is a \mathfrak{g}_0 -submodule of \mathfrak{g}_1 .

The fact that there are more basis elements of type I than of type II will be used in our argument. In choosing the PBW basis for $U = U(\mathfrak{g})$ we will regard the x 's $< y$'s, and y 's $< z$'s, where the x 's are basis elements of \mathfrak{g}_0 .

2.2. We next collect some facts about products in U of type I and type II elements:

- (1) Note that $2y_i^2 = y_i y_i + y_i y_i = [y_i, y_i] = 0$. Similarly $z_i^2 = 0$.
- (2) Also $y_i y_j + y_j y_i = [y_i, y_j] = 0$. Hence the y 's anti-commute (and similarly the z 's anti-commute).
- (3) If $Y = y_{i_1} y_{i_2} \cdots y_{i_q}$ is a monomial in U of elements y_{i_n} of type I, we call q the length of Y , and we write $\ell(Y) = q$. We similarly talk about the length of a monomial of x 's or of z 's.
- (4) Thus any product of y 's of length greater than r is zero (and any product of z 's of length greater than s is zero).

Lemma 2.3. *In $U(\mathfrak{g})$:*

(a) *If $Y = y_{i_1} y_{i_2} \cdots y_{i_q}$ is a product of type I elements of length q , and if $u \in U_0 = U(\mathfrak{g}_0)$, then $Yu = \sum u_\alpha Y_\alpha$, where $u_\alpha \in U_0$, $\ell(u_\alpha) \leq \ell(u)$, and Y_α is a product of y 's with $\ell(Y_\alpha) = \ell(Y) = q$ for all α . If $Y = y_1 \cdots y_r$ is the product of all the type I basis elements, then $Yu = u'Y$ for some $u' \in U_0$. Similarly for $Z = z_{j_1} z_{j_2} \cdots z_{j_p}$.*

(b) *If Y is a product of type I basis elements and Z is a product of type II basis elements, then $ZY = \sum u_\alpha Y_\alpha Z_\alpha$ where $u_\alpha \in U_0$, Z_α is a product of z 's, and $\ell(Y_\alpha) - \ell(Z_\alpha) = \ell(Y) - \ell(Z)$ for all α .*

Proof. For (a) note that in $U(\mathfrak{g})$ if $x \in \mathfrak{g}_0$, and if $y \in \mathfrak{g}_1$ is of type I, then $xy - yx = [x, y] = \sum a_i y_i \in \mathfrak{g}_1$. Then part (a) follows by induction. (The result for the z 's is similar.)

For (b) note that if y, z are of types I and II (respectively) in \mathfrak{g}_1 , then in $U(\mathfrak{g})$ we have $yz + zy = [y, z] \in \mathfrak{g}_0$. Again induction gives the result. \square

Behr [B] showed that $U(b(2))$ is not semiprime ($b(2)$ is not simple). We next show that all $U(b(n))$ are not semiprime. In fact it follows from the proof of the following proposition that if $Y = y_1 \cdots y_r$ is the product of all y 's of type I, then $(YU)^2 = 0$, and hence $U(b(n))$ always has the nilpotent ideal YU .

Proposition 2.4. *The ideal generated by all products of at least $s + 1$ type I elements is in the nil radical, $\langle \{y_{i_1} y_{i_2} \cdots y_{i_{s+1}}\} \rangle \subseteq N(U)$.*

Proof. Let Y be such a product of at least $s + 1$ type I elements. In YUY elements are sums of terms of the form YuY_1Z_1Y where $u \in U_0$, Y_1 is a product of type I elements, and Z_1 is a product of type II elements. Note that by Lemma 2.3(a), YuY_1Z_1Y is a sum of terms of the form $u'Y'Y_1Z_1Y$ for $u' \in U_0$ and Y' a product of type I elements with $\ell(Y') = \ell(Y) \geq s + 1$. Next note that by Lemma 2.3(b) Z_1Y is a sum of terms of the form $u_\alpha Y_\alpha Z_\alpha$, where $u_\alpha \in U_0$ and $\ell(Y_\alpha) - \ell(Z_\alpha) = \ell(Y) - \ell(Z_1) \geq s + 1 - \ell(Z_1) \geq 1$, and hence $\ell(Y_\alpha) \geq 1$. Thus elements of the form $u'Y'Y_1Z_1Y$ are sums of elements of the form $u''Y''Y_1u_\alpha Y_\alpha Z_\alpha$, and by Lemma 2.3(a) these elements are sums of elements of the form $u''Y''Y_1'Y_\alpha Z_\alpha$, where $\ell(Y'') = \ell(Y') \geq s + 1$. Hence elements in YUY are of the form $\sum u_\beta Y_\beta Z_\beta$, where $\ell(Y_\beta) \geq$

$s + 2$. It follows by induction that elements in $Y(UY)^k$ are of the form $\sum u_\beta Y_\beta Z_\beta$, where $\ell(Y_\beta) \geq s + 1 + k$. Consequently, $Y(UY)^{r-s} = 0$, and $Y \in N(U)$. \square

We can now prove the main result of this section. Throughout the following we will use Y_i to denote a monomial in the y 's and Z_j to denote a monomial in the z 's.

Theorem 2.5. *The enveloping algebra $U(b(n))$ has a unique minimal prime ideal.*

Proof. By Behr [B, Corollary, page 21] the elements of U_0^* form a two-sided Ore set of regular elements of U , and localizing at U_0^* yields an Artinian two-sided classical quotient ring Q . Let Q_0 denote the division ring of quotients of U_0 ; then the set of monomials in the basis elements of \mathfrak{g}_1 form a basis for Q over Q_0 . As noted in Corollary 1.5, Q is quasi-Frobenius. By [MR, Proposition 2.1.16, page 47] expansion and contraction are 1-1 correspondences between the prime ideals of Q and the prime ideals of U that do not intersect U_0^* ; furthermore, $N(Q) = N(U)Q = QN(U)$ and $N(U) = N(Q) \cap U$. It is sufficient to show that $N(U)$ is prime, for since $N(U)$ is nilpotent, it would then be the unique minimal prime ideal of U ; hence it is sufficient to show that $N(Q)$ is prime. Since Q is Artinian, this is the case exactly when $Q/N(Q)$ is simple Artinian, or equivalently, when Q has exactly one simple module up to isomorphism. Since Q is quasi-Frobenius, every simple right Q -module is isomorphic to a minimal right ideal of Q (see, for example, [AF, Corollary 31.4, page 340]); hence it will be sufficient to show that all minimal right ideals of Q are isomorphic.

Let Y be the product in U of all the type I basis elements of \mathfrak{g}_1 and let Z be the product of all the type II basis elements of \mathfrak{g}_1 . By Lemma 2.3(a) if $u \in U_0$, then $Yu = u'Y$ for some $u' \in U_0$, and thus $Yu^{-1} = (u')^{-1}Y$; it follows that if $q \in Q_0$, then $Yq = q'Y$ for some $q' \in Q_0$. The same property holds for Z . Consider the right ideal YZQ of Q ; by 2.2.2 the right ideal YZQ does not depend on the order of the factors of Y and Z .

Claim 1. The right ideal YZQ is a minimal right ideal of Q .

Proof of Claim. Take an arbitrary nonzero element YZq of YZQ . We can write $Zq = \sum_{i,j} t_{i,j} Y_i Z_j$ where each $t_{i,j}$ is a nonzero element of Q_0 , and thus have $YZq = \sum_{i,j} Y t_{i,j} Y_i Z_j = \sum_{i,j} t'_{i,j} Y Y_i Z_j$ where $t'_{i,j} \in Q_0$ for each i, j . Since $Y Y_i = 0$ if $\ell(Y_i) > 0$, we can write YZq in the form $YZq = Y \sum_j s_j Z_j$ where $s_j \in Q_0^*$ for all j . Let j_0 be such that Z_{j_0} has no proper subproduct that occurs in the sum $\sum_j s_j Z_j$, and let Z'_{j_0} be the product of all z 's not occurring as a factor of Z_{j_0} . Since the z 's anti-commute, and since the square of any z is zero, we have $Z_j Z'_{j_0} = 0$ for $j \neq j_0$. Consequently, $YZq Z'_{j_0} = Y \sum_j s_j Z_j Z'_{j_0} = Y s_{j_0} Z_{j_0} Z'_{j_0} = \pm Y s_{j_0} Z = Y Z s'_{j_0}$ for some $s'_{j_0} \in Q_0^*$. Thus $YZ \in YZqQ$ for every q , and the right ideal YZQ is minimal.

Claim 2. If I is a minimal right ideal of Q , then I is isomorphic to YZQ .

Proof of Claim. Take $I = qQ$. We can write $q = \sum_{i,j} t_{i,j} Y_{i,j} Z_j$ where for each i, j we have that $t_{i,j}$ is a nonzero element of Q_0 . As in the proof of Claim 1, take j_0 so that Z_{j_0} has no proper subproduct that occurs in the sum, and let Z'_{j_0} be the product of the z 's not occurring as a factor of Z_{j_0} . Multiplying by Z'_{j_0} yields $q' = qZ'_{j_0} = \sum_i t_{i,j_0} Y_i Z = \sum_i \pm s_i Y_{i,j_0} Z$ where $s_i = \pm t_{i,j_0}$ for each i . Observe that $q' \neq 0$ since the terms in the sum are distinct basis elements for Q over Q_0 ; since I is simple, we have $I = q'Q$. Take a common denominator s for the s_i 's, and write $s_i = s^{-1}u_i$ for $s, u_i \in U_0$. Then $I \cong sI = q''Q$ where $q'' = sq' = \sum_i u_i Y_i Z$. By Lemma 2.3(a) we can write $q'' = \sum_j Y_j v_j Z = (\sum_j Y_j v_j)Z$ where each v_j is a

nonzero element of U_0 . Again take j_0 so that no proper subproduct of Y_{j_0} occurs in the sum, and let Y'_{j_0} be the product of all the y 's not occurring as a factor of Y_{j_0} . Multiplying on the left by Y'_{j_0} yields $Y'_{j_0}q'' = Y'_{j_0}Y_{j_0}v_{j_0}Z = \pm Yv_{j_0}Z = YZv'_{j_0}$ for some $v'_{j_0} \in U_0^*$. Observing that $Y'_{j_0}q'' \neq 0$, we have that $Y'_{j_0}q''Q = YZQ$. Hence $I \cong sI \cong Y'_{j_0}sI = YZQ$ as desired.

By the remarks at the beginning of the proof, the result now follows from Claims 1 and 2. □

3. MINIMAL PRIME IDEALS IN ENVELOPING ALGEBRAS OF NILPOTENT LIE SUPERALGEBRAS

In this section we show that if \mathfrak{g} is nilpotent, then $U(\mathfrak{g})$ has a unique minimal prime ideal.

3.1. The *center* of \mathfrak{g} , $Z(\mathfrak{g})$, is defined by $Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$; the higher centers are defined by $Z_i(\mathfrak{g})/Z_{i-1}(\mathfrak{g}) = Z(\mathfrak{g}/Z_{i-1}(\mathfrak{g}))$. Recall that \mathfrak{g} is called *nilpotent* if $Z_\ell(\mathfrak{g}) = \mathfrak{g}$ for some ℓ . Let $\{x_1, x_2, \dots, x_n\}$ be a basis for \mathfrak{g}_0 , and let $\{y_1, y_2, \dots, y_m\}$ be a basis for \mathfrak{g}_1 . In the enveloping algebra $U = U(\mathfrak{g})$ call a monomial $x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_m^{f_m}$ in the PBW basis *even* if $f_1 + f_2 + \cdots + f_m$ is even; recall that each f_i is either 0 or 1 so that a monomial is even if the number of ‘ y -factors’ is even. Call all other monomials *odd*. If $a \in U$, group even and odd monomials together and write $a = a_0 + a_1$; this decomposition of elements yields a Z_2 -grading of U . Let $\sigma : U \rightarrow U$ be defined by $\sigma(a) = \sigma(a_0 + a_1) = a_0 - a_1$; the map σ is an automorphism of U . Furthermore, an ideal I is graded if and only if $\sigma(I) = I$.

Letzter has noted that the following theorem follows from [Le, Remark 3.11, page 254].

Theorem 3.2. *Let \mathfrak{g} be a solvable Lie superalgebra over an algebraically closed field of characteristic 0. If $U(\mathfrak{g})$ does not have a unique minimal prime, then it has exactly two minimal primes P and $\sigma(P)$. Then $P \cap \sigma(P)$ is the unique minimal graded prime ideal and is $N(U)$, the nil radical of U .*

Since U has an Artinian two-sided classical quotient ring Q that is obtained by inverting the nonzero elements of $U_0 = U(\mathfrak{g}_0)$, the monomials in the y 's form a basis for Q over $Q_0 = Q(U_0)$. We can extend the grading to Q ; the automorphism σ extends to Q by $\sigma(s^{-1}a) = s^{-1}\sigma(a)$ where $s \in U_0^*$ and $a \in U$.

While it is possible for Q to have proper idempotents, this is not the case for U .

Proposition 3.3. *Let \mathfrak{g} be a nilpotent Lie superalgebra with enveloping algebra $U = U(\mathfrak{g})$. Then 0 and 1 are the only idempotents of U .*

Proof. Induct on the dimension of \mathfrak{g} . The result is true if \mathfrak{g} has dimension 1. Suppose the result is true for all nilpotent Lie superalgebras with dimension less than that of \mathfrak{g} . Consider $Z(\mathfrak{g})$, which is nonzero since \mathfrak{g} is nilpotent. Since $Z(\mathfrak{g})$ is a graded Lie ideal, it must contain a homogeneous element.

Case 1. Suppose that $Z(\mathfrak{g})$ contains an odd homogeneous element y . Let e be a nonzero idempotent of U . Then $e + yU$ is a nonzero element of U/yU since yU is nilpotent. Thus if e is a proper nonzero idempotent of U , both e and $1 - e$ are nonzero idempotents of U/yU . This cannot be for $U/yU \cong U(\mathfrak{g}/ky)$ has no proper idempotents by induction.

Case 2. Suppose that $Z(\mathfrak{g})$ contains an even homogeneous element z and let e be a proper idempotent of U . We may assume that z is a basis element of \mathfrak{g} . Suppose that $e \in zU$. From the PBW basis of U we see that we can write $e = z^i u$ where u is not in zU . Then $zu = e = e^2 = z^2 u^2$ and we have $u = zu^2 \in zU$, which is a contradiction. Hence $e + zU$ must be a nonzero idempotent of U/zU . Thus $e + zU$ and $(1 - e) + zU$ are proper idempotents of $U/zU \cong U(\mathfrak{g}/kz)$; this is a contradiction by induction. \square

The following proposition gives consequences of U having two minimal prime ideals. Let $\bar{Q} = Q/N(Q)$ and let $\bar{a} = a + N(Q)$. Since $J(Q) = N(Q) = N(U)Q = QN(U)$ (for example, see [CH, Lemma 9.2, page 125]), we have $\sigma(N(Q)) = \sigma(N(U)Q) = N(U)Q = N(Q)$ and hence $N(Q)$ is σ -stable. Thus σ induces an automorphism $\bar{\sigma}$ on \bar{Q} .

Proposition 3.4. *Let $U = U(\mathfrak{g})$ for a nilpotent finite dimensional Lie superalgebra \mathfrak{g} over an algebraically closed field k of characteristic 0. If U has two minimal prime ideals, then there exist elements c and s in U such that*

- (1) c is an odd regular element of U ,
- (2) $s \in U_0$,
- (3) $\bar{c}^2 = \frac{\bar{s}^2}{4}$, and
- (4) $\bar{c}\bar{u}\bar{s} = \bar{s}\bar{u}\bar{c}$ for all $u \in U$. Note that s and c commute modulo $N(Q)$.

Conversely, given such elements c and s , $\frac{1}{2} + cs^{-1} + N(Q)$ is a central idempotent of $Q/N(Q)$.

Proof. Suppose that U has two minimal prime ideals, P and $\sigma(P)$. Since Q is obtained by classically localizing at the two-sided Ore set $S = U_0^*$ all of whose elements are regular in U , there is a correspondence between the prime ideals of Q and the prime ideals of U that do not intersect S [MR, Proposition 2.1.16, page 47]. Since Q is Artinian, Q has Krull dimension 0 and every prime ideal of Q is the expansion of one of the two minimal prime ideals of U . Since U is Noetherian, neither of the minimal prime ideals P nor $\sigma(P)$ can contain a regular element. Consequently Q also has exactly two minimal prime ideals (which are the only prime ideals of Q), namely PQ and $\sigma(P)Q$. The above correspondence shows that if Q does not have a unique minimal prime, then neither can U .

Consider $\bar{Q} = Q/N(Q) = Q/J(Q)$. Then Q has more than one prime ideal if and only if \bar{Q} does, and since \bar{Q} is semisimple Artinian, this is the case if and only if \bar{Q} has a proper central idempotent. In this case \bar{Q} will have exactly two proper central idempotents corresponding to the prime ideals of Q ; call them \bar{e} and $1 - \bar{e}$. Furthermore $\bar{\sigma}(\bar{e}) = 1 - \bar{e}$. Write $\bar{e} = e + J$ where e is an idempotent of Q and write $e = a + b$ where a is even and b is odd. Then $\bar{\sigma}(\bar{e}) = 1 - \bar{e} = 1 - \bar{a} - \bar{b} = \bar{a} - \bar{b}$ and hence $\bar{a} = \frac{1}{2}$. Since \bar{e} is central, so are \bar{a} and \bar{b} . Then $\bar{a}^2 + 2\bar{a}\bar{b} + \bar{b}^2 = \bar{a} + \bar{b}$ which implies that $\frac{1}{4} + \bar{b} + \bar{b}^2 = \frac{1}{2} + \bar{b}$ and consequently that $\bar{b}^2 = \frac{1}{4}$. Write $\bar{b} = \bar{c}\bar{s}^{-1}$ where $s \in U_0$ and c is an odd element of U . Then $\bar{c}^2 = \frac{1}{4}\bar{s}^2$. Note that c must be a regular element of U . For any $u \in U$ we have that $\bar{c}\bar{s}^{-1}\bar{u} = \bar{u}\bar{c}\bar{s}^{-1}$ and in particular \bar{c} and \bar{s} commute. Thus we have $\bar{c}\bar{u}\bar{s} = \bar{s}\bar{u}\bar{c}$ for all $u \in U$.

Calculation shows that the converse follows. \square

The following lemma appeared in Bell-Musson [BM, Lemma 1.10, page 406].

Lemma 3.5. *Let \mathfrak{g} be a nilpotent Lie superalgebra with center $Z(\mathfrak{g}) = kz$ where z is even. Let y be a homogeneous element of $Z_2(\mathfrak{g}) - Z(\mathfrak{g})$ where $Z_2(\mathfrak{g})/Z(\mathfrak{g}) = Z(\mathfrak{g}/Z(\mathfrak{g}))$. Then there exists a homogeneous element x of the same parity as y and a graded ideal \mathfrak{h} of codimension 1 in \mathfrak{g} such that*

- (1) $[y, x] = z$,
- (2) \mathfrak{h} is the centralizer of y in \mathfrak{g} , and
- (3) $\mathfrak{g} = \mathfrak{h} \oplus kx$.

We need the following simple lemma.

Lemma 3.6. *Let \mathfrak{g} be a Lie superalgebra with $\dim(Z(\mathfrak{g}) \cap \mathfrak{g}_0) \geq 2$; say z and z_1 are linearly independent such elements. If u is a nonzero element of U , then there are only finitely many λ 's in k such that $u \in (z + \lambda z_1)U$.*

Proof. Let z_λ denote $z + \lambda z_1$. If $\mu \neq \lambda$, then z_λ is a regular element of $U/z_\mu U \cong U(\mathfrak{g}/kz_\mu)$. Hence if $u = z_\lambda u_1 = z_\mu u_2$ for $u_1, u_2 \in U$, then $u_1 = z_\mu u_3$ for some $u_3 \in U$. Consequently, $u_1 \in z_\mu U$ for every z_μ such that $u \in z_\mu U$ with $\mu \neq \lambda$. Since the high term of u_1 is less than the high term of u in the length lexicographic ordering of monomials, an induction argument yields the result. \square

We need the following lemma of Bell and Musson [BM, Lemma 1.5, page 404].

Lemma 3.7. *Let $A = A_0 \oplus A_1$ be a Z_2 -graded k -algebra with $\sigma : A \rightarrow A$ defined by $\sigma(a_0 + a_1) = a_0 - a_1$. Let δ be a σ -derivation on A , and suppose that there is an even $h \in A$ with $\delta(h) = 0$. Let y' be an odd supercentral element of A with $\delta(y') = 1$, and let S be the Ore extension $A[t; \sigma, \delta]$. Set $B = \delta(y'A)$. Then B is a subalgebra of A such that $A = B \oplus y'A$, and so $A/y'A \cong B$. Moreover, $t^2 - h$ is central in S and $S/(t^2 - h)S \cong A/y'A \otimes M_2(k)$.*

We can now prove the main result of this section.

Theorem 3.8. *Let \mathfrak{g} be a nilpotent Lie superalgebra. Then $U(\mathfrak{g})$ has a unique minimal prime ideal.*

Proof. First assume that k is algebraically closed. The proof will be by induction on the dimension of \mathfrak{g} . The result is clearly true if \mathfrak{g} has dimension 1. Suppose that the result is true for all nilpotent Lie superalgebras with dimension less than that of \mathfrak{g} . Furthermore, assume that U has two minimal prime ideals.

Case 1. There is an odd element $y \in Z(\mathfrak{g})$. Since yU is a nilpotent ideal of U , there is a correspondence between the minimal prime ideals of U and those of U/yU . This contradicts the induction hypothesis, for $U/yU \cong U(\mathfrak{g}/(ky))$.

Case 2. We have $Z(\mathfrak{g}) \subset \mathfrak{g}_0$ and $\dim Z(\mathfrak{g}) \geq 2$. Let c and s be the elements of U given by Proposition 3.4. Take a nonzero $z \in Z(\mathfrak{g})$. It follows from Proposition 3.4 that $\bar{e} = \frac{1}{2} + cs^{-1}$ is a central idempotent of $Q(U/zU)/N(Q(U/zU))$ provided s is not an element of zU , for $U/zU \cong U(\mathfrak{g}/kz)$ would contain $s + zU$ as an element of the Ore set $U_0^*(\mathfrak{g}/kz)$. By Lemma 3.6, z can be chosen so that s is not an element of U . It is conceivable that this could be the zero idempotent; that is, $\frac{1}{2} + cs^{-1} + zU \in N(Q(U/zU))$. In this case $s + 2c + zU$ is an element of the nilpotent ideal $N(U/zU)$. Recall that $Q(U(\mathfrak{g}))$ is spanned by all the monomials in the basis elements of \mathfrak{g}_1 over the division ring $Q(U_0)$, and that $N(Q)^i = N(U)^i Q$; hence $N(U/zU)$ must be nilpotent of index less than or equal to $p = 2^n$ ($n = \dim \mathfrak{g}_1$). Note that $(s + 2c)^p$ and $(s - 2c)^p$ are nonzero since \bar{e} and $1 - \bar{e}$ are proper idempotents of Q . Again by Lemma 3.6 and the fact that k is infinite, we can

choose z so that s is not in zU , $(s + 2c)^p$ is not in zU , and $(s - 2c)^p$ is not in zU . Thus \bar{e} is a proper nonzero central idempotent of $Q(U/zU)/N(Q(U/zU))$. Since $U/zU \cong U(\mathfrak{g}/kz)$, this contradicts the induction hypothesis.

Case 3. We have $Z(\mathfrak{g}) \subset \mathfrak{g}_0$ and $\dim Z(\mathfrak{g}) = 1$. Let x, y be as in Lemma 3.5 and write $\mathfrak{g} = \mathfrak{h} \oplus kx$ where \mathfrak{h} is the centralizer of y . We have three subcases.

Case 3a. Suppose that x and y are both even. Then we have $U(\mathfrak{g}) \cong U(\mathfrak{h})[x; \delta]$. Since \mathfrak{h} is again nilpotent, the induction hypothesis implies that $U(\mathfrak{h})$ has a unique minimal prime ideal. The theorem now follows by [MR, Theorem 1.2.9, page 16 and Proposition 14.2.3, page 495].

Case 3b. Suppose that x and y are both odd and $[y, y] = 0$. Let $h = [x, x]/2$. Then we have $U(\mathfrak{g}) = U(\mathfrak{h})[x; \sigma, \delta]/\langle x^2 - h \rangle$; localizing at the powers of z yields $U(\mathfrak{g})_{(z^i)} = U(\mathfrak{h})_{(z^i)}[x; \sigma, \delta]/\langle x^2 - h \rangle$ where σ and δ have been extended to all of $U(\mathfrak{h})_{(z^i)}$. Hence letting $A = U(\mathfrak{h})_{(z^i)}$ and applying Lemma 3.7 we have $U(\mathfrak{g})_{(z^i)} \cong A/y'A \otimes M_2(k) \cong M_2(A/y'A)$. By induction $U(\mathfrak{h})$ has a unique minimal prime ideal and hence so does $U(\mathfrak{h})_{(z^i)}$. Since y is a supercentral odd element of \mathfrak{h} , the ideal $y'A = yz^{-1}A$ is a nilpotent ideal of A , and thus $A/y'A$ has a unique minimal prime. Hence so does $U(\mathfrak{g})_{(z^i)}$. Since z cannot be contained in any minimal prime of $U(\mathfrak{g})$ (see the proof of Proposition 3.4), U has a unique minimal prime ideal.

Case 3c. Suppose that $[y, y] \neq 0$ for all $y \in Z_2(\mathfrak{g}) - Z(\mathfrak{g})$. Let x be as in Lemma 3.5. Since $y \in Z_2(\mathfrak{g})$, $[y, y] = \lambda z$ for some scalar λ . Suppose that y and y' represent linearly independent cosets in $Z_2(\mathfrak{g})/Z(\mathfrak{g}) = Z_2(\mathfrak{g})/kz$; by Case 3a we may assume that y' is also odd. Let $[y', y'] = \mu z$ and $[y, y'] = \rho z$. Consider $[y + ay', y + ay'] = (\lambda + 2\rho a + \mu a^2)z$. Since k is algebraically closed, there exists $a \in k$ such that $\lambda + 2\rho a + \mu a^2 = 0$. In this case $[y + ay', y + ay'] = 0$, which is a contradiction. Thus we may assume that $\dim Z_2(\mathfrak{g})/Z(\mathfrak{g}) = 1$. Then in Lemma 3.5 we may take $x = y$ and write $\mathfrak{g} = \mathfrak{h} \oplus ky$; recall that \mathfrak{h} is the centralizer of y . If $w \in Z_2(\mathfrak{h})$, then $[w, y] = 0$ and $w \in Z_2(\mathfrak{g})$. Since $\dim Z_2(\mathfrak{g})/Z(\mathfrak{g}) = 1$, we have that $w = \alpha z$ for some scalar α . The nilpotence of \mathfrak{h} implies that $\mathfrak{h} = kz$ and $\mathfrak{g} = kz \oplus ky$. Hence $d(\mathfrak{g}) \neq 0$ and it follows from Bell [Be, Theorem 1.5] that $U(\mathfrak{g})$ is prime.

If k is not algebraically closed, let K be the algebraic closure of k and let $\mathcal{B} = \{b_\alpha : \alpha \in \mathcal{A}\}$ be a basis for K over k . Then \mathcal{B} is a central free basis for $U_K(\mathfrak{g})$ over $U_k(\mathfrak{g})$. Since K is algebraically closed, the preceding part of the proof implies that $U_K(\mathfrak{g})$ has a unique minimal prime ideal P with $P = N(U_K(\mathfrak{g}))$. Let $p = P \cap U_k(\mathfrak{g})$. Suppose that $uU_k(\mathfrak{g})v \subseteq p$ for $u, v \in U_k(\mathfrak{g})$ with u not an element of p . Since $ub_\alpha U_k(\mathfrak{g})v \subseteq pb_\alpha \subseteq P$ for all $\alpha \in \mathcal{A}$, we have $uU_K(\mathfrak{g})v \subseteq P$; consequently, $v \in P \cap U_k(\mathfrak{g}) = p$, and p is a prime ideal. Since p is a nilpotent prime ideal, p is the unique minimal prime ideal of $U_k(\mathfrak{g})$. \square

The following example shows that, while U has no nontrivial idempotents, it is possible that Q contains nontrivial idempotents.

3.9. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the subalgebra of 4×4 matrices, where \mathfrak{g}_0 is the vector space spanned by the matrix units $\{x_1 = e_{1,2}, x_2 = e_{1,3}, x_3 = e_{2,3}\}$, \mathfrak{g}_1 is the vector space spanned by the matrix units $\{y_1 = e_{1,4}, y_2 = e_{4,3}, y_3 = e_{4,2}\}$, and the bracket is defined on basis elements as $[g_i, g_j] = g_i g_j + (-1)^{ij+1} g_j g_i$. In the notation of [S], \mathfrak{g} is a subalgebra of $pl(3, 1)$. The only nonzero brackets involving basis elements are: $[x_1, x_3] = x_2$, $[x_3, y_3] = -y_2$, $[y_1, y_2] = x_2$, and $[y_1, y_3] = x_1$. It is easy to check that:

- (1) $Z_4 = \mathfrak{g}$, and so \mathfrak{g} is a nilpotent Lie superalgebra.

- (2) $y_2y_3Uy_2y_3 = 0$, so U is not semiprime.
- (3) $x_2 \in Z(\mathfrak{g})$.
- (4) $e = y_1y_2x_2^{-1}$ is a nontrivial idempotent of Q .

Let \mathfrak{h} be the k -span of $\{x_2, x_3, y_1, y_2\}$. Then it can be shown that \mathfrak{h} is a nilpotent Lie superalgebra, $U(\mathfrak{h})$ is prime, and $e = y_1y_2x_2^{-1}$ is a proper idempotent of $Q(U(\mathfrak{h}))$.

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