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QUASIDISKS AND THE ZYGMUND PROPERTY

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ABSTRACT. In this paper, we obtain a new characterization of quasidisks by the Zygmund property.

1. INTRODUCTION

Suppose that D is a proper subdomain of the finite complex plane $\mathbb{C}$. For $z \in \mathbb{C}$ and $0 < r < \infty$, let $B(z,r)$ denote the open disk with center z and radius r. For constant $M > 0$ and $f(z)$ analytic in D, we say $f(z) \in MH_2^t$ if the inequality

$$|f(z) - P_1(f,z)| \le M\delta \log \frac{2d}{\delta} \tag{1.1}$$

holds for any $z_1, z_2 \in D$ and $z \in D \cap [B(z_1,d) \cup B(z_2,d)]$, where $d = |z_1 - z_2|$, $\delta = \min\{|z - z_1|, |z - z_2|\}$, and

$$P_1(f,z) = \frac{z_2 - z}{z_2 - z_1} f(z_1) + \frac{z - z_1}{z_2 - z_1} f(z_2). \tag{1.2}$$

Let $H_2^t = \bigcup_{M>0} MH_2^t$. By [1], in the case $D = \{z\colon |z| < 1\}$, H_2^t is the following well-known Zygmund's class Λ_*:

(1.3)

$$\Lambda_* = \left\{ f(z) \text{ analytic in } D \sup_{|h|\le t} \max_{\theta\in|0,2\pi|} |f(e^{i(\theta+h)}) - 2f(e^{i\theta}) + f(e^{i(\theta-h)})| \le M_f t \right\}.$$

Zygmund's class Λ_* has many important applications in approximation theory.

A domain $D \subset \mathbb{C}$ is said to be an (α,β)-John domain, $0 < \alpha \le \beta < \infty$, if there exists $z_0 \in D$ such that every $z \in D$ can be joined to z_0 by a rectifiable curve $\gamma\colon [0,d] \to D$, satisfying:

$$\begin{aligned} &\text{(a) } \gamma(0) = z, \quad \gamma(d) = z_0; \\ &\text{(b) } d \le \beta; \\ &\text{(c) } \operatorname{dist}(\gamma(s), \partial D) \ge \alpha \frac{s}{d} \qquad (0 \le s \le d), \end{aligned} \tag{1.4}$$

where s is the arc-length parameter.

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A domain $D \subset \mathbb{C}$ is said to be an (α, β)-uniform domain, if for each pair of points $z_1, z_2 \in D$, $z_1 \neq z_2$, there is an $(\alpha|z_1 - z_2|, \beta|z_1, z_2|)$-John domain G such that $z_1, z_2 \in G \subset D$.

D is said to be a K-quasidisk if it is the image of a disk or half-plane under a K-quasiconformal mapping $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By [5], we know that if D is a K-quasidisk, then D is an (α, β)-uniform domain for constants α and β that depend only on K.

Quasidisks were characterized in [3] and [4] by the Hardy-Littlewood property (only for unbounded domains) and in [2] by the Schwarzian univalence criterion. In 1992, we obtained the following theorem.

Theorem Z ([6])**.** *Suppose that D is a quasidisk in $\mathbb{C}$. Then necessary and sufficient conditions for $f(z) \in H_2^t$ are that $f(z)$ is analytic in D and satisfies*

$$|f''(z)| = O(\operatorname{dist}(z, \partial D)^{-1}), \qquad z \in D. \tag{1.5}$$

The sketch of the proof is as follows.

Suppose that $f(z) \in H_2^t$. For any $z \in D$, let $0 < r < \operatorname{dist}(z, \partial D)/4$. Then $D_r = B(z, r) \subset D$. Choosing $z_1, z_2 \in D_r$ with $|z_1 - z_2| = 2r$,

$$D_r \subset B(z_1, 2r) \cup B(z_2, 2r) \subset D \tag{1.6}$$

and

$$\overline{D_r} \backslash \{z_1, z_2\} \subset B(z_1, 2r) \cup B(z_2, 2r). \tag{1.7}$$

Setting

$$P_1(f, z) = \frac{z_2 - z}{z_2 - z_1} f(z_1) + \frac{z - z_1}{z_2 - z_1} f(z_2),$$

we have

$$f(z) - P_1(f, z) = \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(\zeta) - P_1(f, \zeta)}{\zeta - z} d\zeta. \tag{1.8}$$

It follows that

$$f''(z) = \frac{1}{\pi i} \int_{\partial D_r} \frac{f(\zeta) - P_1(f, \zeta)}{(\zeta - z)^3} d\zeta. \tag{1.9}$$

By (1.1) we have

$$|f''(z)| \leq \frac{M_f}{\pi r^3} \int_{\partial D_r} \delta(\zeta) \log \frac{4r}{\delta(\zeta)} |d\zeta|, \tag{1.10}$$

where $\delta(\zeta) = \min\{|\zeta - z_1|, |\zeta - z_2|\}$.

Set

$$\zeta = z + re^{i\theta}, \qquad z_k = z + re^{i\theta_k}, \quad k = 1, 2.$$

Then

$$\delta(\zeta) = 2r \min\left\{\sin \frac{\theta - \theta_k}{2}\right\},$$

so

$$|f''(z)| \leq \frac{M_f}{\pi r^3} 16r^2 \int_0^{2\pi} \sin \frac{\theta}{2} \log \frac{2}{\sin \frac{\theta}{2}} d\theta \leq C_0 \frac{1}{r}.$$

Taking $r = \operatorname{dist}(z, \partial D)/8$ yields (1.5).

In order to prove the sufficiency, let $z_1, z_2 \in D$ and $z_1 \neq z_2$. For any $z \in [B(z_1,h) \cup B(z_2,h)] \cap D$, let $\delta = \min\{|z_1 - z|, |z_2 - z|\}$. Since D is an (α,β)-uniform domain, for $k = 1,2$ there exists an $(\alpha|z_k - z|, \beta|z_k, z|)$-John domain $D_k \subset D$ containing z and z_k. Let z_{k_0} be the point in the definition of the $(\alpha|z_k - z|, \beta|z_k, z|)$-John domain D_k, γ_{1k} the corresponding rectifiable arc joining z to z_{k0}, and γ_{2k} the arc joining z_k to z_{k0}. Then we have

$$\begin{aligned} f(z) - P_1(f,z) =& \frac{z_2 - z}{z_2 - z_1}\left[\int_{\gamma_{1k}}(\zeta - z)f''(\zeta)d\zeta - \int_{\gamma_{2k}}(\zeta - z_1)f''(\zeta)d\zeta\right] \\ &- \frac{z - z_1}{z_2 - z_1}\left[\int_{\gamma_{2k}}(\zeta - z_2)f''(\zeta)d\zeta - \int_{\gamma_{1k}}(\zeta - z)f''(\zeta)d\zeta\right] \\ &+ \frac{(z_2 - z)(z - z_1)}{z_2 - z_1}[f'(z_{10}) - f'(z_{20})] \\ =& S_1 + S_2 + S_3. \end{aligned} \tag{1.11}$$

It is easy to see that

$$S_k = O(\delta), \qquad k = 1,2. \tag{1.12}$$

We estimate S_3. By virtue of (c) in (1.4), we have

$$\operatorname{dist}(z_{k0}, \partial D) \geq \alpha|z - z_k|, \qquad k = 1,2. \tag{1.13}$$

In the case

$$|z_{10} - z_{20}| \leq \max_{k=1,2} \operatorname{dist}(z_{k0}, \partial D)/2,$$

we may assume without loss of generality that $\operatorname{dist}(z_{20}, \partial D) \geq \operatorname{dist}(z_{10}, \partial D)$. Then the open disk B_2 with center z_{20} and radius $\operatorname{dist}(z_{20}, \partial D)/2$ is contained in D. Consequently, the distance of each point on $\overline{B}_2$ to ∂D is not less than $\operatorname{dist}(z_{20}, \partial D)/2$. Noting that $z_{10} \in \overline{B}_2$, let σ be the segment from z_{10} to z_{20}. Then we have

$$|f'(z_{10}) - f'(z_{20})| = \left|\int_\sigma f''(\zeta)d\zeta\right| = O(1).$$

Thus

$$S_3 = O(\delta).$$

In the case $|z_{10} - z_{20}| \geq \max_{k=1,2} \operatorname{dist}(z_{k0}, \partial D)/2$, it is not too difficult to show that

$$|f'(z_{10}) - f'(z_{20})| = O\left(\log \frac{|z_{10} - z_{20}|}{\delta}\right).$$

It follows from (1.13) and (1.12) that

$$|f(z) - P_1(f,z)| \leq M\delta \log \frac{2h}{\delta}.$$

This proves that $f(z) \in H_2^t$.

Definition 1. Suppose that D is a proper subdomain of $\mathbb{C}$. We say that D has the Zygmund property if there exists a constant $M > 0$ such that $f(z) \in H_2^t$ whenever $f(z)$ is analytic in D and satisfies $|f''(z)| \leq M \operatorname{dist}(z, \partial D)^{-1}$ in D.

From Theorem Z we know that quasidisks have the Zygmund property. In the present paper we show that, if an unbounded domain $\mathbb{C}$ has the Zygmund property, it is a quasidisk too. We thus obtain a characterization of unbounded quasidisks by the Zygmund property.

Theorem. *Suppose that D is a simply connected domain in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\infty \in \partial D$ and that $D^* = \widehat{\mathbb{C}} \backslash \overline{D}$ is a domain. Then D is a quasidisk if and only if both D and D^* have the Zygmund property.*

2. Some lemmas

Lemma 1 ([3])**.** *Suppose that D is a simply connected subdomain of $\mathbb{C}$ and that $z_0 \in \mathbb{C}$. If there exist points in $D \cap \overline{B}(z_0, r)$ which cannot be joined in $D \cap \overline{B}(z_0, br)$, then there exist points $z_1, z_2 \in D \cap \overline{B}(z_0, r)$ and $w_0 \in \partial B(z_0, br) \backslash D$ such that*

$$|h(z_1) - h(z_2) - 2\pi i| \leq \frac{2}{b-1} \tag{2.1}$$

whenever $h(z)$ is an analytic branch of $\log(z - w_0)$ in D.

Under the conditions of Lemma 1, let z_0, z_1, z_2 and w_0 be points as indicated in its statements. Choose an arc γ in D from z_1 to z_2, and let z' be the first point at which γ meets $\partial B(z_1, \frac{d}{2})$ when γ is traversed from z_1 to z_2, where $d = |z_1 - z_2|$. Denote by γ' the subarc of γ from z_1 to z'.

Given $h(z)$, an analytic branch of $\log(z - w_0)$ in D, let $h_0(z)$ be the analytic branch of $\log(z - w_0)$ in $B(z_0, br)$ satisfying

$$h_0(z_1) = h(z_1). \tag{2.2}$$

If σ is the segment from z_1 to z_2, then plainly

$$|h_0(z_2) - h_0(z_1)| = \left| \int_\sigma h_0'(z) dz \right| \leq \int_\sigma \frac{|dz|}{|z - w_0|} \leq \frac{2}{b-1}. \tag{2.3}$$

Since $\gamma' \subset D \cap B(z_1, \frac{d}{2}) \subset B(z_0, br)$, we infer from (2.2)

$$h(z') = h_0(z'). \tag{2.4}$$

Both $h_0(z)$ and $h(z)$ are analytic branches of $\log(z - w_0)$ in some neighborhood of z_2; we thus have $h_0(z_2) - h(z_2) = 2k\pi i$. Using (2.1), (2.2) and (2.3) we conclude

$$|h_0(z_2) - h(z_2) - 2\pi i| \leq \frac{4}{b-1} \leq \frac{4}{3},$$

$$h_0(z_2) - h(z_2) = 2\pi i. \tag{2.5}$$

For $z \in B(z_0, br)$, let $f_0(z) = (z - w_0) h_0(z)$.

Lemma 2. *Under the conditions of Lemma 1 and with z_0, z_1, z_2, w_0 and z' as indicated in the statement of the lemma and the ensuing discussion, it is the case that*

$$|f_0(z') - P_1(f_0, z')| \leq \frac{3d}{b-2}. \tag{2.6}$$

Proof. Let σ_1 be the segment from z_1 to z', and σ_2 the segment from z_2 to z'. We compute

$$\begin{aligned}
&|f_0(z') - P_1(f_0, z')| \\
&\quad = \left| \frac{z_2 - z'}{z_2 - z_1}[f_0(z') - f_0(z_1)] + \frac{z' - z_1}{z_2 - z_1}[f_0(z') - f_0(z_2)] \right| \\
&\quad = \left| \frac{z_2 - z'}{z_2 - z_1} \int_{\sigma_1} (z - z_1) f_0''(z) dz + \frac{z' - z_1}{z_2 - z_1} \int_{\sigma_2} (z - z_2) f_0''(z) dz \right| \\
&\quad \le \frac{|z_2 - z'|}{|z_2 - z_1|} \int_{\sigma_1} |z - z_1| \frac{|dz|}{|z - w_0|} + \frac{|z' - z_1|}{|z_2 - z_1|} \int_{\sigma_2} |z - z_2| \frac{|dz|}{|z - w_0|} \\
&\quad \le \frac{3}{2} |z_1 - z'| \frac{|z_1 - z'|}{(b-2)r} + \frac{|z_1 - z'||z' - z_2|}{|z_2 - z_1|} \frac{|z_2 - z'|}{(b-2)r} \\
&\quad \le \frac{6}{b-2} |z_1 - z'| \le \frac{3d}{b-2}.
\end{aligned}$$

□

Definition 2 ([2])**.** A set E in $\widehat{\mathbb{C}}$ is said to be a-locally connected if for all $z_0 \in \mathbb{C}$ and $r > 0$, any pair of points in $E \cap \overline{B}(z_0, r)$ can be joined in $E \cap \overline{B}(z_0, ar)$ and any pair of points in $E \backslash B(z_0, r)$ can be joined in $E \backslash B(z_0, \frac{r}{a})$.

Lemma 3 ([2])**.** *Suppose that a domain D in $\mathbb{C}$ is a-locally connected and that ∂D is connected and contains at least two points. Then ∂D is a K-quasiconformal circle, where K depends only on a.*

3. Proof of the Theorem

The necessity of both D and D^* having the Zygmund property is ensured by Theorem Z. We must treat the sufficiency.

As in [2], we only need to prove the following proposition.

Proposition. *Suppose that D is a simply connected proper subdomain of $\mathbb{C}$ which has the Zygmund property. Then there exists a constant $b > 4$, which depends only on the constant M in Definition 1, such that for all $z_0 \in \mathbb{C}$ and $r > 0$, each pair of points in $D \cap \overline{B}(z_0, r)$ can be joined in $D \cap \overline{B}(z_0, br)$.*

Proof. Choose

$$b = \frac{M \log 4 + 8}{\pi} + 2, \tag{3.1}$$

and suppose the conclusion does not hold for some $z_0 \in \mathbb{C}$ and $r > 0$. Fix points z_1, z_2 in $D \cap \overline{B}(z_0, r)$ which cannot be joined in $D \cap \overline{B}(z_0, br)$, and let z' and w_0 be as in section 2.

Consider $h(z)$ an analytic branch of $\log(z - w_0)$ in D, along with the function

$$f(z) = (z - w_0) h(z). \tag{3.2}$$

Then $f(z)$ is analytic in D, where it satisfies

$$|f''(z)| = \frac{1}{|z - w_0|} \le \operatorname{dist}(z, \partial D). \tag{3.3}$$

The hypothesis on D implies that

$$|f(z) - P_1(f,z)| \le M\delta \log \frac{2d}{\delta} \tag{3.4}$$

holds for all $z \in D \cap [B(z_1,d) \cup B(z_2,d)]$. In particular, (3.4) holds for the point $z' \in D \cap [B(z_1,d) \cup B(z_2,d)]$. Noting $\delta = |z'-z_1| = |z'-z_2| = \frac{d}{2}$, we have

$$|f(z') - P_1(f,z')| \le \frac{Md\log 4}{2}. \tag{3.5}$$

Because $f(z_1) = f_0(z_1), f(z') = f_0(z')$ and $f(z_2) = f_0(z_2) + 2\pi i(z_2 - w_0)$, we obtain

$$\begin{aligned} |f(z') - P_1(f,z')| &= |f_0(z') - P_1(f_0,z') + P_1(f-f_0,z')| \\ &\ge \left|\frac{z'-z_1}{z_2-z_1}[f(z_2) - f_0(z_2)]\right| - |f_0(z') - P_1(f_0,z')| \\ &= \pi|z_2 - w_0| - |f_0(z') - P_1(f_0,z')|. \end{aligned}$$

By (2.6) and (3.1), we conclude that

$$\begin{aligned} |f(z') - P_1(f,z')| &\ge \pi(b-2)r - \frac{3d}{b-2} \\ &\ge (M\log 4 + 8)d - \frac{3\pi dM}{\log 4 + 8} \\ &\ge Md\log 4, \end{aligned}$$

which contradicts (3.5).

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