QUASIDISKS AND THE ZYGMUND PROPERTY

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Abstract. In this paper, we obtain a new characterization of quasidisks by the Zygmund property.

1. Introduction

Suppose that $D$ is a proper subdomain of the finite complex plane $\mathbb{C}$. For $z \in \mathbb{C}$ and $0 < r < \infty$, let $B(z, r)$ denote the open disk with center $z$ and radius $r$. For constant $M > 0$ and $f(z)$ analytic in $D$, we say $f(z) \in MH^2_t$ if the inequality

\[ |f(z) - P_1(f, z)| \leq M \delta \log \frac{2d}{\delta} \]

holds for any $z_1, z_2 \in D$ and $z \in D \cap [B(z_1, d) \cup B(z_2, d)]$, where $d = |z_1 - z_2|$, $\delta = \min\{|z - z_1|, |z - z_2|\}$, and

\[ P_1(f, z) = \frac{z_2 - z}{z_2 - z_1} f(z_1) + \frac{z - z_1}{z_2 - z_1} f(z_2). \]

Let $H^2 = \bigcup_{M > 0} MH^2_t$. By [1], in the case $D = \{z: |z| < 1\}$, $H^2$ is the following well-known Zygmund’s class $\Lambda^*$:

\[ \Lambda^* = \left\{ f(z) \text{ analytic in } D : \sup_{|h| \leq \theta \in [0, 2\pi]} |f(e^{i(\theta + h)}) - 2f(e^{i\theta}) + f(e^{i(\theta - h)})| \leq Mf^t \right\}. \]

Zygmund’s class $\Lambda^*$ has many important applications in approximation theory.

A domain $D \subset \mathbb{C}$ is said to be an $(\alpha, \beta)$-John domain, $0 < \alpha \leq \beta < \infty$, if there exists $z_0 \in D$ such that every $z \in D$ can be joined to $z_0$ by a rectifiable curve $\gamma: [0, d] \to D$, satisfying:

\[ (a) \gamma(0) = z, \quad \gamma(d) = z_0; \]

\[ (b) d \leq \beta; \]

\[ (c) \text{dist}(\gamma(s), \partial D) \geq \alpha \frac{s}{d} \quad (0 \leq s \leq d), \]

where $s$ is the arc-length parameter.
A domain $D \subset \mathbb{C}$ is said to be an $(\alpha, \beta)$-uniform domain, if for each pair of points $z_1, z_2 \in D$, $z_1 \neq z_2$, there is an $(\alpha|z_1 - z_2|, \beta|z_1, z_2|)$-John domain $G$ such that $z_1, z_2 \in G \subset D$.

$D$ is said to be a $K$-quasidisk if it is the image of a disk or half-plane under a $K$-quasiconformal mapping $f: \mathbb{C} \to \mathbb{C}$, where $\mathbb{C} = \mathbb{C} \cup \{\infty\}$. By [5], we know that if $D$ is a $K$-quasidisk, then $D$ is an $(\alpha, \beta)$-uniform domain for constants $\alpha$ and $\beta$ that depend only on $K$.

Quasidisks were characterized in [3] and [4] by the Hardy-Littlewood property (only for unbounded domains) and in [2] by the Schwarzian univalence criterion. In 1992, we obtained the following theorem.

**Theorem Z ([6]).** Suppose that $D$ is a quasidisk in $\mathbb{C}$. Then necessary and sufficient conditions for $f(z) \in H^2$ are that $f(z)$ is analytic in $D$ and satisfies

\begin{equation}
|f''(z)| = O(\text{dist}(z, \partial D)^{-1}), \quad z \in D.
\end{equation}

The sketch of the proof is as follows.

Suppose that $f(z) \in H^2$. For any $z \in D$, let $0 < r < \text{dist}(z, \partial D)/4$. Then $D_r = B(z, r) \subset D$. Choosing $z_1, z_2 \in D_r$ with $|z_1 - z_2| = 2r$,

\begin{equation}
D_r \subset B(z_1, 2r) \cup B(z_2, 2r) \subset D
\end{equation}

and

\begin{equation}
\overline{D_r} \setminus \{z_1, z_2\} \subset B(z_1, 2r) \cup B(z_2, 2r).
\end{equation}

Setting

\[P_1(f, z) = \frac{z_2 - z}{z_2 - z_1} f(z_1) + \frac{z - z_1}{z_2 - z_1} f(z_2),\]

we have

\begin{equation}
f(z) - P_1(f, z) = \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(\zeta) - P_1(f, \zeta)}{\zeta - z} d\zeta.
\end{equation}

It follows that

\begin{equation}
f''(z) = \frac{1}{\pi i} \int_{\partial D_r} \frac{f(\zeta) - P_1(f, \zeta)}{(\zeta - z)^3} d\zeta.
\end{equation}

By (1.1) we have

\begin{equation}
|f''(z)| \leq \frac{M_f}{\pi r^3} \int_{\partial D_r} \delta(\zeta) \log \frac{4r}{\delta(\zeta)} |d\zeta|,
\end{equation}

where $\delta(\zeta) = \min\{|\zeta - z_1|, |\zeta - z_2|\}$.

Set

\[\zeta = z + re^{i\theta}, \quad z_k = z + re^{i\theta_k}, \quad k = 1, 2.\]

Then

\[\delta(\zeta) = 2r \min \left\{\frac{\sin \theta - \theta_k}{2}\right\},\]

so

\[|f''(z)| \leq \frac{M_f}{\pi r^3} 16r^2 \int_0^{2\pi} \sin \frac{\theta}{2} \log \frac{2}{\sin \frac{\theta}{2}} d\theta \leq C_0 \frac{1}{r}.\]

Taking $r = \text{dist}(z, \partial D)/8$ yields (1.5).
In order to prove the sufficiency, let $z_1, z_2 \in D$ and $z_1 \neq z_2$. For any $z \in [B(z_1, h) \cup B(z_2, h)] \cap D$, let $\delta = \min\{|z_1 - z|, |z_2 - z|\}$. Since $D$ is an $(\alpha, \beta)$-uniform domain, for $k = 1, 2$ there exists an $(\alpha|z_k - z|, \beta|z_k, z|)$-John domain $D_k \subset D$ containing $z$ and $z_k$. Let $z_{k0}$ be the point in the definition of the $(\alpha|z_k - z|, \beta|z_k, z|)$-John domain $D_k$, $\gamma_{1k}$ the corresponding rectifiable arc joining $z$ to $z_{k0}$, and $\gamma_{2k}$ the arc joining $z_k$ to $z_{k0}$. Then we have

$$f(z) - P_1(f, z) = \frac{z_2 - z}{z_2 - z_1} \int_{\gamma_{1k}} (\zeta - z)f''(\zeta)d\zeta - \int_{\gamma_{2k}} (\zeta - z_1)f''(\zeta)d\zeta - \frac{z - z_1}{z_2 - z_1} \int_{\gamma_{2k}} (\zeta - z_2)f''(\zeta)d\zeta - \int_{\gamma_{1k}} (\zeta - z)f''(\zeta)d\zeta + \frac{(z_2 - z)(z - z_1)}{z_2 - z_1} [f'(z_{10}) - f'(z_{20})] = S_1 + S_2 + S_3.$$  

(1.11)

It is easy to see that

$$S_k = O(\delta), \quad k = 1, 2.$$  

(1.12)

We estimate $S_3$. By virtue of (c) in (1.4), we have

$$\text{dist}(z_{k0}, \partial D) \geq \alpha|z - z_k|, \quad k = 1, 2.$$  

(1.13)

In the case

$$|z_{10} - z_{20}| \leq \max_{k=1,2} \text{dist}(z_{k0}, \partial D)/2,$$

we may assume without loss of generality that $\text{dist}(z_{20}, \partial D) \geq \text{dist}(z_{10}, \partial D)$. Then the open disk $B_2$ with center $z_{20}$ and radius $\text{dist}(z_{20}, \partial D)/2$ is contained in $D$. Consequently, the distance of each point on $\overline{B_2}$ to $\partial D$ is not less than $\text{dist}(z_{20}, \partial D)/2$. Noting that $z_{10} \in \overline{B_2}$, let $\sigma$ be the segment from $z_{10}$ to $z_{20}$. Then we have

$$|f'(z_{10}) - f'(z_{20})| = \left| \int_\sigma f''(\zeta)d\zeta \right| = O(1).$$

Thus

$$S_3 = O(\delta).$$

In the case $|z_{10} - z_{20}| \geq \max_{k=1,2} \text{dist}(z_{k0}, \partial D)/2$, it is not too difficult to show that

$$|f'(z_{10}) - f'(z_{20})| = O \left( \log \frac{|z_{10} - z_{20}|}{\delta} \right).$$

It follows from (1.13) and (1.12) that

$$|f(z) - P_1(f, z)| \leq M\delta \log \frac{2h}{\delta}.$$  

This proves that $f(z) \in H^1_\alpha$.

**Definition 1.** Suppose that $D$ is a proper subdomain of $\mathbb{C}$. We say that $D$ has the Zygmund property if there exists a constant $M > 0$ such that $f(z) \in H^1_\alpha$ whenever $f(z)$ is analytic in $D$ and satisfies $|f''(z)| \leq M \text{dist}(z, \partial D)^{-1}$ in $D$.
From Theorem Z we know that quasidisks have the Zygmund property. In the present paper we show that, if an unbounded domain $C$ has the Zygmund property, it is a quasidisk too. We thus obtain a characterization of unbounded quasidisks by the Zygmund property.

**Theorem.** Suppose that $D$ is a simply connected domain in $\hat{C} = C \cup \{\infty\}$ with $\infty \in \partial D$ and that $D^* = \hat{C} \setminus \overline{D}$ is a domain. Then $D$ is a quasidisk if and only if both $D$ and $D^*$ have the Zygmund property.

2. SOME LEMMAS

**Lemma 1** ([3]). Suppose that $D$ is a simply connected subdomain of $C$ and that $z_0 \in C$. If there exist points in $D \cap \overline{B}(z_0, r)$ which cannot be joined in $D \cap \overline{B}(z_0, br)$, then there exist points $z_1, z_2 \in D \cap \overline{B}(z_0, r)$ and $w_0 \in \partial B(z_0, br) \setminus D$ such that

$$\left|h(z_1) - h(z_2) - 2\pi i\right| \leq \frac{2}{b - 1}$$

whenever $h(z)$ is an analytic branch of $\log(z - w_0)$ in $D$.

Under the conditions of Lemma 1, let $z_0, z_1, z_2$ and $w_0$ be points as indicated in its statements. Choose an arc $\gamma$ in $D$ from $z_1$ to $z_2$, and let $z'$ be the first point at which $\gamma$ meets $\partial B(z_1, \frac{d}{2})$ when $\gamma$ is traversed from $z_1$ to $z_2$, where $d = |z_1 - z_2|$. Denote by $\gamma'$ the subarc of $\gamma$ from $z_1$ to $z'$.

Given $h(z)$, an analytic branch of $\log(z - w_0)$ in $D$, let $h_0(z)$ be the analytic branch of $\log(z - w_0)$ in $B(z_0, br)$ satisfying

$$h_0(z_1) = h(z_1).$$

If $\sigma$ is the segment from $z_1$ to $z_2$, then plainly

$$\left|h_0(z_2) - h_0(z_1)\right| = \left|\int_{\sigma} h'_0(z)dz\right| \leq \int_{\sigma} |dz| \leq \frac{2}{b - 1}.$$

Since $\gamma' \subset D \cap B(z_1, \frac{d}{2}) \subset B(z_0, br)$, we infer from (2.2)

$$h(z') = h_0(z').$$

Both $h_0(z)$ and $h(z)$ are analytic branches of $\log(z - w_0)$ in some neighborhood of $z_2$; we thus have $h_0(z_2) - h(z_2) = 2k\pi i$. Using (2.1), (2.2) and (2.3) we conclude

$$\left|h_0(z_2) - h(z_2) - 2\pi i\right| \leq \frac{4}{b - 1} \leq \frac{4}{3},$$

$$h_0(z_2) - h(z_2) = 2\pi i.$$

For $z \in B(z_0, br)$, let $f_0(z) = (z - w_0)h_0(z)$.

**Lemma 2.** Under the conditions of Lemma 1 and with $z_0, z_1, z_2, w_0$ and $z'$ as indicated in the statement of the lemma and the ensuing discussion, it is the case that

$$\left|f_0(z') - P_1(f_0, z')\right| \leq \frac{3d}{b - 2}.$$
Proof. Let \( \sigma_1 \) be the segment from \( z_1 \) to \( z' \), and \( \sigma_2 \) the segment from \( z_2 \) to \( z' \). We compute
\[
|f_0(z') - P_1(f_0, z')| = \left| \frac{z_2 - z'}{z_2 - z_1} [f_0(z') - f_0(z_1)] + \frac{z' - z_1}{z_2 - z_1} [f_0(z') - f_0(z_2)] \right|
\]
\[
= \left| \frac{z_2 - z'}{z_2 - z_1} \int_{\sigma_1} (z - z_1) f''_0(z)dz + \frac{z' - z_1}{z_2 - z_1} \int_{\sigma_2} (z - z_2) f''_0(z)dz \right|
\]
\[
\leq \left| \frac{z_2 - z'}{z_2 - z_1} \int_{\sigma_1} |z - z_1| \frac{|dz|}{|z - w_0|} + \frac{z' - z_1}{z_2 - z_1} \int_{\sigma_2} |z - z_2| \frac{|dz|}{|z - w_0|} \right|
\]
\[
\leq \frac{3}{2} |z_1 - z'| \frac{|z_1 - z'|}{(b-2)r} + \frac{|z_1 - z'| |z' - z_2| |z_2 - z'|}{|z_2 - z_1| (b-2)r}
\]
\[
\leq \frac{6}{b-2} |z_1 - z'| \leq \frac{3d}{b-2}.
\]

\( \square \)

Definition 2 ([2]). A set \( E \) in \( \hat{\mathbb{C}} \) is said to be \( a \)-locally connected if for all \( z_0 \in \mathbb{C} \) and \( r > 0 \), any pair of points in \( E \cap \overline{B}(z_0, r) \) can be joined in \( E \cap \overline{B}(z_0, ar) \) and any pair of points in \( E \setminus B(z_0, r) \) can be joined in \( E \setminus B(z_0, \frac{r}{3}) \).

Lemma 3 ([2]). Suppose that a domain \( D \) in \( \mathbb{C} \) is \( a \)-locally connected and that \( \partial D \) is connected and contains at least two points. Then \( \partial D \) is a \( K \)-quasiconformal circle, where \( K \) depends only on \( a \).

3. Proof of the Theorem

The necessity of both \( D \) and \( D^* \) having the Zygmund property is ensured by Theorem Z. We must treat the sufficiency.

As in [2], we only need to prove the following proposition.

Proposition. Suppose that \( D \) is a simply connected proper subdomain of \( \mathbb{C} \) which has the Zygmund property. Then there exists a constant \( b > 4 \), which depends only on the constant \( M \) in Definition 1, such that for all \( z_0 \in \mathbb{C} \) and \( r > 0 \), each pair of points in \( D \cap \overline{B}(z_0, r) \) can be joined in \( D \cap \overline{B}(z_0, br) \).

Proof. Choose
\[
b = \frac{M \log 4 + 8}{\pi} + 2,
\]
and suppose the conclusion does not hold for some \( z_0 \in \mathbb{C} \) and \( r > 0 \). Fix points \( z_1, z_2 \) in \( D \cap \overline{B}(z_0, r) \) which cannot be joined in \( D \cap \overline{B}(z_0, br) \), and let \( z' \) and \( w_0 \) be as in section 2.

Consider \( h(z) \) an analytic branch of \( \log(z - w_0) \) in \( D \), along with the function
\[
f(z) = (z - w_0)h(z).
\]
Then \( f(z) \) is analytic in \( D \), where it satisfies
\[
|f''(z)| = \frac{1}{|z - w_0|} \leq \text{dist}(z, \partial D).
\]
The hypothesis on $D$ implies that

\begin{equation}
|f(z) - P_1(f, z)| \leq M \delta \log \frac{2d}{\delta}
\end{equation}

holds for all $z \in D \cap [B(z_1, d) \cup B(z_2, d)]$. In particular, (3.4) holds for the point $z' \in D \cap [B(z_1, d) \cup B(z_2, d)]$. Noting $\delta = |z' - z_1| = |z' - z_2| = \frac{d}{2}$, we have

\begin{equation}
|f(z') - P_1(f, z')| \leq \frac{Md \log 4}{2}.
\end{equation}

Because $f(z_1) = f_0(z_1), f(z') = f_0(z')$ and $f(z_2) = f_0(z_2) + 2\pi i(z_2 - w_0)$, we obtain

\begin{align*}
|f(z') - P_1(f, z')| &= |f_0(z') - P_1(f_0, z')| + P_1(f - f_0, z')| \\
&\geq \left| \frac{z' - z_1}{z_2 - z_1} f_0(z_2) - f_0(z_2) \right| - |f_0(z') - P_1(f_0, z')| \\
&= \pi |z_2 - w_0| - |f_0(z') - P_1(f_0, z')|.
\end{align*}

By (2.6) and (3.1), we conclude that

\begin{align*}
|f(z') - P_1(f, z')| &\geq \pi (b - 2) r - \frac{3d}{b - 2} \\
&\geq (M \log 4 + 8) d - \frac{3\pi d M}{\log 4 + 8} \\
&\geq Md \log 4,
\end{align*}

which contradicts (3.5).

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