

A RESULT ON DERIVATIONS

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ABSTRACT. Let R be a semiprime ring with a derivation d and let U be a Lie ideal of R , $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed positive integer. Then $ad(I) = 0$ for I the ideal of R generated by $[U, U]$ and if R is 2-torsion free, then $ad(U) = 0$. Furthermore, R is a subdirect sum of semiprime homomorphic images R_1 and R_2 with derivations d_1 and d_2 , induced canonically by d , respectively such that $\bar{a}d_1(R_1) = 0$ and the image of U in R_2 is commutative (central if R is 2-torsion free), where \bar{a} denotes the image of a in R_1 . Moreover, if $U = R$, then $ad(R) = 0$. This gives Brešar's theorem without the $(n - 1)!$ -torsion free assumption on R .

In [8] I. N. Herstein proved that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n a fixed integer, then $d = 0$. In [6] A. Giambruno and I. N. Herstein extended this result to arbitrary derivations in semiprime rings. In [2] L. Carini and A. Giambruno proved that if R is a prime ring with a derivation d such that $d(x)^{n(x)} = 0$ for all $x \in U$, a Lie ideal of R , then $d(U) = 0$ when R has no nonzero nil right ideals, $\text{char } R \neq 2$ and the same conclusion holds when $n(x) = n$ is fixed and R is a 2-torsion free semiprime ring. Using the ideas in [2] and the methods in [5] C. Lanski [11] removed both the bound on the indices of nilpotence and the characteristic assumptions on R .

In [1] M. Brešar gave a generalization of the result due to I. N. Herstein and A. Giambruno [6] in another direction. Explicitly, he proved the theorem: Let R be a semiprime ring with a derivation d , $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$ when R is an $(n - 1)!$ -torsion free ring. The present paper is then motivated by Brešar's result and by Lanski's paper [11]. We prove Brešar's result without the assumption of $(n - 1)!$ -torsion free on R . In fact, we study the Lie ideal case as given in [11] and then obtain Brešar's result as the corollary to our main result. More precisely, we shall prove the following

Main Theorem. *Let R be a semiprime ring with a derivation d and let U be a Lie ideal of R , $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed integer. Then $ad(I) = 0$ for I the ideal of R generated by $[U, U]$ and if R is 2-torsion free, then $ad(U) = 0$.*

Furthermore, R is a subdirect sum of semiprime homomorphic images R_1 and R_2 with derivations d_1 and d_2 , induced canonically by d , respectively such that

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$\bar{a}d_1(R_1) = 0$ and the image of U in R_2 is commutative (central if R is 2-torsion free), where \bar{a} denotes the image of a in R_1 .

Corollary. *Let R be a semiprime ring with a derivation d and $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$.*

Throughout this paper let R be a semiprime ring, U a Lie ideal of R , Z the center of R , C the extended centroid of R and d a derivation of R . Given two elements $a, b \in R$, $[a, b]$ will denote the element $ab - ba$; also, for two subsets A and B of R , $[A, B]$ is then the additive subgroup of R generated by all elements $[a, b]$ with $a \in A$, $b \in B$. For any subset S of R , denote by $r_R(S)$ the right annihilator of S in R , that is, $r_R(S) = \{x \in R \mid Sx = 0\}$ and $l_R(S)$ is defined similarly. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $\text{ann}_R(S)$. Note that it is easy to check that every annihilator ideal of the semiprime ring R is invariant under all derivations of R . This fact will be used in our proofs.

We begin this paper with the following key result.

Theorem 1. *Let R be a prime ring with a derivation d and let U be a Lie ideal of R , $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed integer. Then*

- (i) $ad(U) = 0$ unless $\text{char } R = 2$ and $\dim_C RC = 4$.
- (ii) $ad(R) = 0$ if $[U, U] \neq 0$.

Proof. We first prove (i). If U is central, then $d(U) \subseteq Z$ since $d(Z) \subseteq Z$. Since every nonzero central element in the prime ring R is not a zero-divisor, we have $ad(U) = 0$ by our assumption. Assume next that U is a noncentral Lie ideal of R . We assume further that either $\text{char } R \neq 2$ or $\dim_C RC > 4$. By [10, Theorem 13], $[U, U] \neq 0$ and $0 \neq [I, R] \subseteq U$, where I is the ideal of R generated by $[U, U]$. In this case we want to prove $ad(R) = 0$. Suppose on the contrary that $ad(R) \neq 0$.

By our assumption we have $ad([x, y])^n = 0$ for all $x, y \in I$. Thus

$$a([d(x), y] + [x, d(y)])^n = 0$$

for all $x, y \in I$. Denote by Q the two-sided Martindale quotient ring of R . If d is not inner on Q , applying Kharchenko's theorem [9] we have $a([z, y] + [x, w])^n = 0$ for all x, y, z, w , in R . In particular, $a[x, y]^n = 0$ for all $x, y \in R$. Let $x, y, z, t \in R$ and set $u = [x, y]^n$. Then $0 = a[z, uta]^n = a(zuta)^n$ since $au = 0$ and hence $(azut)^{n+1} = 0$. Applying Levitzki's lemma [7, Lemma 1.1] we have $azu = 0$. By the primeness of R it follows that $u = 0$ since $a \neq 0$. That is, $[x, y]^n = 0$ for all $x, y \in R$. Now by [6] or by an easy computation we can conclude that R is commutative, a contradiction. Thus we may always assume that d is Q -inner. So there exists an element $b \in Q$ such that $d(x) = [b, x]$ for all $x \in R$. Since by [3] Q and I satisfy the same generalized polynomial identities (or GPIs in brief), we have $a[b, [x, y]]^n = 0$ for all $x, y \in Q$. Also, since Q remains prime by the primeness of R , replacing R by Q we may assume that $b \in R$ and C is just the center of R . Note that R is a centrally closed prime C -algebra in the present situation [4]. We divide the proof into two cases.

Case 1. Assume that R satisfies a nonzero GPI.

In this case, since R is a centrally closed prime C -algebra, by Martindale's theorem [12] R is a strongly primitive ring. Let ${}_R V$ be a faithful irreducible left R -module with commuting ring $D = \text{End}({}_R V)$. By the density theorem, R acts

densely on V_D . For any given $v \in V$ we claim that v and bv are D -dependent. Assume first that $av \neq 0$. Suppose on the contrary that v and bv are D -independent. By the density of R in $\text{End}(V_D)$ there exist two elements x and y in R such that

$$xv = 0, \quad xbv = bv \quad \text{and} \quad yv = 0, \quad ybv = v.$$

Then $0 = a(b[x, y] - [x, y]b)^n v = av$, a contradiction. Thus v and bv are D -dependent as claimed. Assume next that $av = 0$. Since $a \neq 0$, we have $aw \neq 0$ for some $w \in V$. Then $a(v + w) = aw \neq 0$. Applying the first situation we have

$$bw = w\alpha \quad \text{and} \quad b(v + w) = (v + w)\beta$$

for some $\alpha, \beta \in D$. But v and w are clearly D -independent, and so there exist two elements x, y in R such that

$$xw = 0, \quad xv = v + w \quad \text{and} \quad yw = v + w, \quad yv = v.$$

Then $[x, y]w = v + w$ and $[x, y](v + w) = w$. Hence

$$0 = a(b[x, y] - [x, y]b)^n w = \pm aw(\beta - \alpha)^n,$$

which implies $\alpha = \beta$ and hence $bv = v\alpha$ as claimed.

From the above we have proved that $bv = v\alpha(v)$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. That is, there exists $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. We claim $\delta \in Z(D)$, the center of D . Indeed, if $\beta \in D$, then $b(v\beta) = (v\beta)\delta = v(\beta\delta)$ and on the other hand $b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta)$. Therefore $v(\beta\delta - \delta\beta) = 0$. So $\beta\delta = \delta\beta$, which implies $\delta \in Z(D)$. So $b \in C$ and hence $d = 0$, a contradiction.

Case 2. Assume that R does not satisfy any nonzero GPI.

Denote by $C\{X, Y, \dots\}$ the free C -algebra with indeterminates X, Y, \dots , and by $Q *_C C\{X, Y, \dots\}$ the free product over C of the C -algebra Q and the free C -algebra $C\{X, Y, \dots\}$. Since $a(b[x, y] - [x, y]b)^n = 0$ for all $x, y \in R$, we see that

$$a(b[X, Y] - [X, Y]b)^n = 0 \quad \text{in } Q *_C C\{X, Y, \dots\}$$

since R has no nonzero GPI. Expanding this we see that

$$(1) \quad ab[X, Y](b[X, Y] - [X, Y]b)^{n-1} - a[X, Y]b(b[X, Y] - [X, Y]b)^{n-1} = 0.$$

Suppose for the moment that ab and a are C -independent. Then (1) implies

$$ab[X, Y](b[X, Y] - [X, Y]b)^{n-1} = 0 = a[X, Y]b(b[X, Y] - [X, Y]b)^{n-1}.$$

Since $0 = a[X, Y](b^2[X, Y] - b[X, Y]b)(b[X, Y] - [X, Y]b)^{n-2}$, we have $b^2 = \mu b$ for some $\mu \in C$; otherwise we get $a([X, Y]b)^n = 0$ and hence either $a = 0$ or $b = 0$, a contradiction. Note that by $b^2 = \mu b$ we have

$$b[b, [X, Y]] = b[X, Y](\mu - b) \quad \text{and} \quad (b - \mu)[b, [X, Y]] = (\mu - b)[X, Y]b.$$

Thus

$$0 = a[X, Y]b(b[X, Y] - [X, Y]b)^{n-1} = a[X, Y]b[X, Y](b - \mu)[X, Y]b \cdots [X, Y]t,$$

where its degree in $[X, Y]$ is n and $t = b$ or $(\mu - b)$. Clearly, this implies $b = \mu$, a contradiction.

Next assume that ab and a are C -dependent. Thus $ab = \beta a$ for some $\beta \in C$. Then $a(b - \beta) = 0$. But b and $b - \beta$ induce the same inner derivation of R , replacing b by $b - \beta$ we may assume from the start that $ab = 0$. So we have

$a[X, Y]b(b[X, Y] - [X, Y]b)^{n-1} = 0$. As before, the same argument implies $b \in C$, a contradiction. This gives the proof of (i).

For (ii), since $[U, U] \neq 0$, it follows from the proof of (i) that $ad(R) = 0$. This completes the proof of the theorem.

We are in a position to prove the Main Theorem.

Proof of the Main Theorem. Let P be a prime ideal of R such that $[U, U] \not\subseteq P$ and set $\overline{R} = R/P$. Assume first that $d(P) \subseteq P$. Then d induces a canonical derivation \overline{d} on \overline{R} . By the assumption, $\overline{ad}(\overline{u})^n = 0$ for all $\overline{u} \in \overline{U}$. Note that \overline{U} is a Lie ideal of \overline{R} such that $[\overline{U}, \overline{U}] \neq 0$ since $[U, U] \not\subseteq P$. It follows from Theorem 1 that $\overline{ad}(\overline{R}) = 0$, that is, $ad(R) \subseteq P$.

Assume next that $d(P) \not\subseteq P$. Then for any $t \in P, u \in U$ we have $0 = ad([t, u])^n = a([d(t), u] + [t, d(u)])^n$ and hence $\overline{a}[\overline{d(t)}, \overline{u}]^n = 0$ in \overline{R} . Applying Theorem 1 again we have $\overline{a}[\overline{d(t)}, \overline{R}] = 0$, that is, $a[d(t), R] \subseteq P$. Thus $a[d(P), R] \subseteq P$. In particular, $a[d(PR), R] \subseteq P$ and hence $a[d(P)R, R] \subseteq P$. So $ad(P)[R, R] \subseteq P$ and hence $ad(P)[R^2, R] \subseteq P$, which implies either $ad(P) \subseteq P$ or $[R, R] \subseteq P$. Since $d(P) \not\subseteq P$ and $[U, U] \not\subseteq P$, this implies $a \in P$. In particular, $ad(R) \subseteq P$.

Up to now we have proved that for any prime ideal P of R either $ad(R) \subseteq P$ or $[U, U] \subseteq P$.

Recall that I is the ideal of R generated by $[U, U]$. Then $ad(R)I \subseteq P$ for all prime ideals P of R . Thus $ad(R)I = 0$ by the semiprimeness of R . Set $J = \text{ann}_R(\text{ann}_R(I))$. Clearly, $ad(R)J = 0$. Recall that $d(J) \subseteq J$. Thus $ad(J) \subseteq J \cap \text{ann}_R(J) = 0$. Since $I \subseteq J$, we have $ad(I) = 0$ as desired.

Assume that R is a semiprime 2-torsion free ring. Then $\bigcap P = 0$, where the intersection is taken over all prime ideals P of R such that R/P is 2-torsion free. Let P be a prime ideal of R such that R/P is 2-torsion free. If $d(P) \subseteq P$, then d induces a canonical derivation on R/P and hence, by Theorem 1, $ad(U) \subseteq P$ follows. Suppose that $d(P) \not\subseteq P$. Then by the preceding argument either $ad(R) \subseteq P$ or $[U, U] \subseteq P$. If $ad(R) \subseteq P$, then we are done. Suppose that $[U, U] \subseteq P$. Then $[\overline{U}, \overline{U}] = 0$ in $\overline{R} = R/P$. Since $\text{char } \overline{R} \neq 2$, we must have that \overline{U} is central by [10, Theorem 4]. That is, $[U, R] \subseteq P$. Let $u \in U$ and $x \in R$. Then $0 = ad([u, x])^n = a([u, d(x)] + [d(u), x])^n$ and so $\overline{a}[\overline{d(u)}, \overline{x}]^n = 0$ in \overline{R} . By Theorem 1(ii), $\overline{a}[\overline{d(u)}, \overline{R}] = 0$. Therefore, $a[d(U), R] \subseteq P$. In particular, $a[d(U), R^2] \subseteq P$ and hence $aR[d(U), R] \subseteq P$. We may assume that $a \notin P$. Therefore, $[d(U), R] \subseteq P$. That is, $\overline{d(U)}$ is central in \overline{R} . By the assumption, $\overline{ad}(u)^n = 0$ for all $u \in U$. Since $\overline{d(u)}$ is central, this implies $\overline{ad}(u) = 0$. So $ad(U) \subseteq P$ as desired. Thus we have proved that $ad(U) \subseteq P$ when $\text{char } R/P \neq 2$. Consequently, $ad(U) \subseteq \bigcap P$, where P runs over all prime ideals of R such that R/P is 2-torsion free. Therefore $ad(U) = 0$, since $\bigcap P = 0$.

It remains to prove the last statement of the theorem. Denote by A the intersection of all prime ideals P of R with the property that $ad(R) \subseteq P$ and by B the intersection of all prime ideals P with the property that $[U, U] \subseteq P$. We follow the argument of Lanski [11]. Since $AB = 0$ and $\text{ann}_R(\text{ann}_R(A)) \text{ann}_R(A) = 0$, we obtain that $ad(R) \subseteq A \subseteq \text{ann}_R(\text{ann}_R(A))$ and $[U, U] \subseteq B \subseteq \text{ann}_R(A)$. Set $R_1 = R/\text{ann}_R(\text{ann}_R(A))$ and $R_2 = R/\text{ann}_R(A)$. Then d induces derivations d_1 and d_2 on R_1 and R_2 , respectively, since $\text{ann}_R(\text{ann}_R(A))$ and $\text{ann}_R(A)$ are d -invariant. Now it is clear that $\overline{ad}_1(R_1) = 0$ and the image of U in R_2 is commutative. Finally, if R is 2-torsion free, then we can construct the subdirect sum R of R_1 and R_2 such

that the image of U in R_2 is central. Indeed, in this case we can take only these prime ideals P of R with $\text{char } R/P \neq 2$ in the construction of A and B . By [10, Theorem 4], if $[U, U] \subseteq P$, then $[U, R] \subseteq P$ and hence $[U, R] \subseteq B$. This implies the image of U in R_2 is central. This finishes the proof of the Main Theorem.

We conclude this paper with proving the Corollary, a generalization of Brešar's result.

Proof of the corollary. By the assumption, $ad(x)^n = 0$ for all $x \in R$. By the Main Theorem, $ad(R)[R, R] = 0$. In particular, $ad(R)[R^2, R] = 0$ and hence $ad(R)R[R, R] = 0$. Therefore,

$$[R, ad(R)]R[R, ad(R)] = 0,$$

which implies $ad(R) \subseteq Z$. Let $x \in R$. Then $(ad(x))^n = a^n d(x)^n = a^{n-1}(ad(x)^n) = 0$, since $ad(x)ad(x) = a(ad(x))d(x) = a^2 d(x)^2$. However, since Z is a reduced ring, we have $ad(x) = 0$. That is, $ad(R) = 0$ as desired.

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