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A RESULT ON DERIVATIONS

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ABSTRACT. Let R be a semiprime ring with a derivation d and let U be a Lie ideal of R, $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed positive integer. Then $ad(I) = 0$ for I the ideal of R generated by $[U, U]$ and if R is 2-torsion free, then $ad(U) = 0$. Furthermore, R is a subdirect sum of semiprime homomorphic images R_1 and R_2 with derivations d_1 and d_2, induced canonically by d, respectively such that $\overline{a}d_1(R_1) = 0$ and the image of U in R_2 is commutative (central if R is 2-torsion free), where $\overline{a}$ denotes the image of a in R_1. Moreover, if $U = R$, then $ad(R) = 0$. This gives Brešar's theorem without the $(n-1)!$-torsion free assumption on R.

In [8] I. N. Herstein proved that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n a fixed integer, then $d = 0$. In [6] A. Giambruno and I. N. Herstein extended this result to arbitrary derivations in semiprime rings. In [2] L. Carini and A. Giambruno proved that if R is a prime ring with a derivation d such that $d(x)^{n(x)} = 0$ for all $x \in U$, a Lie ideal of R, then $d(U) = 0$ when R has no nonzero nil right ideals, $\operatorname{char} R \neq 2$ and the same conclusion holds when $n(x) = n$ is fixed and R is a 2-torsion free semiprime ring. Using the ideas in [2] and the methods in [5] C. Lanski [11] removed both the bound on the indices of nilpotence and the characteristic assumptions on R.

In [1] M. Brešar gave a generalization of the result due to I. N. Herstein and A. Giambruno [6] in another direction. Explicitly, he proved the theorem: Let R be a semiprime ring with a derivation d, $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$ when R is an $(n-1)!$-torsion free ring. The present paper is then motivated by Brešar's result and by Lanski's paper [11]. We prove Brešar's result without the assumption of $(n-1)!$-torsion free on R. In fact, we study the Lie ideal case as given in [11] and then obtain Brešar's result as the corollary to our main result. More precisely, we shall prove the following

Main Theorem. *Let R be a semiprime ring with a derivation d and let U be a Lie ideal of R, $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed integer. Then $ad(I) = 0$ for I the ideal of R generated by $[U, U]$ and if R is 2-torsion free, then $ad(U) = 0$.*

Furthermore, R is a subdirect sum of semiprime homomorphic images R_1 and R_2 with derivations d_1 and d_2, induced canonically by d, respectively such that

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$\overline{a}d_1(R_1) = 0$ and the image of U in R_2 is commutative (central if R is 2-torsion free), where $\overline{a}$ denotes the image of a in R_1.

Corollary. *Let R be a semiprime ring with a derivation d and $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed integer, then $ad(R) = 0$.*

Throughout this paper let R be a semiprime ring, U a Lie ideal of R, Z the center of R, C the extended centroid of R and d a derivation of R. Given two elements $a, b \in R$, $[a, b]$ will denote the element $ab - ba$; also, for two subsets A and B of R, $[A, B]$ is then the additive subgroup of R generated by all elements $[a, b]$ with $a \in A$, $b \in B$. For any subset S of R, denote by $r_R(S)$ the right annihilator of S in R, that is, $r_R(S) = \{x \in R | Sx = 0\}$ and $l_R(S)$ is defined similarly. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $\mathrm{ann}_R(S)$. Note that it is easy to check that every annihilator ideal of the semiprime ring R is invariant under all derivations of R. This fact will be used in our proofs.

We begin this paper with the following key result.

Theorem 1. *Let R be a prime ring with a derivation d and let U be a Lie ideal of R, $a \in R$. Suppose that $ad(u)^n = 0$ for all $u \in U$, where n is a fixed integer. Then*
(i) *$ad(U) = 0$ unless $\mathrm{char}\, R = 2$ and $\dim_C RC = 4$.*
(ii) *$ad(R) = 0$ if $[U, U] \neq 0$.*

Proof. We first prove (i). If U is central, then $d(U) \subseteq Z$ since $d(Z) \subseteq Z$. Since every nonzero central element in the prime ring R is not a zero-divisor, we have $ad(U) = 0$ by our assumption. Assume next that U is a noncentral Lie ideal of R. We assume further that either $\mathrm{char}\, R \neq 2$ or $\dim_C RC > 4$. By [10, Theorem 13], $[U, U] \neq 0$ and $0 \neq [I, R] \subseteq U$, where I is the ideal of R generated by $[U, U]$. In this case we want to prove $ad(R) = 0$. Suppose on the contrary that $ad(R) \neq 0$.

By our assumption we have $ad([x, y])^n = 0$ for all $x, y \in I$. Thus

$$a([d(x), y] + [x, d(y)])^n = 0$$

for all $x, y \in I$. Denote by Q the two-sided Martindale quotient ring of R. If d is not inner on Q, applying Kharchenko's theorem [9] we have $a([z, y] + [x, w])^n = 0$ for all x, y, z, w, in R. In particular, $a[x, y]^n = 0$ for all $x, y \in R$. Let $x, y, z, t \in R$ and set $u = [x, y]^n$. Then $0 = a[z, uta]^n = a(zuta)^n$ since $au = 0$ and hence $(azut)^{n+1} = 0$. Applying Levitzki's lemma [7, Lemma 1.1] we have $azu = 0$. By the primeness of R it follows that $u = 0$ since $a \neq 0$. That is, $[x, y]^n = 0$ for all $x, y \in R$. Now by [6] or by an easy computation we can conclude that R is commutative, a contradiction. Thus we may always assume that d is Q-inner. So there exists an element $b \in Q$ such that $d(x) = [b, x]$ for all $x \in R$. Since by [3] Q and I satisfy the same generalized polynomial identities (or GPIs in brief), we have $a[b, [x, y]]^n = 0$ for all $x, y \in Q$. Also, since Q remains prime by the primeness of R, replacing R by Q we may assume that $b \in R$ and C is just the center of R. Note that R is a centrally closed prime C-algebra in the present situation [4]. We divide the proof into two cases.

Case 1. Assume that R satisfies a nonzero GPI.

In this case, since R is a centrally closed prime C-algebra, by Martindale's theorem [12] R is a strongly primitive ring. Let ${}_RV$ be a faithful irreducible left R-module with commuting ring $D = \mathrm{End}({}_RV)$. By the density theorem, R acts densely on V_D. For any given $v \in V$ we claim that v and bv are D-dependent. Assume first that $av \neq 0$. Suppose on the contrary that v and bv are D-independent. By the density of R in $\mathrm{End}(V_D)$ there exist two elements x and y in R such that

$$xv = 0, \quad xbv = bv \qquad \text{and} \qquad yv = 0, \quad ybv = v.$$

Then $0 = a(b[x,y] - [x,y]b)^n v = av$, a contradiction. Thus v and bv are D-dependent as claimed. Assume next that $av = 0$. Since $a \neq 0$, we have $aw \neq 0$ for some $w \in V$. Then $a(v+w) = aw \neq 0$. Applying the first situation we have

$$bw = w\alpha \quad \text{and} \quad b(v+w) = (v+w)\beta$$

for some $\alpha, \beta \in D$. But v and w are clearly D-independent, and so there exist two elements x, y in R such that

$$xw = 0, \quad xv = v + w \qquad \text{and} \qquad yw = v + w, \quad yv = v.$$

Then $[x,y]w = v + w$ and $[x,y](v+w) = w$. Hence

$$0 = a(b[x,y] - [x,y]b)^n w = \pm aw(\beta - \alpha)^n,$$

which implies $\alpha = \beta$ and hence $bv = v\alpha$ as claimed.

From the above we have proved that $bv = v\alpha(v)$ for all $v \in V$, where $\alpha(v) \in D$ depends on $v \in V$. In fact, it is easy to check that $\alpha(v)$ is independent of the choice of $v \in V$. That is, there exists $\delta \in D$ such that $bv = v\delta$ for all $v \in V$. We claim $\delta \in Z(D)$, the center of D. Indeed, if $\beta \in D$, then $b(v\beta) = (v\beta)\delta = v(\beta\delta)$ and on the other hand $b(v\beta) = (bv)\beta = (v\delta)\beta = v(\delta\beta)$. Therefore $v(\beta\delta - \delta\beta) = 0$. So $\beta\delta = \delta\beta$, which implies $\delta \in Z(D)$. So $b \in C$ and hence $d = 0$, a contradiction.

Case 2. Assume that R does not satisfy any nonzero GPI.

Denote by $C\{X, Y, \dots\}$ the free C-algebra with indeterminates $X, Y, \dots$, and by $Q *_C C\{X, Y, \dots\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X, Y, \dots\}$. Since $a(b[x,y] - [x,y]b)^n = 0$ for all $x, y \in R$, we see that

$$a(b[X,Y] - [X,Y]b)^n = 0 \quad \text{in } Q *_C C\{X, Y, \dots\}$$

since R has no nonzero GPI. Expanding this we see that

$$ab[X,Y](b[X,Y] - [X,Y]b)^{n-1} - a[X,Y]b(b[X,Y] - [X,Y]b)^{n-1} = 0. \tag{1}$$

Suppose for the moment that ab and a are C-independent. Then (1) implies

$$ab[X,Y](b[X,Y] - [X,Y]b)^{n-1} = 0 = a[X,Y]b(b[X,Y] - [X,Y]b)^{n-1}.$$

Since $0 = a[X,Y](b^2[X,Y] - b[X,Y]b)(b[X,Y] - [X,Y]b)^{n-2}$, we have $b^2 = \mu b$ for some $\mu \in C$; otherwise we get $a([X,Y]b)^n = 0$ and hence either $a = 0$ or $b = 0$, a contradiction. Note that by $b^2 = \mu b$ we have

$$b[b, [X,Y]] = b[X,Y](\mu - b) \quad \text{and} \quad (b - \mu)[b, [X,Y]] = (\mu - b)[X,Y]b.$$

Thus

$$0 = a[X,Y]b(b[X,Y] - [X,Y]b)^{n-1} = a[X,Y]b[X,Y](b-\mu)[X,Y]b \cdots [X,Y]t,$$

where its degree in $[X,Y]$ is n and $t = b$ or $(\mu - b)$. Clearly, this implies $b = \mu$, a contradiction.

Next assume that ab and a are C-dependent. Thus $ab = \beta a$ for some $\beta \in C$. Then $a(b - \beta) = 0$. But b and $b - \beta$ induce the same inner derivation of R, replacing b by $b - \beta$ we may assume from the start that $ab = 0$. So we have

$a[X,Y]b(b[X,Y]-[X,Y]b)^{n-1}=0$. As before, the same argument implies $b\in C$, a contradiction. This gives the proof of (i).

For (ii), since $[U,U]\neq 0$, it follows from the proof of (i) that $ad(R)=0$. This completes the proof of the theorem.

We are in a position to prove the Main Theorem.

Proof of the Main Theorem. Let P be a prime ideal of R such that $[U,U]\nsubseteq P$ and set $\overline{R}=R/P$. Assume first that $d(P)\subseteq P$. Then d induces a canonical derivation $\overline{d}$ on $\overline{R}$. By the assumption, $\overline{a}\overline{d}(\overline{u})^n=0$ for all $\overline{u}\in\overline{U}$. Note that $\overline{U}$ is a Lie ideal of $\overline{R}$ such that $[\overline{U},\overline{U}]\neq 0$ since $[U,U]\nsubseteq P$. It follows from Theorem 1 that $\overline{a}\overline{d}(\overline{R})=0$, that is, $ad(R)\subseteq P$.

Assume next that $d(P)\nsubseteq P$. Then for any $t\in P, u\in U$ we have $0=ad([t,u])^n=a([d(t),u]+\underline{[t,d(u)]})^n$ and hence $\overline{a}[\overline{d(t)},\overline{u}]^n=0$ in $\overline{R}$. Applying Theorem 1 again we have $\overline{a}[\overline{d(t)},\overline{R}]=0$, that is, $a[d(t),R]\subseteq P$. Thus $a[d(P),R]\subseteq P$. In particular, $a[d(PR),R]\subseteq P$ and hence $a[d(P)R,R]\subseteq P$. So $ad(P)[R,R]\subseteq P$ and hence $ad(P)[R^2,R]\subseteq P$, which implies either $ad(P)\subseteq P$ or $[R,R]\subseteq P$. Since $d(P)\nsubseteq P$ and $[U,U]\nsubseteq P$, this implies $a\in P$. In particular, $ad(R)\subseteq P$.

Up to now we have proved that for any prime ideal P of R either $ad(R)\subseteq P$ or $[U,U]\subseteq P$.

Recall that I is the ideal of R generated by $[U,U]$. Then $ad(R)I\subseteq P$ for all prime ideals P of R. Thus $ad(R)I=0$ by the semiprimeness of R. Set $J=\operatorname{ann}_R(\operatorname{ann}_R(I))$. Clearly, $ad(R)J=0$. Recall that $d(J)\subseteq J$. Thus $ad(J)\subseteq J\cap\operatorname{ann}_R(J)=0$. Since $I\subseteq J$, we have $ad(I)=0$ as desired.

Assume that R is a semiprime 2-torsion free ring. Then $\bigcap P=0$, where the intersection is taken over all prime ideals P of R such that R/P is 2-torsion free. Let P be a prime ideal of R such that R/P is 2-torsion free. If $d(P)\subseteq P$, then d induces a canonical derivation on R/P and hence, by Theorem 1, $ad(U)\subseteq P$ follows. Suppose that $d(P)\nsubseteq P$. Then by the preceding argument either $ad(R)\subseteq P$ or $[U,U]\subseteq P$. If $ad(R)\subseteq P$, then we are done. Suppose that $[U,U]\subseteq P$. Then $[\overline{U},\overline{U}]=0$ in $\overline{R}=R/P$. Since $\operatorname{char}\overline{R}\neq 2$, we must have that $\overline{U}$ is central by [10, Theorem 4]. That is, $[U,R]\subseteq P$. Let $u\in U$ and $x\in R$. Then $0=ad([u,x])^n=a([u,d(x)]+[d(u),x])^n$ and so $\overline{a}[\overline{d(u)},\overline{x}]^n=0$ in $\overline{R}$. By Theorem 1(ii), $\overline{a}[\overline{d(u)},\overline{R}]=0$. Therefore, $a[d(U),R]\subseteq P$. In particular, $a[d(U),R^2]\subseteq P$ and hence $aR[d(U),R]\subseteq P$. We may assume that $a\notin P$. Therefore, $[d(U),R]\subseteq P$. That is, $\overline{d(U)}$ is central in $\overline{R}$. By the assumption, $\overline{a}\overline{d(u)^n}=0$ for all $u\in U$. Since $\overline{d(u)}$ is central, this implies $\overline{a}\overline{d(u)}=0$. So $ad(U)\subseteq P$ as desired. Thus we have proved that $ad(U)\subseteq P$ when $\operatorname{char}R/P\neq 2$. Consequently, $ad(U)\subseteq\bigcap P$, where P runs over all prime ideals of R such that R/P is 2-torsion free. Therefore $ad(U)=0$, since $\bigcap P=0$.

It remains to prove the last statement of the theorem. Denote by A the intersection of all prime ideals P of R with the property that $ad(R)\subseteq P$ and by B the intersection of all prime ideals P with the property that $[U,U]\subseteq P$. We follow the argument of Lanski [11]. Since $AB=0$ and $\operatorname{ann}_R(\operatorname{ann}_R(A))\operatorname{ann}_R(A)=0$, we obtain that $ad(R)\subseteq A\subseteq\operatorname{ann}_R(\operatorname{ann}_R(A))$ and $[U,U]\subseteq B\subseteq\operatorname{ann}_R(A)$. Set $R_1=R/\operatorname{ann}_R(\operatorname{ann}_R(A))$ and $R_2=R/\operatorname{ann}_R(A)$. Then d induces derivations d_1 and d_2 on R_1 and R_2, respectively, since $\operatorname{ann}_R(\operatorname{ann}_R(A))$ and $\operatorname{ann}_R(A)$ are d-invariant. Now it is clear that $\overline{a}d_1(R_1)=0$ and the image of U in R_2 is commutative. Finally, if R is 2-torsion free, then we can construct the subdirect sum R of R_1 and R_2 such that the image of U in R_2 is central. Indeed, in this case we can take only these prime ideals P of R with $\operatorname{char} R/P \neq 2$ in the construction of A and B. By [10, Theorem 4], if $[U, U] \subseteq P$, then $[U, R] \subseteq P$ and hence $[U, R] \subseteq B$. This implies the image of U in R_2 is central. This finishes the proof of the Main Theorem.

We conclude this paper with proving the Corollary, a generalization of Brešar's result.

Proof of the corollary. By the assumption, $ad(x)^n = 0$ for all $x \in R$. By the Main Theorem, $ad(R)[R, R] = 0$. In particular, $ad(R)[R^2, R] = 0$ and hence $ad(R)R[R, R] = 0$. Therefore,

$$[R, ad(R)]R[R, ad(R)] = 0,$$

which implies $ad(R) \subseteq Z$. Let $x \in R$. Then $(ad(x))^n = a^n d(x)^n = a^{n-1}(ad(x)^n) = 0$, since $ad(x)ad(x) = a(ad(x))d(x) = a^2 d(x)^2$. However, since Z is a reduced ring, we have $ad(x) = 0$. That is, $ad(R) = 0$ as desired.

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