ON THE FIXED POINT SETS OF SMOOTH INVOLUTIONS
ON THE PRODUCTS OF SPHERES

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Abstract. In this paper, we have, under some conditions on cohomology, that the fixed point set of a smooth involution on a product of spheres is of constant dimension.

1. Introduction

Throughout this paper, we assume $G = \mathbb{Z}_2$. Let $G$ act smoothly on a smooth closed manifold $M$ with fixed point set $F$. Denote by $M_G$ the Borel construction associated with a $G$ action on $M$, and by $p : M_G \to B_G = \mathbb{R}P^\infty$ the fibre bundle with fibre $M$. It is well known that if $F$ is nonempty, then it is a disjoint union of finite number of smooth closed submanifolds of $M$. In this paper, we study the relations between the dimensions of the components of $F$ and the cohomology of $M$ or $M_G$. We will prove

Theorem 1.1. Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then the fixed point set $F$ is either empty or of constant dimension if one of the following conditions is satisfied:

(i) $H^*(M_G; \mathbb{Z})$ has a generator set $\{1, y_j\}$ as an algebra over $H^*(\mathbb{R}P^\infty; \mathbb{Z})$ with $\text{deg}(y_j)$ odd for all possible $j$;

(ii) $\tilde{H}^*(M^n; \mathbb{Z})$ has no 2-torsions and is algebraically generated by some elements $\{x_i\}$ of odd degrees with $\text{deg}(x_i) + \text{deg}(x_j) > \text{deg}(x_l)$ for $i \neq j$, and $\tau$ induces a trivial $\mathbb{Z}_2$ action on $\tilde{H}^*(M^n; \mathbb{Z})$.

Let $R$ be a principal ideal domain. Recall that $M^n$ is totally nonhomologous to zero in $M_G$ with coefficient in $R$ if the fibre inclusion $j : M^n \to M_G$ induces a surjection in cohomology $H^*(-; R)$ ([3, p373]). Thus by the Leray-Hirsch theorem [3, Theorem 1.4, p372], the condition (i) of Theorem 1.1 is satisfied if $M^n$ is totally nonhomologous to zero in $M_G$ with coefficient in $\mathbb{Z}$, and $\tilde{H}^*(M^n; \mathbb{Z})$ has no 2-torsions, and is algebraically generated by some elements of odd degrees.

Let $X \sim_R Y$ denote two spaces $X$ and $Y$ such that $H^*(X; R)$ and $H^*(Y; R)$ are isomorphic as rings. Denote by $W(M)$ the total Stiefel-Whitney classes of $M$. Note that $W(M) = 1$ if $M$ is a product of some spheres. The statement (i) of the next theorem is an immediate corollary of Theorem 1.1.

Theorem 1.2. Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then $M^n$ is totally nonhomologous to zero in $M_G$ with coefficient in $R$ if

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Theorem 1.2. Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then $F$ is either empty or of constant dimension, if 

(i) $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*(M^n; \mathbb{Z})$, and $M^n \sim_{\mathbb{Z}_2} S^{2n+1} \times S^{2n+1} \times \ldots \times S^{2n+1}$ with $2n_i + 2n_j > 2n - 1$ whenever $i \neq j$ (e.g. $M^n \sim_{\mathbb{Z}_2} (S^{2m+1})^r$), or 

(ii) $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*(M^n; \mathbb{Z}_2)$, $M \sim_{\mathbb{Z}_2} (S^1)^n$ and $W(M) = 1$.

Theorem 1.3. Let $M$ be a smooth closed manifold with $W(M) = 1$. Suppose $\tau$ is a smooth involution on $M$ which induces the trivial $\mathbb{Z}_2$ action on $H^*(M; \mathbb{Z}_2)$.

(i) If $M^{2n} \sim_{\mathbb{Z}_2} (S^2)^n$, then $F$ is nonempty and is of constant dimension. Let $k$ be the dimension of $F$. Then $k$ is even and $F$ has at most $2^{n-k/2}$ components $\{F_i\}$, and for each $F_i$, $H^*(F_i; \mathbb{Z}_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1 \leq j \leq n}$ with $b_{ij} \in H^2(F_i; \mathbb{Z}_2)$ and $b_{ij}^2 = 0$ for all possible $j$. In particular, $H^*(F_i; \mathbb{Z}_2)$ contains a subring which is isomorphic to $H^*((S^2)^k/\mathbb{Z}_2)$.

(ii) Suppose $M \sim_{\mathbb{Z}_2} (S^1)^n$ and $F$ nonempty. Then $F$ is of constant dimension. Let $k$ be the dimension of $F$. Then $F$ has at most $2^{n-k}$ components $\{F_i\}$, and for each $F_i$, $H^*(F_i; \mathbb{Z}_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1 \leq j \leq n}$ with $b_{ij} \in H^2(F_i; \mathbb{Z}_2)$ and $b_{ij}^2 = 0$ for all possible $j$. In particular, $H^*(F_i; \mathbb{Z}_2)$ contains a subring which is isomorphic to $H^*((S^1)^k/\mathbb{Z}_2)$.

We point out, since the statement (i) in [5, Proposition 2.1] (there is a misprint there, $i^*c_k^{(m)} = c_k^{(m)}$ should be $i^*c_k^{(m)} = c_k$) is true if and only if the smooth involution $\tau$ induces the trivial $\mathbb{Z}_2$ action on $H^*((S^1)^n; \mathbb{Z}_2)$, the main theorem we proved there should be modified as follows.

Theorem. Any smooth involution on $(S^1)^n$ with the trivial induced $\mathbb{Z}_2$ action on $H^*((S^1)^n; \mathbb{Z}_2)$ has either empty or constant-dimensional fixed point set $F$.

2. PROOFS OF THE THEOREMS

Let $M^n$ be a smooth closed manifold with a smooth involution $\tau$. Then $\tau$ induces a $\mathbb{Z}_2$-equivariant vector bundle structure on the tangent bundle $T(M^n)$ of $M^n$. Let $S^\infty$ be the infinite-dimensional sphere with a $\mathbb{Z}_2$ action given by the antipodal involution. Consider the product space $S^\infty \times M^n$ with the $\mathbb{Z}_2$ diagonal action. Then projection $S^\infty \times M^n \to M^n$ is equivariant. Pulling back the $\mathbb{Z}_2$-equivariant vector bundle $T(M^n)$ by this projection, we obtain a $\mathbb{Z}_2$-equivariant vector bundle over $S^\infty \times M^n$, which defines a vector bundle over the Borel space $M_G = (S^\infty \times M^n)/\mathbb{Z}_2$ by [1, Proposition 1.6.1, p36]. Denote this vector bundle by $\tilde{T}(M^n)$. Similarly, the diagonal action on $S^\infty \times M^n$, where the $\mathbb{Z}_2$ action on $S^\infty$ is given by the antipodal involution, defines a smooth closed manifold $R^m(\tau) = (S^m \times M^n)/\mathbb{Z}_2$. Let $p$ denote either projection $R^m(\tau) \to RP^m$ or $M_G \to RP^\infty$. Then $(R^m(\tau), p, RP^m)$ is a differentiable fibre bundle over $RP^m$ with fibre $M^n$. Consequently, the tangent bundle of $R^m(\tau)$ splits and

$$T(R^m(\tau)) \cong p^*T(RP^m) \oplus \tilde{T}_m(M^n),$$

where $\tilde{T}_m(M^n)$ is called the tangent bundle along the fibres ([2, p482]). Actually, $\tilde{T}_m(M^n) = i^*(\tilde{T}(M^n))$, where $i : R^m(\tau) \to M_G$ is the natural inclusion. Note that the restriction of $T_m(M^n)$ (or $\tilde{T}(M^n)$) on a specific fibre is exactly the tangent bundle $T(M^n)$.

Suppose $F \neq \emptyset$. Given $x \in F$, define $d_x$ to be the codimension of the component of $F$ containing $x$, and $I(\tau)$ the set of numbers $d_x$. Let $\rho_x$ be the section of $p$ associated with $x \in F$. Consider the induced bundle $\eta_x^{(m)} = \rho_x^*\tilde{T}_m(M^n)$. Observe
that \(d_x\) is the number of the eigenvalues \((-1)\) of the local representation of the group \(Z_2\) induced by tangent map \(d(\tau)\) on the tangent space \(T_x(M^n)\). This implies the induced bundle \(\eta_x^{(m)}\) is the Whitney sum of an \((n - d_x)\)-dimensional trivial bundle and \(d_x\) copies of the Hopf bundle. Therefore \(W(\eta_x^{(m)}) = (1 + a)^{d_x}\), where \(a \in H^1(RP(m);Z_2)\) is a generator. Thus

\[
I(\tau) = \{d_x | x \in F, W(\eta_x^{(m)}) = (1 + a)^{d_x}, 0 \leq d_x \leq n\}
\]

for every \(m > n\).

**Remark 2.1.** Let \(W_j(-)\) be the \(j\)-th Stiefel-Whitney class. Then whenever \(m > n\), we have \(d_x = \max \{|j|W_j(\eta_x^{(m)}) \neq 0\} = \max \{|j|W_j(\eta_x) \neq 0\}\), where \(\eta_x = \rho_x^*\tau(M^n)\). Let \(C(-)\) and \(C_j(-)\) be the total Chern classes and the \(j\)-th Chern class respectively. Since \(\eta_x^{(m)} \otimes C\) is isomorphic to \(\rho_x^*(\tau_m(M^n) \otimes C)\) as complex bundle and

\[
\rho C(\eta_x^{(m)} \otimes C) = (W(\eta_x^{(m)}))^2,
\]

where \(\rho\) is the mod 2 reduction homomorphism, we have

\[
d_x = \max \{|j|C_j(\eta_x^{(m)} \otimes C) \neq 0\} \text{ whenever } m > n
\]

\[
= \max \{|j|C_j(\eta_x \otimes C) \neq 0\}
\]

\[
= \max \{|j|\rho C_j(\eta_x \otimes C) \neq 0\}.
\]

Thus \(I(\tau)\) can be computed by using either Stiefel-Whitney or Chern classes.

Let \(j\) be the inclusion \(M^n \rightarrow MG\) or \(M^n \rightarrow R^m(\tau)\). The following theorem shows some relations between \(I(\tau)\) and the algebraic structure of \(H^*(MG;Z)\) or \(H^*(MG;Z_2)\).

**Theorem 2.2.** Let \(M\) be a smooth closed manifold with a smooth involution \(\tau\). Suppose there is a generator set \(\{c_i\}\) of \(H^*(MG;Z)\) (resp. \(H^*(MG;Z_2)\) ) as an algebra over \(H^*(RP^\infty;Z)\) (resp. \(H^*(RP^\infty;Z_2)\) ). If there is an \(n_i\) such that

\[
(c_i)^{n_i} \in p^*H^*(RP^\infty;Z) \ (\text{resp. } p^*H^*(RP^\infty;Z_2))
\]

for each \(c_i\), then \(\tau\) has either empty or constant-dimensional fixed point set \(F\).

**Proof.** Suppose \(F \neq \phi\). In the case of coefficient \(Z_2\), consider the homomorphism \(\rho_x^* : H^*(MG;Z_2) \rightarrow H^*(RP^\infty;Z)\). Then for each \(c_i\),

\[
\rho_x^*(c_i) = \begin{cases} 
0 & \text{if } c_i^{m_i} = 0, \\
\alpha_i & \text{if } c_i^{m_i} \neq 0,
\end{cases}
\]

which is independent of the choices of \(x \in F\), where \(m_i\) is the degree of \(c_i\) and \(\alpha_i \in H^1(RP^\infty;Z)\) is a generator. By Remark 2.1, \(d_x = \max \{|j|W_j(\eta_x) = \rho_x^*W_j(\tau(M)) \neq 0\}\) is independent of the choices of \(x \in F\). So \(F\) is of constant dimension. In the case of coefficient \(Z\), consider the complex bundle \(\eta_x \otimes C\) and the homomorphism \(\rho_x^* : H^*(MG;Z) \rightarrow H^*(RP^\infty;Z)\). Just as the preceding case, \(\rho_x^*\) is independent of the choices of \(x \in F\). By Remark 2.1 again, \(d_x = \max \{|j|C_j(\eta_x^{(m)} \otimes C) = \rho_x^*C_j(\tau(M) \otimes C) \neq 0\}\) is independent of the choices of \(x \in F\). So \(F\) is also of constant dimension.

**Remark 2.3.** If \(H^*(MG;Z_2)\) (resp. \(H^*(MG;Z)\)) has a generator set \(\{1, x_1, x_2, ..., x_k\}\) as an algebra over \(H^*(RP^\infty;Z_2)\) (resp. \(H^*(RP^\infty;Z)\)), then there are at most \(2^k\) number of different maps \(\rho_x^*\) for \(x \in F\). Therefore \(F\) has at most \(2^k\) number of components which are of different dimensions.
Proof of Theorem 1.1. For (i), \( \rho^*_s(y_j) = 0 \) for all possible \( j \), since \( H^{\text{odd}}(\mathbb{R}P^\infty; \mathbb{Z}) = 0 \). Thus the homomorphism \( \rho^*_s \) is independent of the choices of \( x \in F \) and (i) follows from Remark 2.1.

For (ii), we consider the spectral sequence \( \{ E^{p,q}_r \} \) with

\[
E^{p,q}_2 = H^p(\mathbb{R}P^\infty; H^q(\mathbb{M}^n; \mathbb{Z}))
\]

([3, p370]), which converges to \( H^*(\mathbb{M}G; \mathbb{Z}) \). Here the coefficient \( H^q(\mathbb{M}^n; \mathbb{Z}) \) is a local system which becomes constant because of the trivial induced \( Z_2 \) action on \( H^*(\mathbb{M}^n; \mathbb{Z}) \). Let \( \{ x_i \} \) be the generator set of the ring \( H^*(\mathbb{M}^n; \mathbb{Z}) \) with \( \deg(x_i) \) odd for all \( i \). First note that for each \( x_i \), \( x_i^2 \) must be of order \( \leq 2 \), since the degree of \( x_i \) is odd. Thus we have \( x_i^2 = 0 \), since \( H^q(\mathbb{M}^n; \mathbb{Z}) \) has no 2-torsions. The multiplicative property implies this spectral sequence collapses, since all elements in \( E^{0,2m+1}_2 \) and \( E^{0,0}_2 \) are permanent cocycles, where \( 2n_i + 1 \) is the degree of some \( x_i \). Here note that the only possible nontrivial target for the differential \( d_2 \), on an element of \( E^{2r,2m+1}_2 \) is in \( E^{2r-2,2m-2r+2}_2 \). Since \( \deg(x_j) + \deg(x_i) > \deg(x_i) \) for \( j \neq i \), we see \( E^{2r,2m+1}_2 \) is a permanent cocycle. Now the edge homomorphism \( H^q(\mathbb{M}G; \mathbb{Z}) \to E^{q,q}_2 \to H^q(\mathbb{M}^n; \mathbb{Z}) \), which is precisely the \( j^* : H^*(\mathbb{M}G; \mathbb{Z}) \to H^*(\mathbb{M}^n; \mathbb{Z}) \) ([3, p374]), is surjective, we see that \( H^*(\mathbb{M}G; \mathbb{Z}) \) is an algebra over \( H^*(\mathbb{R}P^\infty; \mathbb{Z}) \) with generator set \( \{ 1, y_i \} \) and \( \deg(y_i) \) odd for all \( i \), and (ii) follows just as (i).

Proof of Theorem 1.2. We only need to prove (ii). Consider the spectral sequence which converges to \( H^*(\mathbb{M}G; \mathbb{Z}) \) with \( E^{p,q}_2 = H^p(\mathbb{R}P^\infty; H^q(\mathbb{M}; \mathbb{Z})) \). Here the local coefficient system \( H^*(\mathbb{M}; \mathbb{Z}) \) again becomes constant because of the trivial induced \( Z_2 \) action on \( H^*(\mathbb{M}; \mathbb{Z}) \). By the multiplicative property, this spectral sequence collapses since all elements of \( E^{0,1}_2 \) and \( E^{1,0}_2 \) are permanent cocycles. By [3, Theorem 1.6, p374], \( M \) is totally nonhomologous to zero in \( \mathbb{M}G \) and hence in \( R^m(\tau) \) with coefficient in \( Z_2 \) for any \( m \geq 0 \). Thus \( H^*(R^m(\tau); \mathbb{Z}) \) is a free \( H^*(\mathbb{R}P^m(\tau); \mathbb{Z}) \) module with a module basis \( \{ x_i \} \), where \( x_i \) runs through all the possible products \( (c_1)^{j_1}(c_2)^{j_2}... (c_n)^{j_n} \). Here \( c_i = 0 \) or 1, and \( \{ c_1, c_2, ..., c_n \} \) are the elements of \( H^1(\mathbb{M}G; \mathbb{Z}) \) such that \( \{ j^*(c_1), ..., j^*(c_n) \} \) make up a basis of the \( Z_2 \) vector space \( H^1(\mathbb{M}; \mathbb{Z}) \).

Let \( V_i \) be the \( i \)-th Wu class of the tangent bundle \( T(R^m(\tau)) \) ([4, p132]). If \( V_i \in p^*H^i(\mathbb{R}P(m); \mathbb{Z}) \) for all \( i \), then \( W_k \in p^*H^k(\mathbb{R}P(m); \mathbb{Z}) \) for all \( k \) by the formula

\[
W_k = \sum_{i+j=k} S^q(i, j, j_1^*, k_1^*),
\]

where \( W_k \) is the \( k \)-th Stiefel-Whitney class of \( T(R^m(\tau)) \). This implies \( W_k(\mathbb{T}_m(\mathbb{M})) \in p^*H^k(\mathbb{R}P(m); \mathbb{Z}) \) for all \( k \) by the facts

\[
W(T(R^m(\tau))) = p^*W(T(\mathbb{R}P(m)))W(\mathbb{T}_m(\mathbb{M}))
\]

and \( W(T(\mathbb{R}P(m))) = (1 + a)^{m+1} \), where \( a \in H^1(\mathbb{R}P(m); \mathbb{Z}) \) is a generator. Therefore by Remark 2.1, \( F \) must be of constant dimension if not empty.

Now suppose \( V_k \) contains a nontrivial summand of the form \( a^{n_1}c_1c_2...c_j \), \( j \geq 1 \). Then we claim \( j < n \). To see this, we notice \( j^*W(\mathbb{T}_m(\mathbb{M})) = W(M) = 1 \) and that \( \mathbb{T}_m(M) \) is \( (n) \)-dimensional. Thus there is no such summands as \( a^{n_1}c_1c_2...c_n \) in \( W(\mathbb{T}_m(M)) \) neither in \( W(T(R^m(\tau))) \). By using the formula (1), we can write \( V_k \)
as the sum
\[ V_k = \sum Sq^{i_1} Sq^{i_2} \ldots Sq^{i_j}(W_j), \quad j_1 + j_2 + \ldots + j_1 + j = k. \]

Since for all \( i, \ (c_i)^2 = \sum b_j c_j + b_0, \) where \( b_0 \) and \( b_j \) are elements in \( p^*H^*(RP(m); Z_2) \),

we have
\[ Sq^{m'}(c_i) = \sum b'_j c_j + b'_0 \]

for some elements \( b'_j \) and \( b'_0 \) in \( p^*H^*(RP(m); Z_2) \). Then by (2), there is no such term \( a^n c_1 c_2 \ldots c_n \) in \( Sq^{m'} Sq^{i_2} \ldots Sq^{i_j}(W_i) \) neither in \( V_k \). Therefore, we may assume \( V_k \) contains a nontrivial summand \( a^n c_1 c_2 \ldots c_j, \quad 0 < j < n. \)

Then by the definition of Wu class ([4]),
\[
1 = \langle V_k(a^{m-n'} c_j + \ldots + c_n), \sigma \rangle = \langle Sq^k(a^{m-n'} c_j + \ldots + c_n), \sigma \rangle = 0,
\]

since \( Sq^k(a^{m-n'} c_j + \ldots + c_n) \) must be zero by the formula (2). Here \( \sigma \in H_{m+n}(R^n(\tau); Z_2) \) is the homology fundamental class of the closed manifold \( R^n(\tau) \).

This contradiction shows \( V_k \in p^*H^*(RP^\infty; Z_2) \) for all \( k \), and (ii) follows. \( \square \)

**Proposition 2.4.** Let \( M \) be a smooth closed manifold with a smooth involution \( \tau \) which induces the trivial \( Z_2 \) action on \( H^*(M; Z_2) \). Suppose \( M \sim Z_2 \) \((S^2)^n \) and \( W(M) = 1. \) Let \( F \neq \phi \) and \( k \) be the constant dimension of \( F. \) If \( Sq^1(x) \in p^*H^*(RP^\infty; Z_2) \) for all \( x \in H^*(M_G; Z_2) \), then \( k \) is even and \( F \) has at most \( 2^{n-k/2} \) components \( \{ F_i \} \), and for each \( F_i, H^*(F_i; Z_2) \) is algebraically generated by some elements \( \{ b_{ij} \}_{i=1}^n \) with \( b_{ij} \in H^2(F_i; Z_2) \) and \( b_{ij}^2 = 0 \) for all \( j. \)

In particular, \( H^*(F_i; Z_2) \) contains a subring which is isomorphic to \( H^*((S^2)^k/2; Z_2) \).

**Proof.** First the spectral sequence which converges to \( H^*(M_G; Z_2) \) with \( E_2^{p,q} = H^p(RP^\infty; H^q(M; Z_2)) \) collapses. By [3, Theorem 1.6, p374], \( M \) is totally nonhomologous to zero with coefficient in \( Z_2. \)

Let \( \{ c_i \}_{i=1}^n \) be elements of \( H^2(M_G; Z_2) \) such that their restrictions on \( M \) form a basis of the \( Z_2 \) vector space \( H^2(M; Z_2). \)

Let \( j_1 : (F_i)_G = RP^\infty \times F_i \rightarrow M_G \) be the inclusion. Then \( j_1^* : H^k(M_G; Z_2) \rightarrow H^k((F_i)_G; Z_2) \) is a surjection for \( k > 2n \) ([3, Theorem 1.5, p374]). Note that \( H^*((F_i)_G; Z_2) \approx H^*((RP^\infty)_G; Z_2) \otimes H^*(F_i; Z_2). \)

Let
\[
j_1^*(c_i) = a \otimes b_{it_1} + 1 \otimes b_{it_2} + a^2 \otimes 1,
\]

where \( b_{it_j} \in H^2(F_i; Z_2) \) for \( j = 1, 2 \) and \( a \in H^1(RP^\infty; Z_2) \) is a generator. Since \( j_1^* \) is onto in high degrees, \( H^*(F_i; Z_2) \) is algebraically generated by the set \( \{ b_{it_1}, b_{it_2} \}_{i=1}^n. \)

Next by the assumed condition, \( Sq^1(c_i) \in p^*H^3(RP^\infty; Z_2) \); thus \( j_1^* Sq^1(c_i) \in p^*H^*(RP^\infty; Z_2). \)

We claim \( b_{it_1} = 0. \) Otherwise,
\[
j_1^* Sq^1(c_i) = Sq^1 j_1^*(c_i) = Sq^1(1 \otimes b_{it_2} + a \otimes b_{it_1} + a^2 \otimes 1)
\]
\[ = 1 \otimes Sq^1 b_{it_2} + a^2 \otimes b_{it_1} + a \otimes (b_{it_1})^2 \]
\[ \notin p^*H^*(RP^\infty; Z_2). \]

This is a contradiction. Now we claim \( (b_{it_2})^2 = 0 \) for each \( t. \) Let \( j : M \rightarrow M_G, \)
Let \( j : F_i \rightarrow (F_i)_G \) and \( j_i : F_i \rightarrow M \) be inclusions. Then the diagram
\[
\begin{array}{ccc}
F_i & \xrightarrow{j} & (F_i)_G \\
\downarrow j_i & & \downarrow j_i \\
M & \xrightarrow{j} & M_G
\end{array}
\]
commutes. Thus we have \( b_{it2} = (j_1j^*)^*(c_i) = (j_1^*)^*j^*(c_i) \) and \( b_{i2}^2 = (j_1^*)^*(j^*(c_i))^2 = (j_1^*)^*(0) = 0. \)

Note that by Theorem 1.2 (ii), \( F \) is of constant dimension. Let \( k \) be the dimension of \( F \); then the generator of \( H^k(F_i; Z_2) \) must be a product of some \( b_{it2} \)’s. Consequently, \( k \) must be even and \( H^*(F_i; Z_2) \) has a subring which is isomorphic to \( H^*((S^2)^k/2; Z_2) \). This together with the equation \( \sum_{j \geq 0} \text{rank } H^j(M; Z_2) = 2^n \) ([3, Theorem 1.6, p374]) shows the number of the components of \( F \) is at most \( 2^{n-k/2} \).

**Proof of Theorem 1.3.** First we prove the statement (i). By the Lefschetz fixed point theorem, \( F \) is nonempty. By Theorem 1.2 (ii), \( F \) is of constant dimension. Note that \( Sq^1 = \rho \beta \), where \( \rho \) and \( \beta \) fit into the Bockstein exact sequence
\[
\rightarrow H^2(M_G; Z_2) \xrightarrow{\beta} H^3(M_G; Z) \xrightarrow{2} H^3(M_G; Z) \xrightarrow{\beta} H^3(M_G; Z_2) \rightarrow.
\]
We claim \( H^{\text{odd}}(M_G; Z) = 0 \). Indeed, the spectral sequence which converges to \( H^*(M_G; Z) \) with \( E_2^{p,q} = H^p(RP^\infty; H^q(M; Z)) \) collapses. Since
\[
H^3(M; Z) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \text{free abelian} & \text{if } q \text{ is even,} \end{cases}
\]
and \( H^{\text{odd}}(RP^\infty; Z) = 0 \), \( H^{\text{odd}}(M_G; Z) \) must be trivial. This implies \( \beta = 0 \) and \( Sq^1(c_i) = \rho \beta(c_i) = 0 \), where \( \{c_i\} \) are as those in the proof of Proposition 2.4. Finally, \( \tau \) induces the trivial action on \( H^*(M; Z) \) implies the triviality of the induced \( Z_2 \) action on \( H^*(M; Z_2) \). So (i) follows from Proposition 2.4.

Next we consider (ii). Similarly to the proof of Proposition 2.4, let \( \{c_1\}_{1 \leq j \leq n} \) be the elements of \( H^1(M_G; Z_2) \) such that \( \{j^*(c_i)\} \) is a basis of the \( Z_2 \) vector space \( H^1(M^n; Z_2) \), and let
\[
j^*(c_i) = 1 \otimes b_{it} + a \otimes 1, \quad a \in H^1(RP^\infty; Z_2), \quad b_{it} \in H^1(F_i; Z_2).
\]
Here we use the notation of Proposition 2.4. Then \( \{1, b_{11}, b_{22}, \ldots, b_{nt}\} \) generate algebraically the \( Z_2 \) algebra \( H^*(F_i; Z_2) \), and just as in the proof of Proposition 2.4, we have \( (b_{it})^2 = 0 \), and (ii) follows as (i).

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