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ON THE FIXED POINT SETS OF SMOOTH INVOLUTIONS ON THE PRODUCTS OF SPHERES

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ABSTRACT. In this paper, we have, under some conditions on cohomology, that the fixed point set of a smooth involution on a product of spheres is of constant dimension.

1. INTRODUCTION

Throughout this paper, we assume $G = Z_2$. Let G act smoothly on a smooth closed manifold M with fixed point set F. Denote by M_G the Borel construction associated with a G action on M, and by $p : M_G \to B_G = RP^\infty$ the fibre bundle with fibre M. It is well known that if F is nonempty, then it is a disjoint union of finite number of smooth closed submanifolds of M. In this paper, we study the relations between the dimensions of the components of F and the cohomology of M or M_G. We will prove

Theorem 1.1. *Let M^n be a smooth closed manifold with a smooth involution τ. Then the fixed point set F is either empty or of constant dimension if one of the following conditions is satisfied:*

(i) *$H^*(M_G; Z)$ has a generator set $\{1, y_j\}$ as an algebra over $H^*(RP^\infty; Z)$ with $\deg(y_j)$ odd for all possible j;*

(ii) *$\widetilde{H}^*(M^n; Z)$ has no 2-torsions and is algebraically generated by some elements $\{x_i\}$ of odd degrees with $\deg(x_i) + \deg(x_j) > \deg(x_l)$ for $i \neq j$, and τ induces a trivial Z_2 action on $\widetilde{H}^*(M^n; Z)$.*

Let R be a principal ideal domain. Recall that M^n is totally nonhomologous to zero in M_G with coefficient in R if the fibre inclusion $j : M^n \to M_G$ induces a surjection in cohomology $H^*(-; R)$ ([3, p373]). Thus by the Leray-Hirsch theorem [3, Theorem 1.4, p372], the condition (i) of Theorem 1.1 is satisfied if M^n is totally nonhomologous to zero in M_G with coefficient in Z, and $\widetilde{H}^*(M^n; Z)$ has no 2-torsions, and is algebraically generated by some elements of odd degrees.

Let $X \sim_R Y$ denote two spaces X and Y such that $H^*(X; R)$ and $H^*(Y; R)$ are isomorphic as rings. Denote by $W(M)$ the total Stiefel-Whitney classes of M. Note that $W(M) = 1$ if M is a product of some spheres. The statement (i) of the next theorem is an immediate corollary of Theorem 1.1.

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Theorem 1.2. *Let M^n be a smooth closed manifold with a smooth involution τ. Then F is either empty or of constant dimension, if*

(i) *τ induces the trivial Z_2 action on $H^*(M^n;Z)$, and $M^n \sim_Z S^{2n_1+1}\times S^{2n_2+1}\times \ldots \times S^{2n_k+1}$ with $2n_i+2n_j>2n_l-1$ whenever $i\neq j$ (e.g. $M^n\sim_Z (S^{2m+1})^r$), or*

(ii) *τ induces the trivial Z_2 action on $H^*(M^n;Z_2)$, $M\sim_{Z_2}(S^l)^n$ and $W(M)=1$.*

Theorem 1.3. *Let M be a smooth closed manifold with $W(M)=1$. Suppose τ is a smooth involution on M which induces the trivial Z_2 action on $H^*(M;Z_2)$.*

(i) *If $M^{2n}\sim_Z (S^2)^n$, then F is nonempty and is of constant dimension. Let k be the dimension of F. Then k is even and F has at most $2^{n-k/2}$ components $\{F_i\}$, and for each F_i, $H^*(F_i;Z_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1\le j\le n}$ with $b_{ij}\in H^2(F_i;Z_2)$ and $b_{ij}^2=0$ for all possible j. In particular, $H^*(F_i;Z_2)$ contains a subring which is isomorphic to $H^*((S^2)^{k/2};Z_2)$.*

(ii) *Suppose $M\sim_{Z_2}(S^1)^n$ and F nonempty. Then F is of constant dimension. Let k be the dimension of F. Then F has at most 2^{n-k} components $\{F_i\}$, and for each F_i, $H^*(F_i;Z_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1\le j\le n}$ with $b_{ij}\in H^1(F_i;Z_2)$ and $b_{ij}^2=0$ for all possible j. In particular, $H^*(F_i;Z_2)$ contains a subring which is isomorphic to $H^*((S^1)^k;Z_2)$.*

We point out, since the statement (i) in [5, Proposition 2.1] (there is a misprint there, $i^*c_k^{(m)}=c_k^{(m_1)}$ should be $i^*c_k^{(m)}=c_k$) is true if and only if the smooth involution τ induces the trivial Z_2 action on $H^*((S^1)^n;Z_2)$, the main theorem we proved there should be modified as follows.

Theorem. *Any smooth involution on $(S^1)^n$ with the trivial induced Z_2 action on $H^*((S^1)^n;Z_2)$ has either empty or constant-dimensional fixed point set F.*

2. Proofs of the theorems

Let M^n be a smooth closed manifold with a smooth involution τ. Then τ induces a Z_2-equivariant vector bundle structure on the tangent bundle $T(M^n)$ of M^n. Let S^∞ be the infinite-dimensional sphere with a Z_2 action given by the antipodal involution. Consider the product space $S^\infty\times M^n$ with the Z_2 diagonal action. Then projection $S^\infty\times M^n \to M^n$ is equivariant. Pulling back the Z_2-equivariant vector bundle $T(M^n)$ by this projection, we obtain a Z_2-equivariant vector bundle over $S^\infty\times M^n$, which defines a vector bundle over the Borel space $M_G=(S^\infty\times M^n)/Z_2$ by [1, Proposition 1.6.1, p36]. Denote this vector bundle by $\bar{T}(M^n)$. Similarly, the diagonal action on $S^m\times M^n$, where the Z_2 action on S^m is given by the antipodal involution, defines a smooth closed manifold $R^m(\tau)=(S^m\times M^n)/Z_2$. Let p denote either projection $R^m(\tau)\to RP(m)$ or $M_G\to RP^\infty$. Then $(R^m(\tau),p,RP(m))$ is a differentiable fibre bundle over $RP(m)$ with fibre M^n. Consequently, the tangent bundle of $R^m(\tau)$ splits and

$$T(R^m(\tau))\cong p^*T(RP(m))\oplus \bar{T}_m(M^n),$$

where $\bar{T}_m(M^n)$ is called the tangent bundle along the fibres ([2, p482]). Actually, $\bar{T}_m(M^n)=i^*(\bar{T}(M^n)$, where $i: R^m(\tau)\to M_G$ is the natural inclusion. Note that the restriction of $\bar{T}_m(M^n)$ (or $\bar{T}(M^n)$) on a specific fibre is exactly the tangent bundle $T(M^n)$.

Suppose $F\neq\phi$. Given $x\in F$, define d_x to be the codimension of the component of F containing x, and $I(\tau)$ the set of numbers d_x. Let ρ_x be the section of p associated with $x\in F$. Consider the induced bundle $\eta_x^{(m)}=\rho_x^*\bar{T}_m(M^n)$. Observe that d_x is the number of the eigenvalues (-1) of the local represetation of the group Z_2 induced by tangent map $d(\tau)$ on the tangent space $T_x(M^n)$. This implies the induced bundle $\eta_x^{(m)}$ is the Whitney sum of an $(n-d_x)$-dimensional trivial bundle and d_x copies of the Hopf bundle. Therefore $W(\eta_x^{(m)}) = (1+a)^{d_x}$, where $a \in H^1(RP(m); Z_2)$ is a generator. Thus

$$I(\tau) = \{d_x | x \in F, W(\eta_x^{(m)}) = (1+a)^{d_x}, 0 \leq d_x \leq n\}$$

for every $m > n$.

Remark 2.1. Let $W_j(-)$ be the j-th Stiefel-Whitney class. Then whenever $m > n$, we have $d_x = \max\{j | W_j(\eta_x^{(m)}) \neq 0\} = \max\{j | W_j(\eta_x) \neq 0\}$, where $\eta_x = \rho_x^* \bar{T}(M^n)$. Let $C(-)$ and $C_j(-)$ be the total Chern classes and the j-th Chern class respectively. Since $\eta_x^{(m)} \otimes C$ is isomorphic to $\rho_x^*(\bar{T}_m(M^n) \otimes C)$ as complex bundle and

$$\rho C(\eta_x^{(m)} \otimes C) = (W(\eta_x^{(m)}))^2,$$

where ρ is the mod 2 reduction homomorphism, we have

$$\begin{aligned} d_x &= \max\{j | C_j(\eta_x^{(m)} \otimes C) \neq 0\} \text{ whenever } m > n \\ &= \max\{j | C_j(\eta_x \otimes C) \neq 0\} \\ &= \max\{j | \rho C_j(\eta_x \otimes C) \neq 0\}. \end{aligned}$$

Thus $I(\tau)$ can be computed by using either Stiefel-Whitney or Chern classes.

Let j be the inclusion $M^n \to M_G$ or $M^n \to R^m(\tau)$. The following theorem shows some relations between $I(\tau)$ and the algebraic structure of $H^*(M_G; Z)$ or $H^*(M_G; Z_2)$.

Theorem 2.2. *Let M be a smooth closed manifold with a smooth involution τ. Suppose there is a generator set $\{c_i\}$ of $H^*(M_G; Z)$ (resp. $H^*(M_G; Z_2)$) as an algebra over $H^*(RP^\infty; Z)$ (resp. $H^*(RP^\infty; Z_2)$). If there is an n_i such that*

$$(c_i)^{n_i} \in p^* H^*(RP^\infty; Z) \ (\textit{resp.} \ \in p^* H^*(RP^\infty; Z_2))$$

for each c_i, then τ has either empty or constant-dimensional fixed point set F.

Proof. Suppose $F \neq \phi$. In the case of coefficient Z_2, consider the homomorphism $\rho_x^* : H^*(M_G; Z_2) \to H^*(RP^\infty; Z_2)$. Then for each c_i,

$$\rho_x^*(c_i) = \begin{cases} 0 & \text{if } c_i^{n_i} = 0, \\ a^{m_i} & \text{if } c_i^{n_i} \neq 0, \end{cases}$$

which is independent of the choices of $x \in F$, where m_i is the degree of c_i and $a \in H^1(RP^\infty; Z_2)$ is a generator. By Remark 2.1, $d_x = \max\{j | W_j(\eta_x) = \rho_x^* W_j(\bar{T}(M)) \neq 0\}$ is independent of the choices of $x \in F$. So F is of constant dimension. In the case of coefficient Z, consider the complex bundle $\eta_x \otimes C$ and the homomorphism $\rho_x^* : H^*(M_G; Z) \to H^*(RP^\infty; Z)$. Just as the preceding case, ρ_x^* is independent of the choices of $x \in F$. By Remark 2.1 again, $d_x = \max\{j | C_j(\eta_x^{(m)} \otimes C) = \rho_x^* C_j(\bar{T}(M) \otimes C) \neq 0\}$ is independent of the choices of $x \in F$. So F is also of constant dimension. ☐

Remark 2.3. If $H^*(M_G; Z_2)$ (resp. $H^*(M_G; Z)$) has a generator set $\{1, x_1, x_2, ..., x_k\}$ as an algebra over $H^*(RP^\infty; Z_2)$ (resp. $H^*(RP^\infty; Z)$), then there are at most 2^k number of different maps ρ_x^* for $x \in F$. Therefore F has at most 2^k number of components which are of different dimensions.

Proof of Theorem 1.1. For (i), $\rho_x^*(y_j) = 0$ for all possible j, since $H^{odd}(RP^\infty; Z) = 0$. Thus the homomorphism ρ_x^* is independent of the choices of $x \in F$ and (i) follows from Remark 2.1.

For (ii), we consider the spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$E_2^{p,q} = H^p(RP^\infty; H^q(M^n; Z))$$

([3, p370]), which converges to $H^*(M_G; Z)$. Here the coefficient $H^q(M^n; Z)$ is a local system which becomes constant because of the trivial induced Z_2 action on $H^*(M^n; Z)$. Let $\{x_i\}$ be the generator set of the ring $H^*(M^n; Z_2)$ with $\deg(x_i)$ odd for all i. First note that for each x_i, x_i^2 must be of order ≤ 2, since the degree of x_i is odd. Thus we have $x_i^2 = 0$, since $H^q(M^n; Z)$ has no 2-torsions. The multiplicative property implies this spectral sequence collapses, since all elements in $E_2^{0,2n_i+1}$ and $E_2^{2,0}$ are permanent cocycles, where $2n_i + 1$ is the degree of some x_i. Here note that the only possible nontrivial target for the differential d_{2r} on an element of $E_{2r}^{0,2n_i+1}$ is in $E_{2r}^{2r,2n_i-2r+2}$. Since $\deg(x_j) + \deg(x_l) > \deg(x_i)$ for $j \neq l$, we see $E_2^{2r,2n_i-2r+2} = 0$ for $r \geq 1$. Therefore every element of $E_2^{0,2n_i+1}$ is a permanent cocycle. Now the edge homomorphism $H^q(M_G; Z) \rightarrow E_2^{o,q} \rightarrow H^q(M^n; Z)$, which is precisely the $j^* : H^*(M_G; Z) \rightarrow H^*(M^n; Z)$ ([3, p374]), is surjective, we see that $H^*(M_G; Z)$ is an algebra over $H^*(RP^\infty; Z)$ with generator set $\{1, y_i\}$ and $\deg(y_i)$ odd for all i, and (ii) follows just as (i). □

Proof of Theorem 1.2. We only need to prove (ii). Consider the spectral sequence which converges to $H^*(M_G; Z_2)$ with $E_2^{p,q} = H^p(RP^\infty; H^q(M; Z_2))$. Here the local coefficient system $H^*(M; Z_2)$ again becomes constant because of the trivial induced Z_2 action on $H^*(M; Z_2)$. By the multiplicative property, this spectral sequence collapses since all elements of $E_2^{0,l}$ and $E_2^{1,0}$ are permanent cocycles. By [3, Theorem 1.6, p374], M is totally nonhomologous to zero in M_G and hence in $R^m(\tau)$ with coefficeint in Z_2 for any $m \geq 0$. Thus $H^*(R^m(\tau); Z_2)$ is a free $H^*(RP(m); Z_2)$ module with a module basis $\{x_i\}$, where x_i runs through all the possible products $(c_1)^{\epsilon_1}(c_2)^{\epsilon_2}...(c_n)^{\epsilon_n}$. Here $\epsilon_i = 0$ or 1, and $\{c_1, c_2, ..., c_n\}$ are the elements of $H^l(M_G; Z_2)$ such that $\{j^*(c_1), ..., j^*(c_k)\}$ make up a basis of the Z_2 vector space $H^l(M; Z_2)$.

Let V_i be the i-th Wu class of the tangent bundle $T(R^m(\tau))$ ([4, p132]). If $V_i \in p^*H^i(RP(m); Z_2)$ for all i, then $W_k \in p^*H^k(RP(m); Z_2)$ for all k by the formula

$$W_k = \sum_{i+j=k} Sq^i(V_j), \tag{1}$$

where W_k is the k-th Stiefel-Whitney class of $T(R^m(\tau))$. This implies $W_k(\bar{T}_m(M)) \in p^*H^k(RP(m); Z_2)$ for all k by the facts

$$W(T(R^m(\tau))) = p^*W(T(RP(m)))W(\bar{T}_m(M))$$

and $W(T(RP(m))) = (1 + a)^{m+1}$, where $a \in H^1(RP(m); Z_2)$ is a generator. Therefore by Remark 2.1, F must be of constant dimension if not empty.

Now suppose V_k contains a nontrivial summand of the form $a^{n'}c_{i_1}c_{i_2}...c_{i_j}$, $j \geq 1$. Then we claim $j < n$. To see this, we notice $j^*W(\bar{T}_m(M)) = W(M) = 1$ and that $\bar{T}_m(M)$ is (nl)-dimensional. Thus there is no such summands as $a^{n'}c_1c_2...c_n$ in $W(\bar{T}_m(M))$ neither in $W(T(R^m(\tau)))$. By using the formula (1), we can write V_k

as the sum

$$V_k = \sum Sq^{j_1} Sq^{j_2} ... Sq^{j_t}(W_j), \qquad j_1 + j_2 + ... + j_t + j = k.$$

Since for all i, $(c_i)^2 = \sum b_j c_j + b_0$, where b_0 and b_j are elements in $p^*H^*(RP(m); Z_2)$, we have

$$Sq^{m'}(c_i) = \sum b'_j c_j + b'_0 \tag{2}$$

for some elements b'_j and b'_0 in $p^*H^*(RP(m); Z_2)$. Then by (2), there is no such term $a^{n'} c_1 c_2 ... c_n$ in $Sq^{j_1} Sq^{j_2} ... Sq^{j_t}(W_j)$ neither in V_k. Therefore, we may assume V_k contains a nontrivial summand $a^{n'} c_1 c_2 ... c_j$, $0 < j < n$. Then by the definition of Wu class ([4]),

$$\begin{aligned} 1 &= \langle V_k(a^{m-n'} c_{j+1} c_{j+2} ... c_n), \sigma \rangle \\ &= \langle Sq^k(a^{m-n'} c_{j+1} c_{j+2} ... c_n), \sigma \rangle \\ &= 0, \end{aligned}$$

since $Sq^k(a^{m-n'} c_{j+1} c_{j+2} ... c_n)$ must be zero by the formula (2). Here $\sigma \in H_{m+nl}(R^m(\tau); Z_2)$ is the homology fundamental class of the closed manifold $R^m(\tau)$. This contradiction shows $V_k \in p^*H^*(RP^\infty; Z_2)$ for all k, and (ii) follows. ☐

Proposition 2.4. *Let M be a smooth closed manifold with a smooth involution τ which induces the trivial Z_2 action on $H^*(M; Z_2)$. Suppose $M \sim_{Z_2} (S^2)^n$ and $W(M) = 1$. Let $F \neq \phi$ and k be the constant dimension of F. If $Sq^1(x) \in p^*H^*(RP^\infty; Z_2)$ for all $x \in H^*(M_G; Z_2)$, then k is even and F has at most $2^{n-k/2}$ components $\{F_i\}$, and for each F_i, $H^*(F_i; Z_2)$ is algebraically generated by some elements $\{b_{ij}\}_{1 \leq j \leq n}$ with $b_{ij} \in H^2(F_i; Z_2)$ and $b_{ij}^2 = 0$ for all j. In particular, $H^*(F_i; Z_2)$ contains a subring which is isomorphic to $H^*((S^2)^{k/2}; Z_2)$.*

Proof. First the spectral sequence which converges to $H^*(M_G; Z_2)$ with $E_2^{p,q} = H^p(RP^\infty; H^q(M; Z_2))$ collapses. By [3, Theorem 1.6, p374], M is totally nonhomologous to zero with coefficient in Z_2. Let $\{c_t\}_{t=1,2,...,n}$ be elements of $H^2(M_G; Z_2)$ such that their restrictions on M form a basis of the Z_2 vector space $H^2(M; Z_2)$. Let $j_1 : (F_i)_G = RP^\infty \times F_i \to M_G$ be the inclusion. Then $j_1^* : H^k(M_G; Z_2) \to H^k((F_i)_G; Z_2)$ is a surjection for $k > 2n$ ([3, Theorem 1.5, p374]). Note that $H^*((F_i)_G; Z_2) \approx H^*(RP^\infty; Z_2) \otimes H^*(F_i; Z_2)$. Let

$$j_1^*(c_t) = a \otimes b_{it1} + 1 \otimes b_{it2} + a^2 \otimes 1,$$

where $b_{itj} \in H^j(F_i; Z_2)$ for $j = 1, 2$ and $a \in H^1(RP^\infty; Z_2)$ is a generator. Since j_1^* is onto in high degrees, $H^*(F_i; Z_2)$ is algebraically generated by the set $\{1, b_{it1}, b_{it2}\}_{t=1,2,...,n}$.

Next by the assumed condition, $Sq^1(c_t) \in p^*H^3(RP^\infty; Z_2)$; thus $j_1^* Sq^1(c_t) \in p^*H^*(RP^\infty; Z_2)$. We claim $b_{it1} = 0$. Otherwise,

$$\begin{aligned} j_1^* Sq^1(c_t) &= Sq^1 j_1^*(c_t) \\ &= Sq^1(1 \otimes b_{it2} + a \otimes b_{it1} + a^2 \otimes 1) \\ &= 1 \otimes Sq^1 b_{it2} + a^2 \otimes b_{it1} + a \otimes (b_{it1})^2 \\ &\notin p^*H^*(RP^\infty; Z_2). \end{aligned}$$

This is a contradiction. Now we claim $(b_{it2})^2 = 0$ for each t. Let $j : M \to M_G$,

$\tilde{j}: F_i \to (F_i)_G$ and $\tilde{j_1}: F_i \to M$ be inclusions. Then the diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\tilde{j}} & (F_i)_G \\ \downarrow{\scriptstyle \tilde{j_1}} & & \downarrow{\scriptstyle j_1} \\ M & \xrightarrow{j} & M_G \end{array}$$

commutes. Thus we have $b_{it2} = (j_1\tilde{j})^*(c_t) = (\tilde{j_1})^* j^*(c_t)$ and $b_{it2}^2 = (\tilde{j_1})^*(j^*(c_t))^2 = (\tilde{j_1})^*(0) = 0$.

Note that by Theorem 1.2 (ii), F is of constant dimension. Let k be the dimension of F; then the generator of $H^k(F_i; Z_2)$ must be a product of some b_{it2}'s. Consequently, k must be even and $H^*(F_i; Z_2)$ has a subring which is isomorphic to $H^*((S^2)^{k/2}; Z_2)$. This together with the equation $\sum_{j\geq 0}$ rank $H^j(M; Z_2) = 2^n$ ([3, Theorem 1.6, p374]) shows the number of the components of F is at most $2^{n-k/2}$. □

Proof of Theorem 1.3. First we prove the statement (i). By the Lefschetz fixed point theorem, F is nonempty. By Theorem 1.2 (ii), F is of constant dimension. Note that $Sq^1 = \rho\beta$, where ρ and β fit into the Bockstein exact sequence

$$\to H^2(M_G; Z_2) \xrightarrow{\beta} H^3(M_G; Z) \xrightarrow{2} H^3(M_G; Z) \xrightarrow{\rho} H^3(M_G; Z_2) \to.$$

We claim $H^{odd}(M_G; Z) = 0$. Indeed, the spectral sequence which converges to $H^*(M_G; Z)$ with $E_2^{p,q} = H^p(RP^\infty; H^q(M; Z))$ collapses. Since

$$H^q(M; Z) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \text{free abelian} & \text{if } q \text{ is even,} \end{cases}$$

and $H^{odd}(RP^\infty; Z) = 0$, $H^{odd}(M_G; Z)$ must be trivial. This implies $\beta = 0$ and $Sq^1(c_i) = \rho\beta(c_i) = 0$, where $\{c_i\}$ are as those in the proof of Proposition 2.4. Finally, τ induces the trivial action on $H^*(M; Z)$ implies the triviality of the induced Z_2 action on $H^*(M; Z_2)$. So (i) follows from Proposition 2.4.

Next we consider (ii). Similarly to the proof of Proposition 2.4, let $\{c_t\}_{1\leq j\leq n}$ be the elements of $H^1(M_G; Z_2)$ such that $\{j^*(c_t)\}$ is a basis of the Z_2 vector space $H^1(M^n; Z_2)$, and let

$$j_1^*(c_t) = 1 \otimes b_{it} + a \otimes 1, \qquad a \in H^1(RP^\infty; Z_2),\ b_{it} \in H^1(F_i; Z_2).$$

Here we use the notation of Proposition 2.4. Then $\{1, b_{1t}, b_{2t}, ..., b_{nt}\}$ generate algebraically the Z_2 algebra $H^*(F_i; Z_2)$, and just as in the proof of Proposition 2.4, we have $(b_{it})^2 = 0$, and (ii) follows as (i). □

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