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APPROXIMATION FROM LOCALLY FINITE-DIMENSIONAL SHIFT-INVARIANT SPACES

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ABSTRACT. After exploring some topological properties of locally finite-dimensional shift-invariant subspaces S of $L_p(\mathbb{R}^s)$, we show that if S provides approximation order k, then it provides the corresponding simultaneous approximation order. In the case S is generated by a compactly supported function in $L_\infty(\mathbb{R})$, it is proved that S provides approximation order k in the $L_p(\mathbb{R})$-norm with $p > 1$ if and only if the generator is a derivative of a compactly supported function that satisfies the Strang-Fix conditions.

1. INTRODUCTION

Let S be a linear space consisting of functions defined on $\mathbb{R}^s$. S is said to be *shift invariant* if $f(\cdot + \alpha)$ lies in S whenever f does, for every $\alpha \in \mathbb{Z}^s$. S is said to be *locally finite-dimensional* if the restriction of S to any bounded subset of $\mathbb{R}^s$ is finite-dimensional. A typical example for such spaces is the linear span S of a finite number of compactly supported functions and their shifts. That is,

$$S = S_0(\Phi) := \operatorname{span}\{\varphi(\cdot + \alpha)\colon \alpha \in \mathbb{Z}^s, \varphi \in \Phi\},$$

with Φ a finite family of compactly supported functions. When Φ consists of one function φ, we denote $S_0(\Phi)$ by $S_0(\varphi)$. $S_0(\Phi)$ is usually called *a finitely generated shift-invariant space*. It is clear that the shift-invariance and the local finite-dimension are purely algebraic properties. In this paper we shall show how to probe the (simultaneous) approximation order provided by S by means of these two algebraic properties.

Let $m \geq 0$ be an integer and $k > 0$. A subspace S of $L_p(\mathbb{R}^s)$ is said to provide *simultaneous approximation order* (m, k) if

$$\inf_{g\in S} \sum_{j=0}^{m} \sum_{|\alpha|=j} h^j \|D^\alpha(f - g(\cdot/h))\|_p \leq C_f h^k \tag{1.1}$$

as $h \to 0+$, for every $f \in W_p^m(\mathbb{R}^s) \cap W_p^k(\mathbb{R}^s)$. Here, C_f is a constant independent of h and D^α is the α-order differentiation operator. By convention, S is said to provide approximation order k when it provides simultaneous approximation order $(0, k)$. The simultaneous approximation order of shift-invariant subspaces generated by a

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finite number of functions has been of interest in Approximation Theory and Finite Element Analysis for a long time and it is well known [1], [11], [12] that $S_0(\varphi)$ provides approximation order k if $\varphi \in L_p(\mathbb{R}^s)$ with $p \geq 1$ is compactly supported and satisfies the so-called Strang-Fix conditions of order k:

(i) $\hat{\varphi}(0) \neq 0$;
(ii) $D^\alpha \hat{\varphi} = 0$ on $2\pi\mathbb{Z}^s \backslash 0$, for all $|\alpha| < k$.

Here, $\hat{\varphi}$ denotes the Fourier transform of φ. The Strang-Fix conditions have been so well reputed because they enable us to determine the approximation order provided by $S_0(\varphi)$, with $\hat{\varphi}(0) \neq 0$, by examining the single generator φ, in spite of the fact that $S_0(\varphi)$ is infinite-dimensional.

It is well known that the above-mentioned Strang-Fix conditions can also be described algebraically as follows. Denote by $\varphi^{*\prime}$ the *discrete convolution mapping with* φ. Namely,

$$\varphi^{*\prime}\colon f \to \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) f(\alpha).$$

Since φ is assumed to be compactly supported, for each $x \in \mathbb{R}^s$, $\varphi^{*\prime} f(x)$ is a sum of finite number of terms. As shown in his paper [11], Shoenberg observed that $S_0(\varphi)$ provides approximation order k if $\varphi^{*\prime}$ maps Π_{k-1} onto Π_{k-1} in the univariate case, where Π_{k-1} is *the linear space of all polynomials of degree* $< k$. As is well known now, the Strang-Fix conditions of order k are equivalent to that $\varphi^{*\prime}$ maps Π_{k-1} onto Π_{k-1}. Therefore, $S_0(\varphi)$ provides approximation order k if φ has the algebraic property that $\varphi^{*\prime}$ maps Π_{k-1} onto Π_{k-1}. In his recent paper [8], Jia has shown the following interesting fact: For any function f defined on $\mathbb{R}^s$, there exists $a\colon \mathbb{Z}^s \to \mathbb{C}^s$ such that $f = \varphi^{*\prime} a$ if and only if $S_0(\varphi)$ locally contains f. Here, $S_0(\varphi)$ is said to *locally contain* f if the restriction of $S_0(\varphi)$ to any compact subset B contains the restriction of f to B. Therefore, if $\hat{\varphi}(0) \neq 0$, then $S_0(\varphi)$ provides approximation order k if and only if $S_0(\varphi)$ locally contains Π_{k-1} [8]. We note that $S_0(\varphi)$ locally containing Π_{k-1} cannot guarantee it to provide approximation order k in general. When $\hat{\varphi}(0) \neq 0$, it is well known that there is a *local approximation scheme* that realizes the approximation order provided by $S_0(\varphi)$ [1].

The condition that $\hat{\varphi}(0) \neq 0$ has been assumed in the past study of approximation order of $S_0(\varphi)$. As shown by Strang and Fix [12], for any compactly supported $\varphi \in L_2(\mathbb{R}^s)$, if $S_0(\varphi)$ provides an approximation order k via a controlled approximation scheme, then this condition is also necessary. But $S_0(\varphi)$ may provide a positive approximation order even if $\hat{\varphi}(0) = 0$. One well-known example is the function

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ -1, & \text{if } 1 < x \leq 2; \\ 0, & \text{else.} \end{cases} \tag{1.2}$$

It is known that $S_0(\varphi)$ indeed provides approximation order 1 in the $L_2(\mathbb{R})$-norm. In the recent paper by de Boor, DeVore, and Ron [4], the authors obtained a necessary and sufficient condition for $S_0(\varphi)$ to provide approximation order k in the $L_2(\mathbb{R}^s)$-norm, for any generator $\varphi \in L_2(\mathbb{R}^s)$. A corresponding result of a necessary and sufficient condition under which $S_0(\varphi)$ provides simultaneous approximation order (m, k) has been presented in [13], for any $\varphi \in W_2^m(\mathbb{R}^s)$. As shown in [13], for any compactly supported *univariate function* $\varphi \in W_2^m(\mathbb{R})$, $S_0(\varphi)$ provides simultaneous approximation order (m, k) in the $L_2(\mathbb{R})$-norm if and only if there exist a neighborhood Ω_α of the origin and a constant C_α such that

$$|\hat{\varphi}(x + 2\pi\alpha)| \leq C_\alpha |x|^k |\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha, \tag{1.3}$$

for all $\alpha \in \mathbb{Z}\backslash 0$. In particular, this implies that if $S_0(\varphi)$ provides approximation order k, then it also provides simultaneous approximation order (m, k).

In the next sections we shall prove that, for any shift-invariant subspace $S \subset W_p^m(\mathbb{R}^s)$ that is locally finite-dimensional, if S provides approximation order k, then it also provides simultaneous approximation order (m, k). In the univariate case, we shall prove that, for any nontrivial compactly supported $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ with $p > 1$, $S_0(\varphi)$ provides approximation order k if and only if φ satisfies (1.3) for all $\alpha \in \mathbb{Z}\backslash 0$, where $p' = p/(p-1)$ is the conjugate number of p. In other words, in this case, $S_0(\varphi)$ provides approximation order k if and only if

$$D^\alpha \hat{\varphi} = 0 \quad \text{on } 2\pi\mathbb{Z}\backslash 0, \forall 0 \leq \alpha < k + m, \tag{1.4}$$

where m is the smallest integer such that $D^m \hat{\varphi}(0) \neq 0$. As we shall see, any compactly supported function $\varphi \in L_p(\mathbb{R})$ satisfies (1.4) if and only if it is the mth-order derivative of a compactly supported function that satisfies the Strang-Fix condition of order $k + m$, with m the smallest integer such that $D^m \hat{\varphi}(0) \neq 0$. Therefore, among the compactly supported functions in $L_p(\mathbb{R})$ are only those that satisfy the Strang-Fix conditions or their derivatives the candidates for generating a shift-invariant space that may provide a positive approximation order.

2. Locally finite-dimensional shift-invariant spaces

In this section we show some nice topological properties owned by every shift-invariant subspace $S \subset L_p(\mathbb{R}^s)$ that is locally finite-dimensional.

Proposition 2.1. *Let $S \subset L_p(\mathbb{R}^s)$ be a shift-invariant subspace with $1 \leq p \leq \infty$ such that S is locally finite-dimensional. For any linear mapping $A\colon S \to L_q(\mathbb{R}^s)$, with $1 \leq q \leq \infty$, if A commutes with integer translations, then there exist two positive constants C_1 and C_2 such that, for all $f \in S$,*

$$\begin{aligned} &C_1 \left(\sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_|} \|f(\cdot + \alpha) - g\|^q_{L_p([0..1]^s)} \right)^{1/q} \\ &\quad \leq \|Af\|_q \leq C_2 \left(\sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_|} \|f(\cdot + \alpha) - g\|^q_{L_p([0..1]^s)} \right)^{1/q} \end{aligned} \tag{2.1}$$

when $q < \infty$,

$$\begin{aligned} &C_1 \sup_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_|} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)} \\ &\quad \leq \|Af\|_q \leq C_2 \sup_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_|} \|f(\cdot + \alpha) - g\|_{L_p([0..1]^s)} \end{aligned} \tag{2.2}$$

when $q = \infty$, where $\ker A_| := \{f \in S\colon Af = 0$ *on* $[0..1]^s\}$.

Proof. It is clear that $\ker A_|$ is a subspace of S. For any $f \in S$ and $g \in S$, the restriction of Af to $[0..1]^s$ equals the restriction of Ag to $[0..1]^s$ if and only if

$$A(f - g) = Af - Ag = 0 \quad \text{on } [0..1]^s.$$

This proves that the restriction of the range of A to $[0..1]^s$ is isomorphic to the restriction of the quotient space $S/\ker A_|$ to $[0..1]^s$. Since S is locally finite-dimensional, the restrictions of $S/\ker A_|$ and S to $[0..1]^s$ are finite-dimensional. As is well known, any linear mapping on a finite-dimensional normed space is bounded and any two norms on a finite-dimensional space are equivalent. Consequently, there exist two positive constants C_1 and C_2 such that

$$C_1 \min_{g\in\ker A_|} \|f-g\|_{L_p([0..1]^s)} \le \|Af\|_{L_q([0..1]^s)} \le C_2 \min_{g\in\ker A_|} \|f-g\|_{L_p([0..1]^s)} \tag{2.3}$$

for all $f \in S$. Since S is shift invariant, $f \in S$ implies that $f(\cdot+\alpha) \in S$ for any $\alpha \in \mathbb{Z}^s$. Also we have that

$$(Af)(\cdot+\alpha) = A(f(\cdot+\alpha)), \quad \forall\alpha \in \mathbb{Z}^s$$

because A commutes with integer translations. When $q = \infty$, (2.2) follows from (2.3) and the fact that S is shift invariant. When $q < \infty$, it follows from (2.3) that, for any $f \in S$,

$$\begin{aligned}\int_{\mathbb{R}^s} |(Af)(x)|^q\,dx &= \sum_{\alpha\in\mathbb{Z}^s} \int_{[0..1]^s} |(Af)(x+\alpha)|^q\,dx \\ &= \sum_{\alpha\in\mathbb{Z}^s} \int_{[0..1]^s} |A(f(x+\alpha))|^q\,dx \\ &\le \sum_{\alpha\in\mathbb{Z}^s} C_2^q \min_{g\in\ker A_|} \left(\int_{[0..1]^s} |f(x+\alpha)-g|^p\,dx\right)^{q/p} \\ &= C_2^q \sum_{\alpha\in\mathbb{Z}^s} \min_{g\in\ker A_|} \|f(\cdot+\alpha)-g\|^q_{L_p([0..1]^s)}.\end{aligned}$$

Analogously we can establish the other inequality of (2.1). ☐

In the case where $p = q$ we obtain that $\|Af\|_p \le C_2\|f\|_p$.

Corollary 2.2. *Let $1 \le p \le \infty$ and $S \subset L_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. If $A\colon S \to L_p(\mathbb{R}^s)$ is a linear mapping that commutes with integer translations, then A is bounded.*

For any polynomial of degree n: $p_n = \sum_{|\alpha|\le n} a_\alpha(\cdot)^\alpha$, by $p_n(D)$ we mean the differential operator

$$p_n(D) := \sum_{|\alpha|\le n} a_\alpha D^\alpha.$$

It is clear that $p_n(D)$ is a linear mapping from $W_p^n(\mathbb{R}^s)$ to $L_p(\mathbb{R}^s)$. For any sufficiently smooth function f and any $t \in \mathbb{R}^s$,

$$D^\alpha(f(\cdot+t)) = (D^\alpha f)(\cdot+t).$$

This shows that $p_n(D)$ commutes with integer translations.

Corollary 2.3. *Let p_n be any polynomial in s-variable of degree n and S a shift-invariant subspace of $W_p^n(\mathbb{R}^s)$ that is locally finite-dimensional. Then there exists a constant C such that $\|p_n(D)f\|_p \le C\|f\|_p$ for all f in S.*

It is well known that

$$\left(\sum_{\alpha\in\mathbb{Z}^s}|a(\alpha)|^q\right)^{1/q} \le \left(\sum_{a\in\mathbb{Z}^s}|a(\alpha)|^p\right)^{1/p}$$

if $1 \le p \le q \le \infty$ and $a \in l_q(\mathbb{Z}^s)$. From the proof of Proposition 2.1 we obtain

Corollary 2.4. *Let $1 \le p < q \le \infty$ and $S \subset L_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional and is locally contained in $L_q(\mathbb{R}^s)$. Then, $S \subset L_q(\mathbb{R}^s)$ and there exists a constant C such that $\|f\|_q \le C\|f\|_p$ for all $f \in s$.*

Corollary 2.5. *Let $S \subset L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional, with $1 \le p < q \le \infty$. Then the closure of S in $L_p(\mathbb{R}^s)$ is contained in the closure of S in $L_q(\mathbb{R}^s)$.*

When $S = S_0(\Phi)$ is generated by a finite number of compactly supported functions in $L_p(\mathbb{R}^s)$, we denote by $S_*(\Phi)$ the closure of $S_0(\Phi)$ in the topology of *pointwise convergence*. Namely, $f \in S_*(\Phi)$ if and only if there is a sequence $s_j \in S_0(\Phi)$ such that $\lim_{j\to\infty} s_j(x) = f(x)$ for almost every $x \in \mathbb{R}^s$. One can verify that $S_*(\Phi)$ consists of all functions of the form

$$\sum_{\varphi\in\Phi}\varphi^{*\prime}a_\varphi,$$

for all $a_\varphi\colon \mathbb{Z}^s \to \mathbb{C}^s$. As shown in [8], f is contained in $S_*(\Phi)$ if and only if f is locally contained in $S_0(\Phi)$. Since the topology of $L_p(\mathbb{R}^s)$ is stronger than that of $S_*(\Phi)$, we have the following corollary that was first observed by Jia.

Corollary 2.6. *For any finite family Φ of compactly supported functions in $L_p(\mathbb{R}^s)$, $S_*(\Phi) \cap L_p(\mathbb{R}^s)$ is a closed subspace of $L_p(\mathbb{R}^s)$.*

Since $S_*(\Phi) \cap L_p(\mathbb{R}^s)$ is also shift-invariant and locally finite-dimensional, from Corollary 2.4 we obtain

Proposition 2.7. *For any $1 \le p < q \le \infty$ and any finite $\Phi \subset L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)$ consisting of compactly supported functions, we have that $S_*(\Phi)\cap L_p(\mathbb{R}^s) \subset S_*(\Phi)\cap L_q(\mathbb{R}^s)$.*

For $S_0(\Phi) \subset L_p(\mathbb{R}^s)$, denote by $S_p(\Phi)$ the closure of $S_0(\Phi)$ in $L_p(\mathbb{R}^s)$. Let φ be a compactly supported function in $L_2(\mathbb{R}^s)$. As proved by de Boor, DeVore, Ron [4], $S_*(\varphi) \cap L_2(\mathbb{R}^s)$ is a subspace of $S_2(\varphi)$. Since $S_*(\varphi) \cap L_2(\mathbb{R}^s)$ contains $S_0(\varphi)$ and is closed, it follows that $S_2(\varphi) = S_*(\varphi) \cap L_2(\mathbb{R}^s)$. We shall show that this has an extension to the case where $1 < p < 2$ and $\varphi \in L_{p'}(\mathbb{R}^s)$. Recall that $p' = p/(p-1)$. For the proof we need

Proposition 2.8. *Let $1 < p < \infty$ and $\Phi \subset L_p(\mathbb{R}^s) \cap L_{p'}(\mathbb{R}^s)$ be a finite family of compactly supported functions. Then, $S_{p'}(\Phi)$ can be identified with the dual space $(S_p(\Phi))^*$ of $S_p(\Phi)$.*

Proof. As $S_p(\Phi)$ is a closed subspace of $L_p(\mathbb{R}^s)$, we know that the dual space of $S_p(\Phi)$ is isomorphic to the quotient space $L_{p'}(\mathbb{R}^s)/(S_p(\Phi))^\perp$ [10]. Since $L_p(\mathbb{R}^s)$ and $L_{p'}(\mathbb{R}^s)$ are reflexive and $S_p(\Phi)$ is a closed subspace of $L_p(\mathbb{R}^s)$, we have that

$$(S_p(\Phi))^{**} = (L_{p'}(\mathbb{R}^s)/(S_p(\Phi))^\perp)^* = (S_p(\Phi))^{\perp\perp} = S_p(\Phi).$$

It suffices to prove the case where $p \le 2$ because, otherwise, $p' \le 2$ and $(S_{p'}(\Phi))^* = (S_p(\Phi))^{**} = S_p(\Phi)$. First we prove that $S_{p'}(\Phi)$ is dense in $(S_p(\Phi))^*$. For any

$f \in (S_p(\Phi))^{**} = S_p(\Phi)$ that annihilates $S_{p'}(\Phi)$, $\int_{\mathbb{R}^s} |f|^2 = 0$ because f lines in $S_p(\Phi) \subset S_{p'}(\Phi)$. So $f = 0$. This proves that $S_{p'}(\Phi)$ is dense in $(S_p(\Phi))^*$. Therefore, $S_p(\Phi) = (S_p(\Phi))^{**} = (S_{p'}(\Phi))^*$ and

$$S_{p'}(\Phi) = (S_{p'}(\Phi))^{**} = (S_p(\Phi))^*. \quad \square \tag{2.4}$$

Theorem 2.9. *Let $1 < p \leq 2$ and $p' = p/(p-1)$. For any compactly supported $\varphi \in L_{p'}(\mathbb{R}^s)$, $S_p(\varphi) = S_*(\varphi) \cap L_p(\mathbb{R}^s)$.*

Proof. By Corollary 2.5 and Proposition 2.7,

$$S_p(\varphi) \subset S_*(\varphi) \cap L_p(\mathbb{R}^s) \subset S_*(\varphi) \cap L_2(\mathbb{R}^s) = S_2(\varphi) \subset S_{p'}(\varphi).$$

This implies the dual space of $S_*(\varphi) \cap L_p(\mathbb{R}^s)$ contains that of $S_p(\varphi)$. As we know, $S_p(\varphi)$ and $S_*(\varphi) \cap L_p(\mathbb{R}^s)$ both are closed in $L_p(\mathbb{R}^s)$. From the reflexivity it follows that some closed subspace of $S_{p'}(\varphi)$ can be identified with $(S_*(\varphi) \cap L_p(\mathbb{R}^s))^*$. From Proposition 2.8 we know that $S_{p'}(\varphi)$ can be identified with the dual space of $S_p(\varphi)$. Hence, $S_p(\varphi)$ and $S_*(\varphi) \cap L_p(\mathbb{R}^s)$ have the same dual space $S_{p'}(\varphi)$. Thus we obtain that

$$S_p(\varphi) = (S_p(\varphi))^{**} = (S_*(\varphi) \cap L_p(\mathbb{R}^s))^{**} = S_*(\varphi) \cap L_P(\mathbb{R}^s). \quad \square$$

3. Application 1: Multivariate approximation

In this section we apply the results obtained in Section 2 to obtain some results about multivariate approximation from shift-invariant spaces that are locally finite-dimensional.

Theorem 3.1. *Let $S \subset W_p^m(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. If S provides approximation order k in the $L_p(\mathbb{R}^s)$-norm, then S also provides simultaneous approximation order (m, k).*

Proof. For any $f \in W_p^m(\mathbb{R}^s)$, there exists $s_h \in S$ such that

$$\|f - s_h(\cdot/h)\|_p \leq C_f h^k$$

with some constant C_f independent of h. Let ψ be any compactly supported function in $W_p^m(\mathbb{R}^s)$ such that $S_0(\psi)$ provides simultaneous approximation order (m, k). For instance, we can choose ψ as a tensor product of some univariate B-spline functions [5]. Let

$$S_+ := S + S_0(\psi).$$

That is, S_+ is the space spanned by S and $S_0(\psi)$. It is clear that S_+ is shift invariant and locally finite-dimensional. Therefore, by Corollary 2.3, there exists a constant C independent of h such that, for any $|\alpha| \leq m$,

$$\|D^\alpha g(\cdot/h)\|_p \leq C h^{-|\alpha|} \|g(\cdot/h)\|_p, \quad \forall g \in S_+. \tag{3.1}$$

Since $S_0(\psi)$ provides simultaneous approximation k, there exists g_h in $S_0(\psi)$ such that

$$\|D^\alpha (f - g_h(\cdot/h))\|_p \leq B_f h^{k-|\alpha|}$$

for all $|\alpha| \le m$, where B_f is some constant independent of h. As $g_h - s_h$ lies in S_+,

$$\begin{aligned}\|D^\alpha(f - s_h(\cdot/h))\|_p &\le \|D^\alpha(f - g_h(\cdot/h))\|_p + \|D^\alpha(g_h(\cdot/h) - s_h(\cdot/h))\|_p \\ &\le B_f h^{k-|\alpha|} + Ch^{-|\alpha|}\|g_h(\cdot/h) - s_h(\cdot/h)\|_p \\ &\le B_f h^{k-|\alpha|} + Ch^{-|\alpha|}(\|f - g_h(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p) \\ &\le (B_f + CB_f + CC_f)h^{k-|\alpha|}. \quad \square\end{aligned}$$

From the above proof we obtain the following

Corollary 3.2. *Let $S \subset W_p^m(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. For any $f \in W_p^m(\mathbb{R}^s) \cap W_p^k(\mathbb{R}^s)$ and any $s_h \in S_0(\Phi)$, if $\|f - s_h(\cdot/h)\|_p = O(h^k)$, then $\|D^\alpha(f - s_h(\cdot/h))\|_p = O(h^{k-|\alpha|})$ for every $|\alpha| \le m$.*

This shows that any approximation scheme having approximation order k in the $L_p(\mathbb{R}^s)$-norm automatically has simultaneous approximation order (m, k) in this case.

Theorem 3.3. *Let $S \subset L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)$ be a shift-invariant subspace with $1 \le p < q \le \infty$, such that S is locally finite-dimensional. If S provides approximation order k in the $L_p(\mathbb{R}^s)$-norm, then S provides approximation order larger than or equal to $k - s(1/p - 1/q)$ in the $L_q(\mathbb{R}^s)$-norm.*

Proof. Let ψ be a compactly supported function in $L_\infty(\mathbb{R}^s)$ such that ψ satisfies the Strang-Fix conditions of order k. Then $S_0(\psi)$ provides approximation order k in any $L_r(\mathbb{R}^s)$-norm for all $1 \le r \le \infty$. Moreover [1], there is a local approximation scheme

$$Q_h \colon W_r^k(\mathbb{R}^s) \to S(\psi) := \psi *' \ell_r(\mathbb{Z}^s)$$

independent of r such that $\|f - (Q_h f)(\cdot/h)\|_r = O(h^k)$.

Let S_+ be the space spanned by S and $S(\psi)$. It is clear that S_+ is also shift invariant and locally finite-dimensional. By Corollary 2.4, there exists a constant C such that $\|f\|_q \le C\|f\|_p$, for all $f \in S_+$. For $f \in W_p^k(\mathbb{R}^s) \cap W_q^k(\mathbb{R}^s)$ and some $s_h \in S$,

$$\begin{aligned}\|f - s_h(\cdot/h)\|_q &\le \|f - (Q_h f)(\cdot/h)\|_q + \|(Q_h f)(\cdot/h) - s_h(\cdot/h)\|_q \\ &= h^{s/q}\|Q_h f - s_h\|_q + O(h^k) \\ &\le h^{s/q} C\|Q_h f - s_h\|_p + O(h^k) \\ &= h^{s/q - s/p} C\|(Q_h f)(\cdot/h) - s_h(\cdot/h)\|_p + O(h^k) \\ &\le Ch^{-s(1/p-1/q)}(\|f - (Q_h f)(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p) + O(h^k) \\ &= O(h^{k-s(1/p-1/q)})\end{aligned}$$

because S provides approximation order k in the $L_p(\mathbb{R}^s)$-norm. $\square$

When $s(1/p-1/q) < 1$, k is an integer, and S provides an integral approximation order in the $L_q(\mathbb{R}^s)$-norm, it follows that S also provides approximation order k in the $L_q(\mathbb{R}^s)$-norm. In particular, if S is known to provide an integral approximation order in the $L_q(\mathbb{R}^s)$-norm for all $q \ge p$, S providing approximation order k in the $L_p(\mathbb{R}^s)$-norm with k an integer implies S providing (at least) approximation order k in the $L_q(\mathbb{R}^s)$-norm, for all $q \geq p$. As we shall see, in the univariate case, the approximation order of any shift-invariant space generated by a compactly supported function must be an integer, provided $p \geq 2$.

4. Application 2: Univariate approximation

In the univariate case, there is an algebraic characterization for S to provide approximation order k even if $\hat{\varphi}(0) = 0$ for every $\varphi \in S \cap L_1(\mathbb{R})$, where S is a shift-invariant subspace of $L_p(\mathbb{R})$ that is locally finite-dimensional. Let $\Phi \subset L_p(\mathbb{R})$ be a finite family of compactly supported functions. Recall that $S_*(\Phi)$ is the closure of $S_0(\Phi)$ in the topology of pointwise convergence. As proved by Jia [7], $S_*(\Phi)$ provides approximation order k if and only if there is a compactly supported $\psi \in S_*(\Phi) \cap L_p(\mathbb{R})$ such that $\psi^{*\prime}$ maps Π_{k-1} onto Π_{k-1}. Namely, $S_*(\Phi)$ provides approximation order k if and only if there exist sequences $a_\varphi \colon \mathbb{Z} \to \mathbb{C}$ such that

$$\psi = \sum_{\varphi \in \Phi} \varphi^{*\prime} a_\varphi \in L_p(\mathbb{R})$$

is compactly supported and satisfies the Strang-Fix conditions of order k. This result reveals an intrinsic property of $S_*(\varphi)$ that provides a positive approximation order, as well as of φ because $S_*(\varphi)$ is the closure of $S_0(\varphi)$ in the topology of pointwise convergence. As pointed out in [7], this follows that the approximation order provided by $S_*(\Phi)$ is an integer. Of another interest is that, when $\Phi \subset L_p(\mathbb{R}) \cap L_q(\mathbb{R})$ with $p < q$, $S_*(\Phi)$ provides approximation order k in the $L_q(\mathbb{R})$-norm if and only if it provides approximation order k in the $L_p(\mathbb{R}^s)$-norm because, as we proved in Section 2, $S_*(\Phi) \cap L_p(\mathbb{R}^s) \subset S_*(\Phi) \cap L_q(\mathbb{R}^s)$ and a compactly supported function in $L_q(\mathbb{R})$ lies in $L_p(\mathbb{R})$.

As $S_*(\Phi)$ is an infinite-dimensional space unless it is trivial, apparently it is nontrivial to determine if $S_*(\Phi)$ contains a compactly supported function that satisfies the Strang-Fix conditions of order k, even if Φ consists of a single compactly supported function in $L_p(\mathbb{R})$. For any given compactly supported function $\varphi \in L_p(\mathbb{R})$, it is more practically interesting to have a necessary and sufficient condition on the generator φ itself for determining the approximation order provided by $S_0(\varphi)$. In the following we shall show that when $S = S_0(\varphi)$ with $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ compactly supported then $S_*(\varphi)$ and $S_0(\varphi)$ provide the same approximation order in the $L_p(\mathbb{R})$-norm for $p > 1$. In particular, we prove that, for any compactly supported $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$, with $p > 1$, $S_0(\varphi)$ provides approximation order k if and only if $\varphi = D^m\psi$ for some compactly supported $\psi \in W_p^m(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k+m$. In other words, $S_0(\varphi)$ provides approximation order k if and only if there exists an integer $m \geq 0$ such that $D^m\hat{\varphi}(0) \neq 0$, and $D^\alpha\hat{\varphi} = 0$ on $2\pi\mathbb{Z}\backslash 0$ for all $0 \leq \alpha < k+m$.

Theorem 4.1. *Let $1 < p \leq \infty$ and $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ be compactly supported. Denote by $S_p(\varphi)$ and $S_*(\varphi)$ the closure of $S_0(\varphi)$ in the $L_p(\mathbb{R})$-norm and in the topology of pointwise convergence, respectively. Then $S_0(\varphi)$ provides approximation order k in the $L_p(\mathbb{R})$-norm if and only if there exists a compactly supported function $\psi \in S_2(\varphi) \cap S_p(\varphi) \cap S_*(\varphi)$ such that ψ satisfies the Strang-Fix conditions of order k.*

Proof. We only need to prove the necessity because $S_0(\varphi)$ and $S_p(\varphi)$ provide the same approximation order. It is clear that if $S_0(\varphi)$ provides approximation order k in the $L_p(\mathbb{R})$-norm, then so does $S_*(\varphi) \cap L_p(\mathbb{R})$. Therefore, $S_*(\varphi)$ contains a compactly supported function $\psi \in L_p(\mathbb{R})$ that satisfies the Strang-Fix conditions of order k. By Theorem 2.16 in [4], ψ lies in $S_2(\varphi)$ if $p \geq 2$. By Corollary 2.5, $S_2(\varphi)$ is contained in $S_p(\varphi)$ when $p \geq 2$. In the case $p < 2$, by Theorem 2.9 and Corollary 2.5, $S_p(\varphi) = S_*(\varphi) \cap L_p(\mathbb{R}) \subset S_2(\varphi)$. □

When $p \geq 2$, we have that $p' \leq p$. Hence, if $p \geq 2$, then the condition that the compactly supported function φ belongs to $L_{p'}(\mathbb{R})$ is automatically satisfied. As an immediate consequence, we obtain that the approximation order provided by $S_0(\varphi)$ is an integer if $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ is compactly supported and $p > 1$.

Corollary 4.2. *Let $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ be a compactly supported function, with $1 < p \leq \infty$. Then, $S_0(\varphi)$ provides approximation order k in the $L_p(\mathbb{R})$-norm if and only if $S_0(\varphi)$ provides approximation order k in the $L_2(\mathbb{R})$-norm.*

Proof. It suffices to prove that $S_0(\varphi)$ providing approximation order k in the $L_2(\mathbb{R})$-norm implies $S_0(\varphi)$ providing approximation order $\geq k$ in the $L_p(\mathbb{R})$-norm.

If $S_2(\varphi)$ provides approximation order k, then $S_2(\varphi) = S_* \cap L_2(\mathbb{R})$ contains a compactly supported function ψ that satisfies the Strang-Fix conditions of order k. When $p > 2$, from that $S_2(\varphi) \subset S_p(\varphi)$ it follows that $S_p(\varphi)$ also provides approximation order k. In the case $p < 2$, we have that $\psi \in S_*(\varphi) \cap L_p(\mathbb{R}^s) = S_p(\varphi)$, because ψ is compactly supported. So $S_p(\varphi)$ provides approximation order k. □

As we know, for a nontrivial compactly supported $\varphi \in L_2(\mathbb{R})$, $S_0(\varphi)$ provides approximation order k if and only if, for each $\alpha \in \mathbb{Z}\backslash 0$, there exist a neighborhood Ω_α of the origin and a constant C_α such that (1.3) holds. So we have

Theorem 4.3. *For any nontrivial compactly supported function $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$, with p satisfying $1 < p \leq \infty$, $S_0(\varphi)$ provides approximation order k if and only if, for each $\alpha \in \mathbb{Z}\backslash 0$, there exist a neighborhood Ω_α of the origin and a constant C_α such that*

$$|\hat{\varphi}(x + 2\pi\alpha)| \leq C_\alpha |x|^k |\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha. \tag{4.1}$$

For any nontrivial compactly supported function $\varphi \in L_1(\mathbb{R})$, there exists an integer $m \geq 0$ such that $D^\alpha \hat{\varphi}(0) = 0$ for all nonnegative integers $\alpha < m$ but $D^m \hat{\varphi}(0) \neq 0$. As one can verify, (4.1) is equivalent to that $D^\beta \hat{\varphi}(2\pi\alpha) = 0$ for all integers $0 \leq \beta < k + m$, where m is the smallest integer such that $D^m \hat{\varphi}(0) \neq 0$. When $m > 0$, it is clear that

$$\varphi_1(x) := \int_{-\infty}^{x} \varphi(t)\, dt$$

is a compactly supported continuous function and for almost every $x \in \mathbb{R}$ we have that $D\varphi_1(x) = \varphi(x)$. Thus we obtain

$$\widehat{\varphi_1}(x) = \frac{\hat{\varphi}(x)}{ix}, \quad \forall x \neq 0$$

and $\widehat{\varphi_1}(0) = -iD\hat{\varphi}(0)$. When $m \geq 1$, define

$$\varphi_m = \int_{-\infty}^{x} \frac{(x-t)^{m-1}}{(m-1)!} \varphi(t)\, dt.$$

by induction we can prove that φ_m is compactly supported, $D^m \varphi_m = \varphi$, and

$$\widehat{\varphi_m}(x) = \frac{\hat{\varphi}(x)}{(ix)^m}, \quad \forall x \neq 0. \tag{4.2}$$

Note that $\lim_{x\to 0}\widehat{\varphi_m}(x) = (-i)^m D^m\hat{\varphi}(0) \neq 0$.

Corollary 4.4. *Let $p > 1$, $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$ be a compactly supported function, and m be the smallest integer such that $D^m\hat{\varphi}(0) \neq 0$. Then, $S_0(\varphi)$ provides approximation order k in the $L_p(\mathbb{R})$-norm if and only if $\varphi = D^m\psi$ for some compactly supported function $\psi \in W_p^m(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k+m$.*

Corollary 4.5. *For $p > 1$ and any compactly supported function $\varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R})$, $S_0(\varphi)$ provides approximation order $k \geq 1$ if and only if it locally contains Π_{k-1}.*

Proof. We only need to prove the sufficiency because the necessity has been proved by Jia [8]. Since $S_0(\varphi)$ locally contains a nontrivial subspace Π_{k-1}, φ is not trivial. So, $\varphi = D^m\psi$ for some compactly supported $\psi \in W_p^m(\mathbb{R})$ that satifies $\hat{\psi}(0) \neq 0$. Note that $D^m S_0(\psi) = S_0(\varphi)$. It follows that $S_0(\psi)$ locally contains Π_{k+m-1}. Since $\hat{\psi}(0) \neq 0$, we know that $S_0(\psi)$ locally contains Π_{k+m-1} if and only if ψ satisfies the Strang-Fix conditions of order $k+m$. Therefore, $S_0(\varphi)$ provides approximation order k. □

Example. Let φ be the function defined by (1.2). It is clear that φ is bounded and is the first-order derivative of the following B-spline:

$$\psi(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1; \\ 2-x, & \text{if } 1 < x \leq 2; \\ 0, & \text{else.} \end{cases} \tag{4.3}$$

One can verify that ψ satisfies the Strang-Fix conditions of order 2. Thus we know that $S_0(\varphi)$ does provide approximation order 1 in the $L_p(\mathbb{R})$-norm for $p > 1$.

As we know, if $\hat{\varphi}(0) = 0$, then $S_0(\varphi)$ cannot provide any positive approximation order in the $L_1(\mathbb{R})$-norm. When $\varphi \in L_1(\mathbb{R})$ is compactly supported and $\hat{\varphi}(0) \neq 0$, it is well known that $S_0(\varphi)$ provides approximation order $k > 0$ if and only if φ satisfies the Strang-Fix conditions of order k. So the approximation order provided by $S_0(\varphi)$ in the $L_1(\mathbb{R})$-norm is an integer.

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