

DISCRETE VALUATION OVERRINGS OF NOETHERIAN DOMAINS

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ABSTRACT. We show that, given a chain $0 = P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals in a Noetherian domain R , there exist a finitely generated overring T of R and a saturated chain of primes in T contracting term by term to the given chain. We further show that there is a discrete rank n valuation overring of R whose primes contract to those of the given chain.

Let R be an integral domain (with 1). It is well known that if $0 = P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of prime ideals in R , then there exist a valuation overring V of R and a chain of primes $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_n$ in V with $Q_i \cap R = P_i$ for $i = 1, \dots, n$. On the other hand, Chevalley showed [C] that if R is Noetherian and P is any nonzero prime of R , then there is a discrete rank 1 valuation overring of R centered on P . D. D. Anderson has asked whether these two results can be “combined”: given a Noetherian domain R and a chain as above, is there a discrete rank n valuation overring of R whose primes contract to those of the given chain? (Recall that a finite-dimensional valuation ring V is discrete if PV_P is principal for each prime ideal P of V .)

We show that this question has an affirmative answer. In fact, we prove the following stronger result.

Theorem. *Let $0 = P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of primes in the Noetherian domain R , and let s_1, \dots, s_n be a sequence of integers with $1 \leq s_i \leq \text{ht}(P_i/P_{i-1})$. Then there exist an overring T of R and a chain of primes $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_n$ in T such that $Q_i \cap R = P_i$ and $\text{ht}(Q_i/Q_{i-1}) = s_i$ for $i = 1, \dots, n$. Moreover, T can be taken to be either a finitely generated extension of R or a discrete valuation ring of rank $s = \sum s_i$.*

We use “ \subset ” to denote proper containment. For a prime ideal P of a (commutative) ring R , an *upper to P* is a prime ideal U of $R[X]$ for which $U \cap R = P$ and $U \neq P[X]$. An overring of a domain is understood to have the same quotient field as the base domain. Other terminology is standard as in [G].

Lemma 1. *Let P be a prime ideal in a Noetherian ring R , and let $a, b \in R \setminus P$. If there is no prime p for which $P \subset p$, $\text{ht}(p/P) = 1$, and $a, b \in p$, then $\text{rad}(P, aX - b)R[X]$ is an upper to P in $R[X]$.*

Proof. Let U be a prime of $R[X]$ minimal over $(P, aX - b)R[X]$. By the principal ideal theorem, $\text{ht}(U/PR[X]) = 1$. Hence either U is an upper to P or $U = p[X]$,

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where $p = U \cap R$. In the latter case, we have $P \subset p$, $\text{ht}(p/P) = 1$, and $a, b \in p$, contradicting the hypothesis. Thus any prime minimal over $(P, aX - b)R[X]$ is an upper to P , and since $aX - b$ has degree 1, only one upper to P can contain it. That upper is clearly $\text{rad}(P, aX - b)R[X]$. \square

We now prove the first case of our theorem; that is, we construct a finitely generated overring T of R such that T contains the indicated chain of primes. Our proof may be regarded as a refinement of the proof of Chevalley's result presented by Kaplansky in [K2].

Proof. Choose an r with $1 \leq r \leq n$ for which $\text{ht}(P_r/P_{r-1}) \geq 2$. We will produce a simple overring $R[u]$ of R and a chain of primes $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_{r-1} \subset Q_r \subset \cdots \subset Q_n$ in $R[u]$ such that $Q_i \cap R = P_i$ for $0 \leq i \leq n$, $\text{ht}(Q_i/Q_{i-1}) = \text{ht}(P_i/P_{i-1})$ for $i \neq r$, and $\text{ht}(Q_r/Q_{r-1}) = \text{ht}(P_r/P_{r-1}) - 1$. The first case of the theorem will follow from an iteration of this process.

Pick $a \in P_r \setminus P_{r-1}$ and, for $0 \leq i \leq r-1$, let $S_i = \{p \in \text{Spec}(R) \mid P_i \subset p, a \in p, \text{ and } \text{ht}(p/P_i) = 1\}$. Note that each S_i is finite. Also, since $\text{ht}(P_r/P_i) \geq 2$, $P_r \not\subset p$ for $p \in S_i$. By prime avoidance, we may choose $b \in P_r \setminus \bigcup S_i$. By Lemma 1, $U_i = \text{rad}(P, aX - b)R[X]$ is an upper to P_i for $0 \leq i \leq r-1$. Since $a, b \in P_r$, we see that $U_{r-1} \subset P_r[X]$, and we have the following chain of primes in $R[X]/U_0$:

$$U_0/U_0 \subset U_1/U_0 \subset \cdots \subset U_{r-1}/U_0 \subset P_r[X]/U_0 \subset P_{r+1}[X]/U_0 \subset \cdots \subset P_n[X]/U_0.$$

Since U_0 is an upper to zero containing $aX - b$, $R[X]/U_0$ can be identified with the simple overring $R[u]$, where $u = b/a$. We claim that the chain (in $R[u]$ which identifies with the chain) above satisfies our requirements. It is clear that it contracts to the original chain in R . Therefore, to complete the argument, it suffices to show that $\text{ht}(U_i/U_{i-1}) = \text{ht}(P_i/P_{i-1})$ for $1 \leq i \leq r-1$, that $\text{ht}(P_r[X]/U_{r-1}) = \text{ht}(P_r/P_{r-1}) - 1$, and that $\text{ht}(P_i[X]/P_{i-1}[X]) = \text{ht}(P_i/P_{i-1})$ for $r+1 \leq i \leq n$. The last of these three statements is clear, and the first two are easy exercises using the generalized principal ideal theorem [K1, Theorem 154] (using the fact that $U_i/P_i[X]$ is the radical of a principal ideal in the ring $R[X]/P_i[X] \simeq (R/P_i)[X]$). This completes the argument for the first case of our theorem. \square

The second case of the theorem will be proved in two steps. The next lemma covers the situation where the given chain $0 = P_0 \subset P_1 \subset \cdots \subset P_n$ is saturated in R .

Lemma 2. *Let R be a Noetherian domain with quotient field K , let $0 = P_0 \subset P_1 \subset \cdots \subset P_n$ be a saturated chain of primes in R , and let L be a finite extension field of K . Then there exists a discrete rank n valuation domain V with quotient field L such that, if $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_n$ are the primes of V , then $Q_i \cap R = P_i$ and the quotient field of V/Q_i is a finite extension of the quotient field of R/P_i for each $i = 1, \dots, n$.*

Proof. First, assume $n = 1$. Since localizing at P_1 does not change the quotient field of R/P_1 , we may assume (for the case under consideration) that R is local with maximal ideal P_1 (and $\dim(R) = 1$). By [N, 33.10], the integral closure R' of R is a Krull domain, and $R'_Q/QR'_Q \simeq R'/Q$ is a finite extension of R/P_1 , where Q is a maximal ideal of R' . Of course, R'_Q is a discrete rank 1 valuation domain. Since L is a finite extension of K , there is a discrete rank 1 valuation domain

(V, Q_1) with quotient field L such that $V \cap K = R'_Q$ and $Q_1 \cap R' = QR'_Q$ [G, Section 19] and such that V/Q_1 is a finite extension of R'_Q/QR'_Q (and hence also of R/P_1) [G, Lemma 41.4]. This takes care of the case $n = 1$. Inductively, assume that W is a discrete rank $n - 1$ valuation ring with quotient field L such that, if $0 = Q_0 \subset \cdots \subset Q_{n-1}$ are the primes of W , then $Q_i \cap R = P_i$ and the quotient field of W/Q_i is a finite extension of that of R/P_i for each $i = 1, \dots, n - 1$. In particular, W/Q_{n-1} is a finite extension of the quotient field of R/P_{n-1} . Then since P_n/P_{n-1} is a height one prime of the Noetherian domain R/P_{n-1} , by the case $n = 1$, there is a discrete rank 1 valuation ring (\bar{V}, \bar{Q}_n) with quotient field W/Q_{n-1} such that $\bar{Q}_n \cap (R/P_{n-1}) = P_n/P_{n-1}$ and such that \bar{V}/\bar{Q}_n is a finite extension of the quotient field of R/P_{n-1} . Let V be the inverse image of \bar{V} under the natural map $W \rightarrow W/Q_{n-1}$. It is not difficult to verify that this V does what is required. \square

We now prove the second case of our theorem. That is, we construct a discrete valuation overring T of R and an appropriate chain of primes in T .

Proof. By the first case of the theorem, there exist a finitely generated overring S of R and a chain of primes $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_n$ in S such that $Q_i \cap R = P_i$ and $\text{ht}(Q_i/Q_{i-1}) = s_i$ for $i = 1, \dots, n$. For each i , insert between Q_{i-1} and Q_i a subchain of length s_i ; this produces a saturated chain of primes of length $s = \sum s_i$ from 0 to Q_n in S . Now apply Lemma 2 to obtain the required T . \square

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