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ALMOST DISJOINT PERMUTATION GROUPS

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ABSTRACT. A permutation group G on a set E of (infinite) cardinality κ is *almost disjoint* if no element of G except the identity has κ fixed points, i.e., if G is an almost disjoint family of subsets of $E \times E$. We show how almost disjoint permutation groups can be constructed from almost disjoint families of sets.

Permutations are composed from left to right; $\mathrm{Sym}(E)$ is the group of all permutations of E; the set of fixed points of a permutation π is denoted by $\mathrm{fix}(\pi)$. The letters δ, κ, λ will denote infinite cardinals.

A family $\mathcal{S}$ of sets is δ-*almost disjoint* if $|X \cap Y| < \delta$ whenever $X, Y \in \mathcal{S}$ and $X \neq Y$. A family of κ-element sets is *almost disjoint* if it is κ-almost disjoint. A permutation group G on an infinite set E is δ-*almost disjoint* if $|\mathrm{fix}(\pi)| < \delta$ for every $\pi \in G \setminus \{1\}$; this amounts to saying that, if we regard the permutations as subsets of $E \times E$, then G is a δ-almost disjoint family of sets. The group G is *almost disjoint* if it is κ-almost disjoint for $\kappa = |E|$.

In this note, we show that some questions about the existence of almost disjoint permutation groups reduce to questions about almost disjoint families of sets, which have been thoroughly investigated by set theorists: see Baumgartner [1] or Williams [4] for a survey of results in this area.

Theorem 1. *For any uncountable cardinals $\delta \leq \kappa \leq \lambda$, the following statements are equivalent*:

(i) *there is a δ-almost disjoint subgroup G of* $\mathrm{Sym}(\kappa)$ *with* $|G| = \lambda$;

(ii) *there is a δ-almost disjoint family $\mathcal{S}$ of κ-element subsets of κ with $|\mathcal{S}| = \lambda$.*

Proof. Only the implication (ii) $\Rightarrow$ (i) requires proof. Let I and T be sets, $|I| = \lambda$, $|T| = \kappa$. Choose sets $S_i \subseteq T$ $(i \in I)$ so that $|S_i| = \kappa$ and $|S_i \cap S_j| < \delta$ for $i \neq j$. For each $i \in I$, choose an involution $\theta_i \in \mathrm{Sym}(T)$ so that $S_i\theta_i = T \setminus S_i$, and then define an involution $\pi_i \in \mathrm{Sym}(T \times \mathbb{Z})$ by setting

$$(t, n)\pi_i = \begin{cases} (t\theta_i,\ n+1) & \text{if } t \in S_i, \\ (t\theta_i,\ n-1) & \text{if } t \notin S_i. \end{cases}$$

Let G be the subgroup of $\mathrm{Sym}(T \times \mathbb{Z})$ generated by $\{\pi_i : i \in I\}$. As $|T \times \mathbb{Z}| = \kappa$ and $|G| = \lambda$, we only have to show that G is δ-almost disjoint. It will suffice to prove the following.

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Claim. Let $m \geq 1$. If $i_1, \ldots, i_m \in I$ and $i_j \neq i_{j+1}$ for $1 \leq j < m$, then $|\operatorname{fix}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m})| < \delta$.

We prove the claim by induction on m. It is clear for $m = 1$, since π_i has no fixed points. Suppose $m > 1$. We may assume that $i_m \neq i_1$, as otherwise we have $m > 2$ and the induction hypothesis applies to the conjugate permutation $\pi_{i_2}\cdots\pi_{i_{m-1}}$. Let $i_{m+1} = i_1$; then $i_j \neq i_{j+1}$ for $1 \leq j \leq m$. Let

$$R = \bigcup_{j=1}^{m} (S_{i_j} \cap S_{i_{j+1}})\theta_{i_j}\cdots\theta_{i_2}\theta_{i_1}.$$

We will show that $\operatorname{fix}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}) \subseteq R \times \mathbb{Z}$; this will finish the proof, since $|R \times \mathbb{Z}| < \delta$.

Suppose $(t_0, n_0) \in \operatorname{fix}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m})$; we have to show that $t_0 \in R$. For $1 \leq j \leq m+1$, let $(t_j, n_j) = (t_0, n_0)\pi_{i_1}\pi_{i_2}\cdots\pi_{i_j} = (t_{j-1}, n_{j-1})\pi_{i_j}$. Then $n_m = n_0, n_{m+1} = n_1$, and for $1 \leq j \leq m+1$ we have

$$n_j - n_{j-1} = \begin{cases} +1 & \text{if } t_{j-1} \in S_{i_j}, \\ -1 & \text{if } t_{j-1} \notin S_{i_j}. \end{cases}$$

Pick $j \in \{1, 2, \ldots, m\}$ with $n_j = \min\{n_1, n_2, \ldots, n_m\}$. Then $n_j < n_{j+1}$, whence $t_j \in S_{i_{j+1}}$; and $n_j < n_{j-1}$, whence $t_{j-1} \notin S_{i_j}$, whence $t_j = t_{j-1}\theta_{i_j} \in S_{i_j}$. Thus we have $t_0\theta_{i_1}\theta_{i_2}\cdots\theta_{i_j} = t_j \in S_{i_j} \cap S_{i_{j+1}}$, and so $t_0 \in (S_{i_j} \cap S_{i_{j+1}})\theta_{i_j}\cdots\theta_{i_2}\theta_{i_1} \subseteq R$. □

The following corollary answers a question of mine [2, Question 1.5], which was motivated by some off-hand remarks of P. M. Neumann. The corollary follows from Theorem 1 together with Sierpiński's result [3, p. 241] that, if $2^\omega = \omega_1$, then there are 2^{ω_1} almost disjoint uncountable subsets of ω_1. By a consistency result of Baumgartner [1, Theorem 5.6], the continuum hypothesis is indispensible here.

Corollary. *If $2^\omega = \omega_1$, then $\operatorname{Sym}(\omega_1)$ has an almost disjoint subgroup of cardinality 2^{ω_1}.*

Theorem 2. *For any infinite cardinals κ and λ, the following statements are equivalent*:

(i) *there is an almost disjoint subgroup G of $\operatorname{Sym}(\kappa)$ with $|G| = \lambda$;*
(ii) *there is an almost disjoint family $\mathcal{S}$ of κ-element subsets of κ with $|\mathcal{S}| = \lambda$.*

Proof. In view of Theorem 1, all we have left to prove is that $\operatorname{Sym}(\omega)$ has an almost disjoint subgroup of cardinality 2^ω. For each $n \in \omega$, let $(B_n, +)$ be the free Boolean group (any nontrivial locally finite variety of groups will do) generated by an $(n+1)$-element set A_n. Let E be the disjoint union of the sets B_n. As shown by Sierpiński [3, p. 242], we can define a subset F of the Cartesian product $\prod_{n<\omega} A_n$, with $|F| = 2^\omega$, so that $|f \cap g| < \omega$ whenever $f, g \in F$ and $f \neq g$. For each $f \in F$, define $\pi_f \in \operatorname{Sym}(E)$ so that $x\pi_f = x + f(n)$ for $x \in B_n$. Let G be the subgroup of $\operatorname{Sym}(E)$ generated by $\{\pi_f : f \in F\}$; it is easy to verify that G is almost disjoint. □

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