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UNIONS OF LOEB NULLSETS

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Abstract. The union of every point-finite, completely measurable family of Loeb nullsets is itself a Loeb nullset, provided the nonstandard model satisfies a simple set-theoretic condition. One application of this result is that every Loeb measurable function into a metric space has a lifting.

1. Introduction

Recall Lusin's Theorem:

Let f be a measurable function from a Radon probability space $(X, \mathcal{B}, P)$ to a second countable topological space. Then for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that the restriction of f to K is continuous and $P(K) > 1 - \epsilon$.

(By **Radon** we mean that X is topological, $\mathcal{B}$ contains all Borel sets, and P is compact–inner–regular, that is, $P(E) = \sup\{P(K) \mid K \subseteq E, K \text{ compact }\}$ for every $E \in \mathcal{B}$.)

More generally, a function f from a topological probability space $(X, \mathcal{B}, P)$ to a topological space Y is **Lusin measurable** provided for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that the restriction of f to K is continuous and $P(K) > 1 - \epsilon$. Fremlin [F] has proved that if $(X, \mathcal{B}, P)$ is Radon and Y is metric, then f is Lusin measurable.

A corresponding result to Lusin's Theorem in nonstandard measure theory is due originally to Robert Anderson ([SB], Theorem 2.1.4):

Every measurable function from a Loeb probability space to a second countable topological space has a lifting.

The correspondence is made precise in [R2], where it is proved that a function is Lusin measurable if and only if it admits a "two–legged" lifting. In addition, necessary and sufficient conditions for the existence of a lifting are given there; this yields a new sufficient condition for a function to be Lusin measurable.

Prikry and Kupka show in [KuP] that Fremlin's result follows easily once one knows that every point–finite, completely measurable family of Radon nullsets is a Radon nullset (see below). Prikry and Koumoullis [KoP] have extended this latter result to arbitrary compact probability spaces. (See §2 for definitions.) An analogous result about unions of Loeb nullsets, and a corresponding generalization of Anderson's theorem to general metric spaces, would be an immediate consequence of the standard theory, provided Loeb measures were compact.

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In [R1] it is shown that every compact probability space is the image of a Loeb space under a measurable map (Lemma 5.1, below); this suggests that Loeb spaces might indeed be compact. However, in a recent paper J. Aldaz [A] gives an example of a Loeb space which is *not* compact, rendering both the unions–of–nullsets result and the corresponding lifting result nontrivial for Loeb spaces.

This paper gives proofs for both these results. In fact, the unions-of-Loeb-nullsets result is strictly stronger than the corresponding standard result (see Corollary 5.2, below). (Weaker results, requiring the much stronger hypothesis that the nonstandard model is "fully" saturated, appeared in [R3].)

2. Preliminaries

For X a set, write $\mathcal{P}(X)$ for the power set of X, $\mathcal{P}_n(X)$ for $\{E \in \mathcal{P}(X) \mid \mathrm{card}(E) = n\}$, and $\mathcal{P}_{\mathrm{fin}}(X)$ for $\{E \in \mathcal{P}(X) \mid E \text{ finite}\} = \bigcup_{n=0}^{\infty} \mathcal{P}_n(X)$.

Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$. $\mathcal{F}$ is **point–finite** if for every $x \in X$, $\{E \in \mathcal{F} \mid x \in E\}$ is finite. $\mathcal{F}$ has the **finite intersection property** if $E_1 \cap \cdots \cap E_n \neq \emptyset$ whenever $\{E_1, \cdots, E_n\} \subseteq \mathcal{F}$, $n \in \mathbb{N}$. $\mathcal{F}$ is **compact** provided that for every subfamily $\mathcal{K} \subseteq \mathcal{F}$ with the finite intersection property, $\bigcap \mathcal{K} \neq \emptyset$.

Suppose now that $(X, \mathcal{B}, P)$ is a probability space. A family $\mathcal{F} \subseteq \mathcal{B}$ is **completely measurable** provided $\bigcup \mathcal{K} \in \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{F}$.

The probability space $(X, \mathcal{B}, P)$ is **compact** provided P is inner–regular with respect to a compact family $\mathcal{K} \subseteq \mathcal{B}$.

The reader is assumed to be familiar with nonstandard analysis in general, and Loeb measures in particular, and is referred to [C] or [SB] for background. All nonstandard models will be assumed $\aleph_1$-saturated.

Suppose that X is internal and $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$. Call $\mathcal{A}$ κ–**terraced** provided (i) $\kappa = \mathrm{card}(\mathcal{A})$, and (ii) For some nondecreasing sequence $\{\mathcal{A}_i\}_{i<\kappa}$ with each $\mathcal{A}_i \subseteq \mathcal{A}$ compact, $\mathcal{A} = \bigcup_{i<\kappa} \mathcal{A}_i$. For $A \in \mathcal{A}$ let $\mathrm{Level}(A) = \min\{i < \kappa \mid A \in \mathcal{A}_i\}$. (Note this definition depends on choice of terracing; a particular one will always be assumed already chosen.)

Suppose α and κ are cardinals; say that α is **not measurably cofinal in** κ, $\mathrm{NMC}(\alpha, \kappa)$, if either α is not a measurable cardinal (see below) or α is not cofinal in κ. Note that $\mathrm{NMC}(\alpha, \kappa)$ holds for every α precisely when the cofinality of κ is not measurable.

Lemma 2.1. *There exist nonstandard models such that for every internal X and every internal $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$ with $\kappa = \mathrm{card}(\mathcal{A})$ infinite, $\mathcal{A}$ is κ–terraced and $\mathrm{NMC}(\alpha, \kappa)$ for all α.*

Proof. Suppose the nonstandard model satisfies the special model axiom SMA_β (see [R4] or [J]). This means that for any first–order language $\mathcal{L}$ with fewer than β relation, function, and constant symbols, and every $\mathcal{L}$–model $\mathfrak{A} = (A, \cdots)$ with both A and every interpretation of every relation and function symbol internal, $\mathfrak{A}$ is a special model (see [CK] for definitions). It is shown in [R4] that all internal infinite sets have the same external cardinality κ when SMA_β holds, $\beta \geq \aleph_0$.

Suppose that X and $\mathcal{A} \subseteq {}^*\mathcal{P}(X)$ are internal, and that $\mathcal{A}$ is infinite. Let $\mathcal{L}$ be the first–order language with unary predicate symbol $Q(x)$ and binary predicate $E(x, y)$, and let $\mathfrak{A} = (A, \cdots)$ be the $\mathcal{L}$–structure with $A = X \cup \mathcal{A}$, $Q^{\mathfrak{A}} = \mathcal{A}$, and $E^{\mathfrak{A}} = \{(x, y) \in X \times \mathcal{A} \mid x \in y\}$. By SMA_β, there is an elementary chain $\mathfrak{A}_0 \preceq$

$\mathfrak{A}_1 \preceq \cdots$ with $\mathfrak{A}_i$ i^+–saturated and $\mathfrak{A} = \bigcup_{i<\kappa} \mathfrak{A}_i$. Let $\{a_i\}_{i<\kappa}$ be any enumeration of $\mathcal{A}$, and put $\mathcal{A}_i = Q^{\mathfrak{A}_i} \cap \{a_j\}_{j\leq i}$. Evidently $\mathcal{A} = \bigcup_{i<\kappa} \mathcal{A}_i$. It remains to show that $\mathcal{A}_i$ is compact.

Suppose $\mathcal{K} \subseteq \mathcal{A}_i$ has the finite intersection property. Consider the set $\Gamma(x)$ of $\mathcal{L} \cup A_i$–formulas of form $E(x, K)$, $K \in \mathcal{K}$. This set has cardinality $\leq i$, and since $\mathcal{K}$ has the finite intersection property, $\Gamma(x)$ is finitely consistent, so by i^+–saturation of $\mathfrak{A}_i$ there is an $a \in A_i$ with $\mathfrak{A}_i \models E(a, K)$ for *all* $K \in \mathcal{K}$, i.e., $a \in \cap\mathcal{K}$, proving compactness.

Thus *any* nonstandard model where (i) SMA_β holds for some $\beta \geq \aleph_0$, and (ii) κ is not cofinally measurable (for example, κ regular and not measurable) will work; this proves the lemma.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is an internal, *finitely–additive probability space. Denote by $(\Omega, \mathcal{A}_L, \mu_L)$ the (standard) probability space generated from $(\Omega, \mathcal{A}, \mu)$ by the Loeb construction. Recall that a measurable function f from Ω to a topological space Y **has a lifting** F provided $F : \Omega \longrightarrow^* Y$ is internal and $F(\omega) \approx f(\omega)$ μ_L–almost everywhere.

The main results can now be stated; the proofs are deferred to §4.

Theorem 2.2. *Let* $(\Omega, \mathcal{A}, \mu)$ *be an internal* **finitely–additive probability space, and let* $\mathcal{E} = \{A_i\}_{i<\alpha}$ *be a point–finite, completely measurable family of* μ_L*–nullsets. Suppose that* $\mathcal{A}$ *is* κ*–terraced, and that* $\mathrm{NMC}(\alpha, \kappa)$ *holds. Then* $\mu_L(\bigcup_{i<\alpha} A_i) = 0$.

Corollary 2.3. *Suppose* $(\Omega, \mathcal{A}, \mu)$ *is an internal* **finitely–additive probability space, where* $\mathcal{A}$ *is* κ*–terraced, and that* $\mathrm{NMC}(\alpha, \kappa)$ *holds for all* α. *Let* f *be a measurable function from* Ω *to a metric space* Y. *Then for some* $\Omega' \subseteq \Omega$ *with* $\mu_L(\Omega') = 1$, *the restriction of* f *to* Ω' *has second countable range (and so* f *has a lifting, by Anderson's lifting theorem).*

3. Useful standard lemmas

The first lemma is due to Bernstein; see ([KuP], Proposition 3.5) for a proof.

Lemma 3.1. *There is a subset* $\mathbb{B}$ *of* $[0, 1]$ *with* $\mathrm{card}(\mathbb{B}) = 2^{\aleph_0}$ *such that every closed subset of* $\mathbb{B}$ *is at most countable.*

Let X be an arbitrary set, and $\nu : \mathcal{P}(X) \longrightarrow [0, \infty)$. A set $E \in \mathcal{P}(X)$ is an **atom** for ν if $\nu(E) \neq 0$ and $\{\nu(B),\ \nu(E - B)\} = \{0,\ \nu(E)\}$ for every $B \subseteq E$. If ν has no atoms it is **atomless**. If α is an infinite cardinal and ν is a finite measure on $(\alpha, \mathcal{P}(\alpha))$ for which (i) α is an atom, (ii) singletons are nullsets, and (iii) $\nu(\bigcup \mathcal{E}) = 0$ whenever $\mathcal{E}$ is a family of nullsets with $\mathrm{card}(\mathcal{E}) < \alpha$, then α is a **measurable cardinal**.

The following is a form of Ramsey's Theorem due to Hajnal and Erdös; see [K] for a proof.

Lemma 3.2. *If* α *is a measurable cardinal and* $\varphi : \mathcal{P}_{\mathrm{fin}}(\alpha) \longrightarrow \{0, 1\}$, *then there is a subset* $\widehat{\alpha} \subseteq \alpha$ *such that* $\mathrm{card}(\widehat{\alpha}) = \alpha$ *and such that* φ *is constant on* $\mathcal{P}_n(\widehat{\alpha})$ *for every* n.

The final lemma in this section is ([KoP], Lemma 4).

Lemma 3.3. *If $\mathcal{F}$ is an uncountable family of nonnullsets in a finite measure space, then for some x, $\{A \in \mathcal{F} \mid x \in A\}$ is infinite. (In other words, there is no point-finite uncountable family of nonnullsets in a finite measure space.)*

4. Proof of main results

Proof of Theorem 2.2. Otherwise, let $\mathcal{E} = \{A_i\}_{i<\alpha}$ be a counterexample, that is, a point–finite completely measurable family of Loeb nullsets with $\mu_L(\bigcup_{i<\alpha} A_i) > 0$. Without loss of generality, α has the least cardinality of any counterexample, and $\mu_L(\bigcup_{i<\alpha} A_i) = 1$. Put $A = \bigcup_{i<\alpha} A_i$.

For $E \subseteq \alpha$ put $\nu(E) = \mu_L(\bigcup_{i\in E} A_i)$. Consider two cases:

Case 1. ν is atomless on $(\alpha, \mathcal{P}(\alpha))$. Inductively define partitions P_λ, $\lambda < \omega_1$, of α as follows. Put $P_0 = \{\alpha\}$. If $\beta < \omega_1$ is a limit ordinal, let P_β consist of all nonempty intersections of the form $\bigcap_{\lambda<\beta} E_\lambda$, where $E_\lambda \in P_\lambda$ for all $\lambda < \beta$. Given P_λ, construct $P_{\lambda+1}$ by replacing each E in P_λ by itself if $\nu(E) = 0$, otherwise by disjoint sets E_1, E_2 where $E_1 \cup E_2 = E$ and $0 < \nu(E_1) \leq \nu(E_2) \leq \nu(E)$. (We may take $\nu(E_1) \neq 0$ since from the definition of ν, $\nu(E_1 \cup E_2) \leq \nu(E_1) + \nu(E_2)$.)

Let $\alpha = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$, with $E_\lambda \in P_\lambda$ for all $\lambda < \omega_1$. (Call such a sequence *adapted*.) Suppose (for a contradiction) that $\nu(E_\lambda) \neq 0$ for every λ. Put $B_\lambda = E_\lambda \setminus E_{\lambda+1}$. By the construction, $\nu(B_\lambda) > 0$ for every λ; but then $\{\bigcup_{i\in B_\lambda} A_i\}_{\lambda<\omega_1}$ is a point-finite uncountable collection of nonnull sets, contradicting Lemma 3.3.

It follows that every adapted sequence is eventually constant, so there are at most $\aleph_0$ of them. Let $\{E_\tau\}_{\tau\in T}$ be the set of intersections of adapted sequences; we may take $T \subseteq \mathbb{B}$. For $\tau \in T$ put $A_\tau = \bigcup_{i\in E_\tau} A_i$; note that $\mu_L(A_\tau) = \nu(E_\tau) = 0$, and that $\{E_\tau\}_{\tau\in T}$ forms a partition of α.

For $x \in A$ let $f(x)$ be the smallest (in the sense of the ordering on $\mathbb{B}$ inherited from $\mathbb{R}$) of the finitely many τ with $x \in A_\tau$. Note that for $0 \leq r \leq 1$, $f^{-1}([0, r)) = \bigcup_{\tau<r} A_\tau$, so $f : A \longrightarrow [0, 1]$ is Loeb measurable.

It follows that f has a lifting, and for some internal $K \subseteq A$ with $\mu_L(K) > 0$ and some internal $F : K \longrightarrow^* [0, 1]$, $F(x) \approx f(x)$ for all $x \in K$. Since $f(K) = \mathrm{st}(F(K))$ is closed ([SB], Proposition 2.1.1) and $f(K) \subseteq \mathbb{B}$, $f(K)$ is at most countable, and

$$\begin{aligned} 0 &< \mu_L(K) \\ &\leq \mu_L(f^{-1}(f(K))) \\ &= \mu_L(\bigcup_{\tau\in f(K)} f^{-1}(\tau)) \\ &\leq \mu_L(\bigcup_{\tau\in f(K)} A_\tau) \qquad \text{(since } f(x) = \tau \text{ only if } x \in A_\tau) \\ &= 0 \qquad \text{(since } f(K) \text{ countable and } \mu_L(A_\tau) = 0). \end{aligned}$$

This is a contradiction, and proves the theorem in this case.

Case 2. ν has an atom in $(\alpha, \mathcal{P}(\alpha))$. Without loss of generality, α *is* the atom. By minimality of α, if $E \subseteq \alpha$ and $\operatorname{card}(E) < \alpha$, then $\nu(E) = 0$. It is easy to verify in this case that ν is actually a measure, so α is a measurable cardinal.

For each $\lambda < \alpha$, put $B_\lambda = \bigcup_{i > \lambda} A_i$. Note $\mu_L(B_\lambda) > 0$, so there is an internal $K_\lambda \subseteq B_\lambda$ with $\mu_L(K_\lambda) > 0$. Choose this K_λ so that $\operatorname{Level}(K_\lambda)$ is minimal.

Define $\varphi : \mathcal{P}_{\text{fin}}(\alpha) \longrightarrow \{0,1\}$ by

$$\varphi(E) = \begin{cases} 1 & \text{if } E = \emptyset \text{ or } \bigcap_{\lambda \in E} K_\lambda \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.2 there is a subset $\widehat{\alpha} \subseteq \alpha$, $\operatorname{card}(\widehat{\alpha}) = \alpha$, such that φ is constant on each $\mathcal{P}_n(\widehat{\alpha})$. By Lemma 3.3, $\varphi \equiv 1$ on $\mathcal{P}_n(\widehat{\alpha})$ whenever $n > 1$, so $\{K_\lambda\}_{\lambda \in \widehat{\alpha}}$ has the finite intersection property.

Let $\mathcal{O} : \widehat{\alpha} \longrightarrow \kappa$ be the map $\mathcal{O}(\lambda) = \operatorname{Level}(K_\lambda)$. Since B_λ decreases in λ and K_λ is chosen to minimize $\operatorname{Level}(K_\lambda)$ for each λ, $\mathcal{O}$ is nondecreasing on $\widehat{\alpha}$.

Define a function $\varphi_1 : \mathcal{P}_{\text{fin}}(\widehat{\alpha}) \longrightarrow \{0,1\}$ by

$$\varphi_1(E) = \begin{cases} 1 & \text{if } \mathcal{O}(\lambda_1) = \mathcal{O}(\lambda_2)\ \forall \lambda_1, \lambda_2 \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3.2 again, there is $\hat{\hat{\alpha}} \subseteq \widehat{\alpha}$ with $\operatorname{card}\left(\hat{\hat{\alpha}}\right) = \alpha$ and φ_1 constant on $\mathcal{P}_n\left(\hat{\hat{\alpha}}\right)$ for every n.

Suppose (for a contradiction) that $\varphi_1 \equiv 1$ on $\mathcal{P}_2\left(\hat{\hat{\alpha}}\right)$; then for some $\gamma < \kappa$, $\gamma = \mathcal{O}(\lambda) = \operatorname{Level}(K_\lambda)$ for all $\lambda \in \hat{\hat{\alpha}}$. Since $\{K_\lambda\}_{\lambda \in \widehat{\alpha}}$ has the finite intersection property and $\{K \in {}^*\mathcal{P}(\Omega) \mid \operatorname{Level}(K) = \gamma\}$ is a compact family, there is an $x \in \bigcap_{\lambda \in \hat{\hat{\alpha}}} K_\lambda$. By point–finiteness of $\{A_i\}_{i<\alpha}$, there is a largest $\beta < \alpha$ with $x \in A_\beta$. Let $\lambda \in \hat{\hat{\alpha}}$, $\lambda > \beta$; then $x \notin \bigcup_{i \geq \lambda} A_i = B_\lambda$, but $x \in K_\lambda \subseteq B_\lambda$, a contradiction.

Therefore, $\varphi_1 \equiv 0$ on $\mathcal{P}_2(\hat{\hat{\alpha}})$, so $\mathcal{O}$ is strictly increasing on $\hat{\hat{\alpha}}$. The argument used in the last paragraph shows that $\{\mathcal{O}(\lambda)\}_{\lambda \in \hat{\hat{\alpha}}}$ is unbounded in κ, so $\hat{\hat{\alpha}}$, hence α, is cofinal in κ, contradicting the hypothesis on κ. The theorem is proved.

Proof of Corollary 2.3. By paracompactness of Y, for every $n \in \mathbb{Z}^+$ there is a point–finite open cover $\mathcal{U}_n$ of Y by $\frac{1}{n}$–balls. By Lemma 3.3, we can write $\mathcal{U}_n = \mathcal{U}_n^1 \cup \mathcal{U}_n^2$, where (i) $\mathcal{U}_n^1$ is countable, (ii) $\mathcal{U}_n^1 \cap \mathcal{U}_n^2 = \emptyset$, and (iii) for every $u \in \mathcal{U}_n^2$, $f^{-1}(u)$ is a μ_L–nullset. Put $\Omega_n = f^{-1}(\bigcup \mathcal{U}_n^1)$; by Theorem 2.2, $f^{-1}(\Omega \smallsetminus \Omega_n) \subseteq \bigcup_{\mu \in \mathcal{U}_n^2} f^{-1}(\mu)$ is a μ_L–nullset. Put $\Omega' = \bigcap_n \Omega_n$; it is easy to confirm that Ω' satisfies the conditions of the lemma.

5. A standard corollary

Lemma 5.1. *Let $(X, \mathcal{B}, P)$ be a (standard) compact probability space, where the nonstandard model is* $\operatorname{card}(\mathcal{B})^+$*–saturated. Then there is a Loeb space $(\Omega, \mathcal{A}_L, \mu_L)$ and a measurable function $\varphi : \Omega \longrightarrow X$, so that $P = \mu_L \circ \varphi^{-1}$.*

Proof. See [R1].

Corollary 5.2. *Let $(X, \mathcal{B}, P)$ be a compact probability space, and $\{A_i\}_{i<\alpha}$ a point–finite completely measurable family of P–nullsets. Then $P(\bigcup_{i<\alpha} A_i) = 0$.*

Proof. There exist nonstandard models satisfying SMA_β for β arbitrarily large; it follows from Lemma 2.1 that there is a nonstandard model which is $\mathrm{card}(\mathcal{B})^+$–saturated, and for some κ every internal family of internal sets is κ–terraced and satisfies $\mathrm{NMC}(\alpha, \kappa)$. Let $(\Omega, \mathcal{A}_L, \mu_L)$ and $\varphi : \Omega \longrightarrow X$ be as given in Lemma 5.1. Then $\{\varphi^{-1}(A_i)\}_{i<\alpha}$ is a point–finite completely measurable family of μ_L–nullsets, so $P(\bigcup_{i<\alpha} A_i) = \mu_L(\varphi^{-1}(\bigcup_{i<\alpha} A_i)) = \mu_L(\bigcup_{i<\alpha} \varphi^{-1}(A_i)) = 0$ by Theorem 2.2.

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