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A COUNTEREXAMPLE TO THE DIFFERENTIABILITY OF THE BERGMAN KERNEL FUNCTION

SO-CHIN CHEN

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ABSTRACT. In this paper we prove the following main result. Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ with $n \geq 2$. Suppose that there exists a complex variety sitting in the boundary bD; then we have

$$K_D(z,w) \notin C^\infty(\overline{D} \times \overline{D} - \Delta(bD)).$$

In particular, the Bergman kernel function associated with the Diederich-Fornaess worm domain is not smooth up to the boundary in joint variables off the diagonal of the boundary.

1. INTRODUCTION

In several complex variables it has been shown that the Bergman kernel function is closely related to the boundary regularity of holomorphic mappings between two domains. Therefore, it is always a fundamental question to investigate the boundary regularity of the Bergman kernel function associated with a smoothly bounded pseudoconvex domain.

Historically, it was first proved by Kerzman [10], based on Kohn's work, that the Bergman kernel function associated to a smoothly bounded strictly pseudoconvex domain D can be extended smoothly to $\overline{D} \times \overline{D} - \Delta(bD)$, where $\Delta(bD) = \{(z,z) | z \in bD\}$. Later it was generalized independently by Bell [2] and Boas [3] to the following.

Theorem 1.1. *Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$. If Γ_1 and Γ_2 are disjoint open subsets of bD consisting of points of finite type in the sense of D'Angelo* [8]*, then the Bergman kernel function associated to D extends smoothly to $\Gamma_1 \times \Gamma_2$.*

Theorem 1.2. *Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$. Suppose that condition R holds on D. Let Γ_1 and Γ_2 be two disjoint open subsets of bD such that Γ_1 consists of points of finite type. Then the Bergman kernel function associated to D extends smoothly to $\Gamma_1 \times \Gamma_2$.*

Here condition R means that the Bergman projection P, the orthogonal projection from $L^2(D)$ onto the closed subspace $H^2(D)$ of square-integrable holomorphic functions, maps $C^\infty(\overline{D})$ continuously into itself.

Based on the above theorems it is natural to conjecture that a similar extension phenomenon might probably hold on any smoothly bounded pseudoconvex domain in $\mathbf{C}^n$. However, in this article we are going to prove the following main result.

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Main Theorem. *Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ with $n \geq 2$. Suppose that there exists a complex variety sitting in the boundary bD; then we have*

$$K_D(z,w) \notin C^\infty(\overline{D} \times \overline{D} - \Delta(bD)),$$

where $\Delta(bD) = \{(z,z) | z \in bD\}$.

2. Two examples

In this section we will present two examples. First we fix a smooth real-valued function $\lambda : \mathbf{R} \to \mathbf{R}$ as in [9] with the following properties:

(i) $\lambda(x) = 0$ if $x \leq 0$.
(ii) $\lambda(x) > 0$ if $x > 1$.
(iii) $\lambda''(x) \geq 100\lambda'(x)$ for all x.
(iv) $\lambda''(x) > 0$ if $x > 0$.
(v) $\lambda'(x) > 100$ if $\lambda(x) > \frac{1}{2}$.

Example A. For any $r > 1$, define the domain Ω_r by

$$\Omega_r = \{(z,w) \in \mathbf{C}^2 | \rho_r(z,w) < 0\},$$

where $\rho_r(z,w) = |z|^2 - 1 + \lambda(|w|^2 - r^2)$.

Then we have

Lemma 2.1. *The domain Ω_r defined as above is a smoothly bounded pseudoconvex domain in $\mathbf{C}^2$ satisfying that*

(a) *Ω_r is convex.*
(b) *Condition R holds on Ω_r.*
(c) *Ω_r is strictly pseudoconvex everywhere except on the set*

$$M_r = \{(z,w) \in \mathbf{C}^2 \| z| = 1 \text{ and } 0 \leq |w| \leq r\}.$$

Condition R on Ω_r follows from either (a) or the fact that Ω_r is Reinhardt. For instance, see Chen [6], [7], Boas and Straube [4] or Straube [12]. Obviously, the boundary contains a family of complex discs parametrized by the unit circle. Hence by our main theorem we have

$$K_{\Omega_r}(z,w) \notin C^\infty(\overline{\Omega}_r \times \overline{\Omega}_r - \Delta(b\Omega_r)).$$

But we would like to provide an elementary proof of the main theorem for Ω_r here which gives a more geometric insight into the problem.

Let $\Delta_r = \{w \in \mathbf{C} \| w| < \sqrt{r^2+1}\}$, and denote by Δ the unit disc in the complex plane. Put $D_r = \Delta \times \Delta_r$. D_r is a pseudoconvex domain in $\mathbf{C}^2$, and obviously we have $\Omega_r \subseteq D_r$. It is also clear that $\Lambda = \{z^m w^n | m,n \in \mathbf{N} \cup \{0\}\}$ forms a complete orthogonal basis for both $H^2(\Omega_r)$ and $H^2(D_r)$. So we have

$$c^r_{mn} = \int_{\Omega_r} |z^m w^n|^2 d\sigma \leq \int_{D_r} |z^m w^n|^2 d\sigma = d^r_{mn} \text{ for all } m,n.$$

Both equalities are definitions. Now consider any point sequence $\{z_j\}_{j=1}^\infty$ in Δ such that $\{z_j\}$ approaches a boundary point, say z_0. Pick any two positive numbers p and q such that $0 < p < q < r$. So now we have two point sequences $\{(z_j,p)\}$ and $\{(z_j,q)\}$ in both Ω_r and D_r which approach the boundary points (z_0,p) and (z_0,q) respectively. Clearly we have $(z_0,p) \neq (z_0,q)$. Next, by using the explicit formula

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for the Bergman kernel function $K_\Delta(z,w)$ on the unit disc in the complex plane, we obtain

$$\begin{aligned} K_{\Omega_r}((z_j,p),(z_j,q)) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{c_{mn}^r}|z_j|^{2m}\cdot(pq)^n \\ &\geq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{d_{mn}^r}|z_j|^{2m}\cdot(pq)^n \\ &= K_\Delta(z_j,z_j)K_{\Delta_r}(p,q) \\ &= \frac{1}{\pi}\cdot\frac{1}{(1-|z_j|^2)^2}K_{\Delta_r}(p,q). \end{aligned}$$

It follows that

$$\lim_{j\to\infty} K_{\Omega_r}((z_j,p),(z_j,q)) = \infty.$$

Hence the Bergman kernel function $K_{\Omega_r}(z,w)$ is not bounded near $z=(z_0,p)$ and $w=(z_0,q)$. In particular, it shows that $K_{\Omega_r}(z,w)\notin C^\infty(\overline{\Omega}_r\times\overline{\Omega}_r-\Delta(b\Omega_r))$. This completes the proof of the main theorem on Ω_r.

Example B (Diederich-Fornaess worm domain)**.** Now define the domain Ω_r, for any $r>1$, by

$$\Omega_r=\{(z,w)\in\mathbf{C}^2|\rho_r(z,w)<0\},$$

where $\rho_r(z,w)=|z+\exp(iln|w|^2)|^2-1+\lambda(\frac{1}{|w|^2}-1)+\lambda(|w|^2-r^2)$. Then we obtain the famous Diederich-Fornaess worm domain [9]. These domains Ω_rs are smoothly bounded pseudoconvex in $\mathbf{C}^2$ and are strictly pseudoconvex everywhere except on the annulus

$$A_r=\{(z,w)\in\mathbf{C}^2|z=0\text{ and }1\leq|w|\leq r\}.$$

Therefore, again by our main theorem we see that

$$K_{\Omega_r}(z,w)\notin C^\infty(\overline{\Omega}_r\times\overline{\Omega}_r-\Delta(b\Omega_r)).$$

We should point out that it has been shown in Barrett [1] that the Bergman projection on these domains does not preserve the Sobolev k-space $W^k(\Omega_r)$ if k is large enough. However, it is still not known whether the condition R holds on Ω_r or not.

3. Proof of the main theorem

Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ with $n\geq 2$, and let V be a complex variety sitting in the boundary bD. Let $p\in V$ be a regular point of V. Denote by $\vec{n}$ the unit outward normal at p. Since the boundary of D is smooth, there exists small $\delta,\varepsilon_0>0$ such that $w-\varepsilon\,\vec{n}\in D$ for all $w\in bD\cap B(p;\delta)$ and all $0<\varepsilon<\varepsilon_0$. Now let us consider a small complex disc $\overline{\Delta}\subseteq bD\cap B(p;\delta)\cap V$ centered at p. More precisely, Δ is the holomorphic embedding of the unit disc in the complex plane in the boundary, and the origin is mapped to p.

Suppose now that $K_D(z,w)\in C^\infty(\overline{D}\times\overline{D}-\Delta(bD))$. Then we have

$$\sup_{w\in b\Delta}|K_D(p,w)|\leq M<+\infty,$$

for some $M>0$. On the other hand, it was proved in P. Pflug [11] that

$$\lim_{\varepsilon\to 0}K_D(p-\varepsilon\,\vec{n},p-\varepsilon\,\vec{n})=+\infty.$$

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Then by the maximum modulus principle we get

$$\sup_{w\in b\Delta_\varepsilon} |K_D(p-\varepsilon \overrightarrow{n}, w)| \geq K_D(p-\varepsilon \overrightarrow{n}, p-\varepsilon \overrightarrow{n}),$$

where $\Delta_\varepsilon = \Delta - \varepsilon \overrightarrow{n} \subseteq D$. It follows that

$$\sup_{w\in b\Delta} |K_D(p,w)| = \lim_{\varepsilon\to 0} \sup_{w\in b\Delta_\varepsilon} |K_D(p-\varepsilon \overrightarrow{n}, w)| = +\infty.$$

This gives the desired contradiction, and the proof of the main theorem is now completed.

4. Concluding remark

In view of the main theorem proved above, it is natural to ask: How does the Bergman kernel function $K_D(z,w)$ behave in joint variables near the boundary, if the domain D satsfies property (P) introduced in Catlin [5]? So far we have no clue to answer this question.

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Institute of Applied Mathematics, National Tsing Hua University, Hsinchu 30043, Taiwan, Republic of China

E-mail address: scchen@am.nthu.edu.tw

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