

TOTAL CURVATURE OF BRANCHED MINIMAL SURFACES

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ABSTRACT. An intrinsic, and much simpler, proof of a generalization of Jorge and Meeks' total curvature formula for complete minimal surfaces is given.

Let $X : M \rightarrow \mathbb{R}^3$ be a complete minimal surface with finite total curvature. Osserman proved that conformally $M = S_k - \{p_1, \dots, p_n\}$, $n \geq 1$, where S_k is a closed Riemann surface of genus k . See, for example, [4, Theorem 9.1, page 81]. Each p_i corresponds to an end E_i of M . Jorge and Meeks [2] proved that there is an integer $I_i \geq 1$ corresponding to E_i , such that the total curvature of M is given by

$$(1) \quad \int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^n I_i \right),$$

where $\chi(M) = 2(1 - k) - n$ is the Euler characteristic of M .

The proof of Jorge and Meeks involves a detailed study of the behaviour of the image $X(M)$ at each end. Since the Gauss curvature is an intrinsic quantity, it is natural to look for an intrinsic proof. In this note we give such an intrinsic, and much simpler, proof of a generalization of (1).

The Enneper-Weierstrass representation of a branched complete minimal surface of finite total curvature $X : M \rightarrow \mathbb{R}^3$ is given by

$$(2) \quad X(p) = \operatorname{Re} \int_{p_0}^p \left(\frac{1}{2}(1 - g^2), \frac{i}{2}(1 + g^2), g \right) \eta,$$

where $g : M = S_k - \{p_1, \dots, p_n\} \rightarrow \mathbb{C} \cup \{\infty\}$ is a meromorphic function and η is a holomorphic 1-form on M . Both g and η can be extended to S_k as a meromorphic function and 1-form respectively; see [4, Theorem 9.1, page 81]. Note that since the proof given there only involves the neighbourhoods of the punctures p_i , it works for branched minimal surfaces as well.

Locally, $\eta = f(z)dz$, where $z = x + iy$. The metric induced by X is given by

$$(3) \quad ds^2 = \Lambda^2(dx^2 + dy^2),$$

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where

$$(4) \quad \Lambda = \frac{1}{2}|f|(1 + |g|^2);$$

see [4, pages 47, 65]. From (4) it is clear that q is a branch point only when η vanishes at q . Hence all branch points are isolated, and if η is a meromorphic 1-form on S_k , there is only a finite number of branch points.

Therefore, given g and η as above, define a metric with isolated degenerate points, $h_{ij} = \Lambda\delta_{ij}$ by (3) and (4). We can study the intrinsic geometry of the branched complete Riemannian manifold (M, h) even though the mapping X in (2) may not be well defined. When X is well defined, it is a branched complete minimal surface.

Let U_i be a disk coordinate neighbourhood of p_i such that $z(p_i) = 0$. Let J_i be the order of Λ at p_i , i.e., J_i is an integer such that in U_i ,

$$\lim_{z \rightarrow 0} |z|^{J_i} \Lambda(z) = C_i > 0,$$

for $1 \leq i \leq n$. Since (M, h) is complete, $J_i \geq 1$. In fact, if X in (2) is well defined, it is well known that $J_i \geq 2$. See [3, page 135].

Suppose $q_i, 1 \leq i \leq m$, are branch points of M . Let V_i be a disk coordinate neighbourhood of q_i such that $z(q_i) = 0$. Let $K_i > 0$ be the branch order of Λ , i.e.,

$$\lim_{z \rightarrow 0} |z|^{-K_i} \Lambda(z) = F_i > 0, \quad \text{in } V_i.$$

Theorem 1. *The total curvature of (M, h) is given by*

$$(5) \quad \int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^n (J_i - 1) + \sum_{i=1}^m K_i \right).$$

Proof. Let $R > 0$ be such that $D_R^i := \{|z| < R\} \subset U_i, 1 \leq i \leq n$ and $D_R^i := \{|z| < R\} \subset V_{i-n}, n + 1 \leq i \leq n + m$. When R is small enough, $D_R^i \cap D_R^j = \emptyset$ for $i \neq j$.

Let $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula, we have

$$(6) \quad \int_{M_R} K dA + \sum_{i=1}^{n+m} \int_{\partial D_R^i} \kappa_g ds = 2\pi\chi(M_R) = 2\pi(\chi(M) - m).$$

If $g(p_i) \neq \infty$, then $\eta = z^{-J_i} f_i(z) dz$ where f_i is a holomorphic function in U_i and $f_i(0) \neq 0$. Write $z = re^{it}$. By Minding's formula, see [1, Volume I, pages 33-34], the geodesic curvature on ∂D_R^i is given by

$$\kappa_g \Lambda = -\frac{1}{R} + \frac{\partial \log \Lambda}{\partial \nu},$$

where ν is the interior unit normal of ∂D_R^i . Now $\Lambda = 1/2|z|^{-J_i}|f_i|(1 + |g|^2)$, so

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1 + |g|^2)}{\partial r}$$

and

$$\int_{\partial D_R^i} \kappa_g ds = \int_0^{2\pi} \kappa_g \Lambda R dt = \int_0^{2\pi} \left(\frac{J_i - 1}{R} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1 + |g|^2)}{\partial r} \right) R dt.$$

Since

$$\int_0^{2\pi} \frac{\partial \log |f_i|}{\partial r} R dt = \int_{D_R^i} \Delta(\log |f_i|) dx dy = 0$$

and $\partial \log(1 + |g|^2)/\partial r$ is bounded, we have

$$\lim_{R \rightarrow 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

If $g(p_i) = \infty$, then $g = z^{-m_i} g_i(z)$, $m_i > 0$, and $\eta = z^{-J_i+2m_i} f_i(z) dz$, where h_i and g_i are holomorphic functions in U_i and $h_i(0) \neq 0$, $g_i(0) \neq 0$. Then

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i - 2m_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1 + |g|^2)}{\partial r}.$$

Since

$$\begin{aligned} \frac{\partial \log(1 + |g|^2)}{\partial r} &= \frac{1}{1 + r^{-2m_i} |g_i|^2} \left(-2m_i r^{-2m_i-1} |g_i|^2 + r^{-2m_i} \frac{\partial |g_i|^2}{\partial r} \right), \\ - \int_0^{2\pi} \frac{\partial \log(1 + |g|^2)}{\partial r} R dt &= \int_0^{2\pi} \frac{2m_i R^{-2m_i} |g_i|^2}{1 + R^{-2m_i} |g_i|^2} dt - \int_0^{2\pi} \frac{R^{-2m_i} \frac{\partial |g_i|^2}{\partial r}}{1 + R^{-2m_i} |g_i|^2} R dt \\ &\rightarrow 4m_i \pi \text{ as } R \rightarrow 0. \end{aligned}$$

We still have

$$\lim_{R \rightarrow 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

Similarly, for the branch points q_i , if $g(q_i) \neq \infty$, then $\eta = z^{K_i} f_i(z) dz$ where f_i is a holomorphic function defined in V_i and $f_i(0) \neq 0$. A similar calculation gives

$$\int_{\partial D_R^{i+n}} \kappa_g ds = - \int_0^{2\pi} \left(\frac{K_i + 1}{R} + \frac{\partial \log |f_i|}{\partial r} + \frac{\partial \log(1 + |g|^2)}{\partial r} \right) R dt.$$

Hence

$$\lim_{R \rightarrow 0} \int_{\partial D_R^{i+n}} \kappa_g ds = -2\pi(K_i + 1).$$

If $g(q_i) = \infty$, then $g(z) = z^{-m_i} g_i(z)$ and $\eta = z^{K_i+2m_i} f_i(z)$; a similar calculation still gives us the same limit.

Note that

$$\lim_{R \rightarrow 0} \int_{M_R} K dA = \int_M K dA.$$

Letting $R \rightarrow 0$ in (6), we get (5). The proof is complete. □

Remark 1. If X in (2) is well defined, then h is induced by X . In this case, the order of Λ at an end is invariant under a rotation in \mathbb{R}^3 , and so we can assume that $g(p_i) = 0$. By the Enneper-Weierstrass representation (2) and the definition of I_i in [2], we see that $J_i - 1 = I_i$. Therefore, when M is a regular minimal surface, (5) gives (1).

The calculation also works for boundary branch points. Let M be a compact domain of a Riemann surface with a C^2 boundary $\Gamma = \partial M$. Suppose that g and η are a given meromorphic function and 1-form, respectively, and h is the Riemannian metric with isolated degenerate points defined by (3) and (4). Let $q_i \in M$ ($1 \leq i \leq m$) be the interior branch points with branch order K_i and $s_i \in M$ ($1 \leq i \leq n$) be the boundary branch points with branch order L_i . Then

Theorem 2. *The total curvature of (M, h) is given by*

$$(7) \quad \int_M K dA = 2\pi \left(\chi(M) + \sum_{i=1}^m K_i \right) + \pi \sum_{i=1}^n L_i - \int_{\Gamma} \kappa_g ds.$$

A sketch of the proof of (7) is as follows:

Define D_R^i as before and $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula,

$$\int_{M_R} K dA + \int_{\partial M_R} \kappa ds + \sum_{i=1}^n (\alpha_R^i + \beta_R^i) = 2\pi(\chi(M) - m),$$

where α_R^i and β_R^i are the exterior angles near the boundary branch points and

$$\lim_{R \rightarrow 0} \alpha_R^i = \frac{\pi}{2}, \quad \lim_{R \rightarrow 0} \beta_R^i = \frac{\pi}{2}.$$

Then (7) follows by

$$\begin{aligned} \lim_{R \rightarrow 0} \int_{\partial D_R^i \cap \partial M_R} \kappa ds &= \lim_{R \rightarrow 0} \int_{\epsilon_R^i}^{\delta_R^i} \left(\frac{-1}{R} - \frac{\partial \log \Lambda}{\partial r} \right) R dt = \lim_{R \rightarrow 0} (\epsilon_R^i - \delta_R^i)(1 + L_i) \\ &= -\pi(1 + L_i), \end{aligned}$$

for the boundary branch points.

Remark 2. If X in (2) is well defined, then X is a minimal surface and h is induced by X . In this case, (7) is the same as the formula in [1, Volume II, page 128].

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