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ON THE WEAK UNIFORM CONVEXITY OF $Q(R)$

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Abstract. We will discuss the geometry of the unit sphere in the Banach space of integrable holomorphic quadratic differentials on a Riemann surface and answer some questions posed by L.R. Goldberg (Proc. Amer. Math. Soc. **118** (1993), 1179–1185).

1

For a given hyperbolic Riemann surface R, that is, covered by the unit disk Δ, we denote by $Q(R)$ the Banach space of all holomophic quadratic differentials $\phi(z)\,dz^2$ on R with the L^1 norm

$$\|\phi\| = \iint_R |\phi(z)|\,dx\,dy < \infty.$$

It is well known that $Q(R)$ is of finite dimension if and only if R is of finite type. Let $S(R) = \{\phi \in Q(R) : \|\phi\| = 1\}$ denote the unit sphere in $Q(R)$. We are going to study the geometry of $S(R)$, answering some questions of Goldberg [5].

In general, if X is a Banach space, then X^* is the Banach space of bounded real-linear functionals $l : X \to \mathbf{R}$. Now define X to be smooth if for every unit vector x in X there is a unique l in X^* with $l(x) = \|l\| = 1$. Note that our definition of smoothness is taken from Diestel [1] and differs from the definition in Goldberg [5]; however, Diestel proved that the two definitions are equivalent.

It is well known that $Q(R)$ is smooth (see [2] or [4]). If $\psi \in S(R)$, the unique functional ψ_* in $Q(R)^*$ with $\psi_*(\psi) = \|\psi_*\| = 1$ is

$$\psi_*(\phi) = \mathrm{Re} \iint_R \phi(z)\frac{\overline{\psi(z)}}{|\psi(z)|}\,dx\,dy, \quad \phi \in Q(R).$$

It is also well known that $Q(R)$ is strictly convex, meaning (as usual) that every point of $S(R)$ is an extreme point of the closed unit ball.

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A Banach space X is said to be uniformly convex if for all sequences (x_n) and (y_n) in the closed unit ball, $\|x_n+y_n\| \to 2$ implies $\|x_n-y_n\| \to 0$. It is obvious that a finite dimensional strictly convex Banach space is uniformly convex (in particular $Q(R)$ is uniformly convex if R has finite type), but the infinite dimensional case is more complicated. In her study of the convexity of $Q(\Delta)$, Goldberg uses two weaker notions.

Following Goldberg [5] we say that X is uniformly convex at the unit vector x if for all sequences (x_n) in the closed unit ball, $\|x_n + x\| \to 2$ implies $x_n \to x$. If X is smooth, x is a unit vector, and x_* is the unique functional of norm one such that $x_*(x) = 1$, we say that X is weakly uniformly convex at x if for all sequences (x_n) in the unit sphere, $x_*(x_n) \to 1$ implies $x_n \to x$.

If $x_*(x_n) \to 1$, then $x_*(x_n + x) \to 2$, so $\|x_n + x\| \to 2$. Therefore uniform convexity at x implies weak uniform convexity at x. Uniform convexity trivially implies uniform convexity at each unit vector x, so if R is a finite Riemann surface, then $Q(R)$ is weakly uniformly convex at every point of $S(R)$. That solves Problem 3 of Goldberg [5].

To study the weak uniform convexity of $Q(R)$ in the infinite dimensional case we use Hamilton sequences. Let l be a linear functional on $Q(R)$. By definition, a Hamilton sequence for l is a sequence (ϕ_n) in $S(R)$ such that $l(\phi_n) \to \|l\|$. The Hamilton sequence (ϕ_n) is called degenerate if $\phi_n \to 0$ locally uniformly in R.

Now we can state our main result.

Theorem 1. *The Banach space $Q(R)$ is weakly uniformly convex at $\psi \in S(R)$ if and only if the linear functional ψ_* does not possess any degenerate Hamilton sequence.*

Remark. Theorem 1 gives a characterization of the flat points in $S(R)$ where $Q(R)$ is not weakly uniformly convex; this gives an answer to Goldberg's Problem 2. A solution to her Problem 1 will be given in §3.

The paper is organized as follows: We prove Theorem 1 in §2. In §3, we obtain some consequences by using Strebel's Frame Mapping Condition from the theorey of quasiconformal mappings. Finally, in §4, we discuss the uniform convexity of the Banach space $Q(R)$.

2

We first point out that Hamilton sequences obey a well known principle which is implict in Strebel [7] and also Harrington-Ortel [6]. The principle is this: If l is a nontrivial linear functional on $Q(R)$, then either l has a degenerating Hamilton sequence or else every Hamilton sequence converges (in norm) to the unique ϕ in $S(R)$ such that $l(\phi) = \|l\|$. These two possibilities are obviously mutually exclusive.

Now we give the proof of Theorem 1.

Assume first that $Q(R)$ is weakly uniformly convex at $\psi \in S(R)$. Let (ϕ_n) be any Hamilton sequence for ψ_*. Since $\psi_*(\phi_n) \to 1$, the weak uniform convexity implies that $\phi_n \to \psi$, so every Hamilton sequence for ψ_* converges in norm to ψ, and no Hamilton sequence for ψ_* is degenerate.

Conversely, suppose $\psi \in S(R)$ and no Hamilton sequence for ψ_* is degenerate. We shall prove that $Q(R)$ is weakly uniformly convex at ψ. Let (ϕ_n) be a sequence in $S(R)$ such that $\psi_*(\phi_n) \to 1$. Since (ϕ_n) is a Hamilton sequence for ψ_*, the above-stated principle implies that $\phi_n \to \psi$ as required.

3

For an extremal quasiconformal mapping $f : R \to R'$, we denote by $H(f)$ the boundary dilatation of f (see Strebel [8] or Gardiner [4,§6.8] for the definition) and by $K_0(f)$ its maximal dilatation. For a given $\psi \in S(R)$, choose k in the open interval $(0, 1)$ and denote by f_ψ the extremal quasiconformal mapping on R whose Beltrami coefficient is $k\overline{\psi}/|\psi|$.

Theorem 2. *If $\psi \in S(R)$, then $Q(R)$ is weakly uniformly convex at ψ if and only if $H(f_\psi) < K_0(f_\psi)$.*

Proof. By Strebel's Frame Mapping Condition (see [4] or [8]) and a recent result of Earle-Li [3], if $\psi \in S(R)$, then $H(f_\psi) < K_0(f_\psi)$ if and only if the linear functional $k\psi_*$ defined by the Beltrami differential of f_ψ has no degenerate Hamilton sequence, that is, ψ_* has no degerenate Hamilton sequence. Therefore, $Q(R)$ is weakly uniformly convex at ψ if and only if $H(f_\psi) < K_0(f_\psi)$, by Theorem 1.

Corollary. *Let $\psi \in S(\Delta)$ have a holomorphic extension to a neighbourhood of the closed disk with no zeroes on the unit circle. Then $Q(\Delta)$ is weakly uniformly convex at ψ.*

Proof. Under the hypothesis, the restriction of f_ψ to a neighbourhood of the unit circle is a real analytic diffeomorphism, so $H(f_\psi) = 1 < K_0(f_\psi)$.

Remark. The Corollary applies in particular to $\psi = \frac{n+2}{2\pi} z^n dz^2 \in S(\Delta)$ and solves Goldberg's Problem 1.

4

Goldberg [5] shows that $Q(\Delta)$ is nowhere uniformly convex. For any Riemann surface R, we have

Theorem 3. (i) *If $\dim Q(R) < \infty$, $Q(R)$ is uniformly convex.*
(ii) *If $\dim Q(R) = \infty$, $Q(R)$ is nowhere uniformly convex.*

Proof. We have already remarked in §1 that since $Q(R)$ is strictly convex, it is uniformly convex whenever it is finite dimensional.

Now we suppose that $\dim Q(R) = \infty$. We modify Goldberg's discussion (see [5]). Fix $\psi \in S(R)$ and $\varepsilon > 0$. Choose some compact set $K \subset R$ such that $\iint_K |\psi(z)|\,dx\,dy > 1 - \varepsilon$. Since $\dim Q(R) = \infty$, there must exist a degenerate sequence (ϕ_n) in $S(R)$, that is, $\phi_n \to 0$ locally uniformly in R. When n is sufficiently large, $\iint_{S-K} |\phi_n(z)|\,dx\,dy > 1 - \varepsilon$. Therefore,

$$\begin{aligned}\|\phi_n \pm \psi\| &= \iint\limits_K |\phi_n \pm \psi|\,dx\,dy + \iint\limits_{S-K} |\phi_n \pm \psi|\,dx\,dy \\ &\geq \iint\limits_K |\psi|\,dx\,dy - \iint\limits_K |\phi_n|\,dx\,dy + \iint\limits_{S-K} |\phi_n|\,dx\,dy - \iint\limits_{S-K} |\psi|\,dx\,dy \geq 2 - 4\varepsilon.\end{aligned}$$

Thus, $Q(R)$ is not uniformly convex at $\psi \in S(R)$.

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