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ON THE EXISTENCE OF SOLUTIONS OF NONLINEAR EQUATIONS

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Abstract. Results on the existence of solutions are derived for asymptotically quasilinear, nonlinear operator equations. Applications are given to implicit nonlinear integral equations.

1. Introduction

The purpose of this paper is to study the existence of a solution for the operator equation

$$L(x) = N(x) + h\,, \tag{1.1}$$

where $L\colon H \to H$ is continuous, $N\colon H \to H$ is continuous compact, $h \in H$ and H is a real Hilbert space with the inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We assume that L is asymptotically linear at infinity and N is asymptotically quasilinear at infinity. The asymptotic quasilinearity of N at infinity roughly speaking means the approximability of N as $x \to \infty$ by a family $\mathcal{M}$ of bounded linear maps. We are motivated by the papers [6], [9] for introducing this notion. Under some conditions on $\mathcal{M}$ we associate to $\mathcal{M}$ a generalized Leray–Schauder degree in the sense of the papers [2], [10]. Then we show that (1.1) is solvable for any $h \in H$. This is a generalization of the well–known results about asymptotically linear operators (see [4], [7], [8], [15]).

We also study the case when the operators L and N leave a wedge in H invariant. We find sufficient conditions for the existence of a solution of (1.1) in that wedge. This is an extension of similar results of the papers [1], [5], [9], [11].

Applications are given to implicit nonlinear integral equations. We are motivated by the papers [12]–[14] to study implicit integral equations, because boundary value problems of differential equations are studied in [14] when highest–order derivatives are not solvable as well as there are certain stimulations from physics in [12], [13]. Finally, we note that results of this paper can be also applied to problems studied in [2], [10].

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2. Solvability of (1.1)

The following definitions will be needed in the sequel (see [2, p. 946]).

A mapping $f\colon H \to H$ is:

– *monotone* (denote $f \in MON$), if $(f(u)-f(v), u-v) \geq 0$ for all u, $v \in H$;

– *pseudomonotone* ($f \in PM$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$ (weak convergence) and $\overline{\lim}(f(u_n), u_n - u) \leq 0$, it follows that $f(u_n) \rightharpoonup f(u)$ and $(f(u_n), u_n) \to (f(u), u)$;

– *of class* S_+ ($f \in S_+$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$ and $\overline{\lim}(f(u_n), u_n - u) \leq 0$, it follows that $u_n \to u$;

– *compact* ($f \in COMP$), if it is continuous and for any bounded sequence $\{u_n\}$ in H the sequence $\{f(u_n)\}$ has a convergent subsequence;

– *completely continuous* ($f \in CC$), if for any sequence $\{u_n\}$ in H with $u_n \rightharpoonup u$, it follows that $f(u_n) \to f(u)$;

– *bounded*, if it takes any bounded set of H into a bounded set.

We note that the following relations hold between the above definitions

$$\begin{aligned}
&CC \subset COMP, \quad S_+ \subset PM, \quad MON \subset PM\,;\\
&f_1 \in S_+,\ f_2 \in COMP \Rightarrow f_1 - f_2 \in S_+\,;\\
&f_1 \in PM,\ f_2 \in CC \Rightarrow f_1 - f_2 \in PM\,;\\
&f \in PM \Rightarrow f + \varepsilon \mathbb{I} \in S_+ \,\forall\, \varepsilon > 0\,,\\
&\text{where } \mathbb{I}\colon H \to H \text{ is the identity map}\,.
\end{aligned} \tag{2.0}$$

In what follows, we shall assume that the mappings are bounded and continuous.

Definition 2.1. A map $N\colon H \to H$ is said to be asymptotically quasilinear at infinity if there is a mapping $N_\infty\colon H \to 2^H$ such that

i) $N_\infty(u)$ is a closed nonnempty subset of H for any $u \in H$;

ii) $\alpha N_\infty(u) = N_\infty(\alpha u) \,\forall\, \alpha \geq 0$, $\forall\, u \in H$;

iii) $\overline{\bigcup_{u \in S} N_\infty(u)}$ is a compact subset for $S = \{u \in H \mid \quad |u| = 1\}$;

iv) N_∞ is upper–semicontinuous;

v) $\forall\, \varepsilon > 0\, \exists\, R > 0$ such that $\forall\, u$, $|u| \geq R$, $\exists\, w_u \in N_\infty(u)$ satisfying

$$\frac{|N(u) - w_u|}{|u|} \leq \varepsilon\,.$$

N_∞ is said to be the asymptote of N.

Remark 2.2. Let us take

$$N_\infty(u) = \{Ku\} \quad \forall\, u \in H$$

for a compact linear map $K\colon H \to H$. Then the assumptions i)–iv) of Definition 2.1 are satisfied trivially. The assumption v) expresses the usual asymptotic linearity at infinity. So a mapping $L\colon H \to H$ is said to be asymptotically linear at infinity if there is a bounded linear map (the so–called asymptote of L) $L_\infty\colon H \to H$ such that

$$\lim_{|u| \to \infty} \frac{|L(u) - L_\infty u|}{|u|} = 0\,.$$

Theorem 2.3. *Suppose L, N are asymptotically linear, quasilinear at infinity with the asymptotes $L_\infty \in S_+$, N_∞, respectively. Assume $L - N \in PM$ and there is a compact linear map $A\colon H \to H$ such that*

$$0 \notin L_\infty u - \lambda N_\infty(u) - (1-\lambda)Au \quad \forall\, u \in S, \forall\, \lambda \in [0,1]\,. \tag{2.1}$$

Then (1.1) *has a solution for any $h \in H$.*

Proof. For simplicity, we consider the case $h = 0$. First, we show the existence of $\delta > 0$, $R_0 > 0$ such that

$$\begin{aligned} &\big|\lambda L(u) - \lambda N(u) + (1-\lambda)(L_\infty u - Au)\big| \geq \delta \\ &\forall\, u,\ |u| \geq R_0, \quad \forall\, \lambda \in [0,1]\,. \end{aligned} \tag{2.2}$$

For this purpose, we compute

$$\begin{aligned} &\lambda L(u) - \lambda N(u) - (1-\lambda)Au + (1-\lambda)L_\infty u \\ &= |u|\left(L_\infty \frac{u}{|u|} - \lambda\frac{w_u}{|u|} - (1-\lambda)A\frac{u}{|u|} + \lambda\frac{L(u) - L_\infty u}{|u|} + \lambda\frac{w_u - N(u)}{|u|}\right), \end{aligned} \tag{2.3}$$

where w_u is from Definition 2.1. Now we show that there is $c_1 > 0$ such that

$$\begin{aligned} &\big|L_\infty z - \lambda w_z - (1-\lambda)Az\big| \geq c_1 \\ &\forall\, z \in S, \forall\, w_z \in N_\infty(z), \quad \forall\, \lambda \in [0,1]\,. \end{aligned} \tag{2.4}$$

If it is not true, then there is a sequence $\lambda_i \in [0,1]$, $z_i \in S$, $w_{z_i} \in N_\infty(z_i)$ such that $\lambda_i \to \lambda_0$ and

$$L_\infty z_i - \lambda_i w_{z_i} - (1-\lambda_i)Az_i \to 0\,.$$

We can assume $z_i \rightharpoonup z_0$ (the weak convergence) and $w_{z_i} \to \bar{w}$ (see the assumption iii) of Definition 2.1). So we have both $Az_i \to Az_0$, since A is compact linear, and

$$L_\infty z_i - \lambda_0 \bar{w} - (1-\lambda_0)Az_0 \to 0\,.$$

Since $L_\infty \in S_+$, we can assume $z_i \to z_0$. This gives

$$L_\infty z_0 - \lambda_0 \bar{w} - (1-\lambda_0)Az_0 = 0\,. \tag{2.5}$$

Finally, we know $w_{z_i} \to \bar{w}$, $w_{z_i} \in N_\infty(z_i)$ and $z_i \to z_0$. The assumptions i) and iv) of Definition 2.1 imply $\bar{w} \in N_\infty(z_0)$. Since (2.5) contradicts (2.1), (2.4) is proved.

We return to (2.3). By the assumption ii) of Definition 2.1 we have

$$\frac{w_u}{|u|} \in N_\infty\left(\frac{u}{|u|}\right)$$

in (2.3). We see now that (2.4) gives (2.2) for R_0 sufficiently large.

According to the assumptions $L - N \in PM$, $L_\infty \in S_+$, $A \in COMP$ and (2.0), we have that

$$\lambda(L - N) + (1 - \lambda)(L_\infty - A) + \varepsilon\mathbb{I} \in S_+ \quad \forall\,\lambda \in [0, 1]\,,$$

where $\mathbb{I}\colon H \to H$ is the identity map and $\varepsilon > 0$. So the following relations hold:

$$\deg(L - N + \varepsilon\mathbb{I}, B_{R_0}, 0) = \deg(L_\infty - A, B_{R_0}, 0) \neq 0\,,$$

where $B_{R_0} = \{u \in H \mid \quad |u| < R_0\}$, $\varepsilon > 0$ is sufficiently small and deg is the generalized Leray–Schauder degree in the sense of the papers [2], [10]. This gives the solvability of $L(u) + \varepsilon u = N(u)$ in B_{R_0} for any $\varepsilon > 0$ sufficiently small. By passing to the limit $\varepsilon \to 0_+$ and using $L - N \in PM$, we obtain the solvability of $L(u) = N(u)$. The proof is finished.

Remark 2.4. If $N_\infty(\cdot)$ are, moreover, convex and $Au \in N_\infty(u)\,\forall\,u \in H$, then (2.1) is simplified to

$$0 \notin L_\infty u - N_\infty(u) \quad \forall\,u \in S\,.$$

This case happens in the following way. Let $\mathcal{M}$ be a nonempty compact convex subset of the set of all compact linear maps from H into H. We put

$$N_\infty(u) = \{Ku \mid K \in \mathcal{M}\} \quad \forall\,u \in H\,.$$

It is clear that the assumptions i)–iv) of Definition 2.1 hold as well as the convexity of $N_\infty(\cdot)$. Now we can take $A = K_0$ for any fixed $K_0 \in \mathcal{M}$. So we can associate the degree $\deg(L_\infty - K_0, B_1, 0)$ to $\mathcal{M}$. Here $B_1 = \{u \in H \mid \quad |u| < 1\}$. We note that $\deg(L_\infty - K_0, B_1, 0) \neq 0$ is constant in $K_0 \in \mathcal{M}$. If the remaining assumptions of Theorem 2.3 hold, then (1.1) is solvable for any $h \in H$.

Finally, if N is asymptotically linear at infinity, i.e. $\mathcal{M} = \{B\}$ for a compact linear map B, then (2.1) is simplified to

$$0 \neq L_\infty u - Bu \quad \forall\,u \neq 0\,.$$

Then Theorem 2.3 is an extension of [7, Theorem XI.1] and the proof of this theorem is related to a proof of that one.

Now we look for special solutions of (1.1). Let $C \subset H$ be a wedge, i.e. C is a closed, nonempty, convex subset of H such that $\lambda C \subset C\,\forall\,\lambda \geq 0$. We know (see [3, p. 71]) that there is a continuous retraction $\eta\colon H \to C$, the so–called metric projection, such that $\eta(\lambda x) = \lambda\eta(x)\,\forall\lambda \geq 0$, $\forall x \in H$ and $|\eta(x)| \leq |x|\,\forall x \in H$.

We are interested in the existence of solutions of (1.1) in C. So in the rest of this section, we solve the equation

$$L(x) = N(x), \quad x \in C\,, \tag{2.6}$$

where $N \in COMP$ satisfies $N(C) \subset C$ and $L \in MON$ is such that $(L+\varepsilon\mathbb{I})(C) = C$ for any $\varepsilon > 0$ sufficiently small. Here $\mathbb{I}$ is the identity map.

We suppose

$(H1)$ L is asymptotically linear at infinity with the asymptote $L_\infty \in S_+$ satisfying $L_\infty x = 0 \Rightarrow x = 0$ and $L_\infty(C) = C$.

$(H2)$ N is asymptotically quasilinear at infinity with the asymptote N_∞ satisfying $N_\infty(u) \subset C\,\forall\,u \in C$.

Theorem 2.5. *Assume that* $(H1)$, $(H2)$ *hold and* $L - N \in PM$. *If for any* $x \in C$, $0 < \lambda \leq 1$, *the condition* $0 \in L_\infty x - \lambda N_\infty(x)$ *implies* $x = 0$, *then* (2.6) *has a solution.*

Proof. We solve

$$L(x) + \varepsilon x = N(\eta(x)) \tag{2.7}$$

for $\varepsilon > 0$ small. We know partly that $L + \varepsilon I$ is strongly monotone (i.e. $\big((L(x_1) + \varepsilon x_1) - (L(x_2) + \varepsilon x_2), x_1 - x_2\big) \geq \varepsilon |x_1 - x_2| \,\forall\, x_1,\, x_2 \in H$), so it is invertible (see [3, p. 100]) and partly that $(L + \varepsilon \mathbb{I})(C) = C$. Hence any solution of (2.7) belongs to C.

Now we verify the assumption (2.1) with $A = 0$ for the map $L + \varepsilon \mathbb{I} - N(\eta)$ with $\varepsilon > 0$ sufficiently small, i.e. we intend to apply Theorem 2.3 to (2.7). For this purpose, we claim that the asymptote of $N(\eta)$ is $N_\infty(\eta)$. Indeed, the boundedness of N and the assumptions of Definition 2.1 imply that for any $\omega > 0$ there is a constant $c(\omega) > 0$ such that for any $u \in H$ there is $w_u \in N_\infty(u)$ satisfying

$$|N(u) - w_u| \leq \omega |u| + c(\omega)\,.$$

Since for any bounded $B \subset H$, the compactness of $\overline{N_\infty(B)}$ follows from the assumptions i)–iv) of Definition 2.1, we can clearly take

$$c(\omega) = \sup \Big\{ |N(u)| + |w| \quad \Big| \quad u \in H;\, |u| \leq R;\, w \in N_\infty(u) \Big\}\,,$$

where R is established by the assumption v) of Definition 2.1 when ω is considered instead of ε.

Hence we have

$$|N(\eta(u)) - w_{\eta(u)}| \leq \omega |\eta(u)| + c(\omega) \leq \omega |u| + c(\omega)\,.$$

Lastly, the validity of the assumptions i)–iv) of Definition 2.1 for $N_\infty\big(\eta(\cdot)\big)$ can be easily verified. The claim is proved.

If (2.1) is not true for this case with $A = 0$, then there is a sequence $\varepsilon_i > 0$, $u_i \in S$, $\lambda_i \in [0, 1]$ such that $\varepsilon_i \to 0$, $u_i \rightharpoonup u_0$, $\lambda_i \to \lambda_0$ and

$$0 \in L_\infty u_i + \varepsilon_i u_i - \lambda_i N_\infty(\eta(u_i))\,. \tag{2.8}$$

So there is $w_i \in N_\infty(\eta(u_i))$ such that

$$0 = L_\infty u_i + \varepsilon_i u_i - \lambda_i w_i\,.$$

This equality gives

$$L_\infty u_i - \lambda_0 w_i \to 0\,.$$

By the assumption iii) of Definition 2.1, we can assume $w_i \to w$. Hence

$$L_\infty u_i - \lambda_0 w \to 0\,.$$

By the condition $L_\infty \in S_+$ we can assume $u_i \to u_0$. Since $w_i \in N_\infty(\eta(u_i))$, we obtain both $w \in N_\infty(\eta(u_0))$ (see the assumptions i) and iv) of Definition 2.1) and

$L_\infty u_0 = \lambda_0 w$. The assumptions $(H1-2)$ imply the invertibility of L_∞ and $w \in C$. Hence we obtain $u_0 \in C$. Finally, we see that the equation (2.8) gives

$$0 \in L_\infty u_0 - \lambda_0 N_\infty(u_0)$$

for some $u_0 \in S \cap C$. This contradicts our assumptions. So (2.1) holds for (2.7). Hence Theorem 2.3 is applicable to (2.7) for any $\varepsilon > 0$ sufficiently small. The proof of Theorem 2.3 gives a constant $M > 0$ such that (2.7) has a solution x_ε, $|x_\varepsilon| \le M$. We already know $x_\varepsilon \in C$. By using both the pseudomonotony of $L - N$ and the weak closeness of C, we obtain the desired solution. The proof is finished.

Remark 2.6. If $L \in MON$ holds also in Theorem 2.3, then $L - N + \varepsilon\mathbb{I} \in S_+$ for any $\varepsilon > 0$ (see (2.0)). Now, let the assumption $L - N \in PM$ be dropped in Theorem 2.5 and let this assumption be replaced by $L \in MON$ in Theorem 2.3. Then the assumptions of both Theorems 2.3 and 2.5, modified in this way, imply only *the almost solvability* of $L(u) = N(u)$ in the sense that

$$0 \in \overline{(L-N)(H)}, \quad \text{respectively} \quad 0 \in \overline{(L-N)(C)}\,.$$

Indeed, by the ends of the proofs of Theorems 2.3 and 2.5, there is a constant $M > 0$ such that both equations $L(u) + \varepsilon u = N(u)$ and $L(u) + \varepsilon u = N(u)$, $u \in C$, have solutions x_ε, $|x_\varepsilon| \le M$, for any $\varepsilon > 0$ sufficiently small. So the claim is proved.

Remark 2.7. The assumption $L - N \in PM$ in Theorems 2.3 and 2.5 is satisfied provided that $L \in PM$ and $N \in CC$.

3. Implicit integral equations

In this section, we illustrate the above abstract results for certain integral equations of the type

$$p(x, u(x)) = \int_0^1 G(x,t) m(t, u(t))\, dt + f(x), \quad x \in [0,1]\,, \tag{3.1}$$

where p, $m \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $G \in L_2([0,1] \times [0,1], \mathbb{R})$ and $f \in C([0,1], \mathbb{R})$. Moreover, we suppose that p is nondecreasing in $u \in \mathbb{R}$. We are motivated by the papers [12]–[14] to study implicit integral equations.

We assume that there are constants $\alpha > 0$, $\beta_2 \ge \beta_1$ satisfying

$$\lim_{|u|\to\infty} |p(x,u) - \alpha u| / |u| = 0 \quad \text{uniformly in } x \in [0,1]\,; \tag{3.2}$$

$$\beta_1 \le \liminf_{|u|\to\infty} m(x,u)/u \le \limsup_{|u|\to\infty} m(x,u)/u \le \beta_2 \quad \text{uniformly in } x \in [0,1]\,. \tag{3.3}$$

Theorem 3.1. *Assume that* $(3.2-3)$ *hold and, moreover, suppose that* p *is increasing in* u*, i.e.*

$$\big(p(x,u_1) - p(x,u_2)\big)(u_1 - u_2) > 0 \quad \forall\, x \in [0,1],\, u_1 \ne u_2\,.$$

If there is a constant $\delta > 0$ such that the equation

$$\alpha u(x) = \int_0^1 G(x,t)\phi(t)u(t)\,dt$$

has no nonzero solution for any $\phi \in L_2([0,1],\mathbb{R})$ satisfying $\beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta$, then (3.1) *has a solution.*

Proof. We apply Theorem 2.3 and Remark 2.4 by putting

$$H = L_2([0,1],\mathbb{R}), \quad L(u) = p(\cdot,u),$$

$$N(u) = \int_0^1 G(\cdot,t)m(t,u(t))\,dt, \quad L_\infty u = \alpha u,$$

$$\mathcal{M} = \left\{ u \to \int_0^1 G(\cdot,t)\phi(t)u(t)\,dt \quad \Big| \quad \phi \in L_2([0,1],\mathbb{R});\ \beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta \right\},$$

$$N_\infty(u) = \left\{ Ku \mid K \in \mathcal{M} \right\}.$$

It is clear that $L \in MON$, $N \in COMP$. We also know that the strict monotony of p in u implies $L \in S_+$ (see [2], [10]). So $L - N \in PM$ (see (2.0)).

Now we verify the assumptions of Definition 2.1. The assumption i) follows partly from the weak compactness of the set

$$\left\{ \phi u \quad \Big| \quad \phi \in L_2([0,1],\mathbb{R});\ \beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta \right\}$$

in $L_2([0,1],\mathbb{R})$ for any $u \in L_2([0,1],\mathbb{R})$ fixed, and partly from the compactness of the map $u \to \int_0^1 G(\cdot,t)u(t)\,dt$. So $N_\infty(\cdot)$ are compact subsets. The assumption ii) is clear. Since the set

$$\left\{ \phi u \quad \Big| \quad (\phi,u) \in L_2([0,1],\mathbb{R}) \times L_2([0,1],\mathbb{R});\ |u|_{L_2} = 1,\ \beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta \right\}$$

is bounded in $L_2([0,1],\mathbb{R})$ and the map $u \to \int_0^1 G(\cdot,t)u(t)\,dt$ is compact, the assumption iii) is proved. The assumption iv) is clear, since $N_\infty(\cdot)$ are compact subsets and $\mathcal{M}$ is a set of uniformly bounded linear mappings. To prove v), we have by (3.3) that m can be decomposed as

$$m(x,u) = m_1(x,u)u + m_2(x,u)\,,$$

where m_1, m_2 are continuous, $m_2(\cdot,\cdot)$ is bounded on $[0,1]\times\mathbb{R}$ and $\beta_1 - \delta \le m_1(\cdot,\cdot) \le \beta_2 + \delta$. Now it is clear that we can take

$$w_u = \int_0^1 G(x,t)m_1\big(t,u(t)\big)u(t)\,dt \quad \forall\, u \in L_2([0,1],\mathbb{R})$$

in the assumption v) of Definition 2.1. So N_∞ is the asymptote of N. Similarly, by using (3.2) we can show that L_∞ is the asymptote of L.

It remains to verify

$$0 \notin L_\infty u - N_\infty(u) \quad \forall\, u \in S$$

for this case. If it is not true, then there is $u \in L_2([0,1],\mathbb{R})$ such that

$$\alpha u(x) = \int_0^1 G(x,t)\phi(t)u(t)\,dt \tag{3.4}$$

for some $\phi \in L_2([0,1],\mathbb{R})$ satisfying $\beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta$. But the assumption of this theorem implies $u = 0$. So (2.1) is valid. We see that all assumptions of Theorem 2.3 are satisfied. The proof is finished.

Theorem 3.2. *Assume that asymptotic behaviors* $(3.2-3)$ *hold only for* $u \to +\infty$. *Suppose in addition that*

$$p(\cdot,0) = 0, \quad f(\cdot) \ge 0, \quad G(\cdot,\cdot) \ge 0, \quad m(\cdot,u) \ge 0\,\forall\, u \in [0,+\infty)\,; \\ p \text{ is increasing in } u\,.$$

If there is a constant $\delta > 0$ *such that the equation*

$$\alpha u(x) = \int_0^1 G(x,t)\phi(t)u(t)\,dt \tag{3.5}$$

has no nonzero nonnegative solution for any $\phi \in L_2([0,1],\mathbb{R})$ *satisfying* $0 \le \phi(\cdot) \le \beta_2 + \delta$, *then* (3.1) *has a nonnegative solution.*

Proof. We follow the above proof. Since we are interested in nonnegative solutions of (3.1), we can assume, by modifying p, m for $u \to -\infty$, the validity of (3.2–3) with $\beta_1 = 0$. We take

$$C = \Big\{ u \in L_2([0,1],\mathbb{R}) \quad \Big| \quad u \ge 0 \text{ almost everywhere on } [0,1] \Big\}.$$

It is clear that $N(C) \subset C$, $(L + \varepsilon\mathbb{I})(C) = C$. Finally, the remaining assumptions of Theorem 2.5 are verified in the same way as above. The difference is only that we now take

$$\mathcal{M} = \left\{ u \to \int_0^1 G(\cdot,t)\phi(t)u(t)\,dt \quad \Big| \quad \phi \in L_2([0,1],\mathbb{R});\, 0 \le \phi(\cdot) \le \beta_2 + \delta \right\}.$$

The proof is finished.

By applying Remark 2.6 we obtain the following

Theorem 3.3. *If the strict monotony of* p *in* u *is dropped in Theorems* $3.1 - 2$, *i.e.* p *is only nondecreasing in* u, *then* (3.1) *is almost solvable in these theorems.*

We complete this paper by presenting two examples.

Example 1. Consider the equation

$$\big(p(x,u)\big)' = m(x,u), \quad u(0) = a\,, \tag{E1}$$

where p, m possess the properties in (3.1).

Theorem 3.4. *If* $(3.2-3)$ *hold and* p *is increasing in* u, *then* (E1) *has a solution, i.e. there is* $u \in C([0,1],\mathbb{R})$ *such that* $p(\cdot,u) \in C^1([0,1],\mathbb{R})$ *and* u *satisfies* (E1).

Proof. We apply Theorem 3.1, since (E1) has the form

$$p(x,u) = \int_0^x m(t,u(t))\,dt + p(0,a)\,. \tag{3.6}$$

The linear equation in Theorem 3.1, for this case, has the form

$$\alpha u(x) = \int_0^x \phi(t)u(t)\,dt\,,$$

where $\phi \in L_2([0,1],\mathbb{R})$ satisfies $\beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta$. By using the Gronwall lemma, this equation has only the zero solution. The proof is finished.

Example 2. Consider the equation

$$\big(p(x,u)\big)'' = m(x,u), \quad u(0) = u(1) = 0\,, \tag{E2}$$

where p, m possess the properties in (3.1).

Theorem 3.5. *Assume that* $(3.2-3)$ *hold with*

$$\beta_1 > -\pi^2(n+1)^2\alpha, \quad \beta_2 < -\pi^2 n^2\alpha$$

for a nonnegative integer number n. *If* p *is increasing in* u, *then* (E2) *has a solution; i.e. there is* $u \in C([0,1],\mathbb{R})$ *such that* $p(\cdot,u) \in C^2([0,1],\mathbb{R})$ *and* u *satisfies* (E2).

Proof. By observing

$$(p(x,u))'' = (p(x,u) - p(0,0) - x(p(1,0) - p(0,0)))''\,,$$

we consider that $p(0,0) = p(1,0) = 0$. We apply Theorem 3.1, since (E2) has the form

$$p(x,u) = \int_0^1 G(x,t)m(t,u(t))\,dt\,, \tag{3.7}$$

where G is the Green function of $w'' = z$, $w(0) = w(1) = 0$. The linear equation in Theorem 3.1, for this case, has the form

$$\alpha u(x) = \int_0^1 G(x,t)\phi(t)u(t)\,dt\,,$$

where $\phi \in L_2([0,1],\mathbb{R})$ satisfies $\beta_1 - \delta \le \phi(\cdot) \le \beta_2 + \delta$. This equation is equivalent to

$$\alpha u'' = \phi u, \quad u(0) = u(1) = 0\,. \tag{3.8}$$

Since $\beta_1 > -\pi^2(n+1)^2\alpha$ and $\beta_2 < -\pi^2 n^2\alpha$, it is well–known (see [3]) that the equation (3.8) has only the zero solution for a sufficiently small constant $\delta > 0$. The proof is finished.

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