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ON THE PERTURBATION THEORY OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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ABSTRACT. Let X be a real Banach space. Let $T : X \supset D(T) \to 2^X$ be m-accretive with $(T+I)^{-1}$ compact. Let $C : X \supset D(T) \to X$ be such that $C(I+\lambda T)^{-1} : X \to X$ is condensing for some $\lambda \in (0,1)$. Let $p \in X$ and assume that there exists a bounded open set $G \subset X$ and $z \in D(T) \cap G$ such that $C(D(T) \cap \overline{G})$ is bounded and

$$(*) \qquad \langle u + Cx - p, j \rangle \geq 0,$$

for all $x \in D(T) \cap \partial G,\ u \in Tx,\ j \in J(x-z)$. Then $p \in (T+C)(D(T) \cap \overline{G})$. A basic homotopy result of the degree theory for $I - A$, with A condensing and $D(A)$ possibly unbounded, is used to improve and/or extend recent results by Hirano and Kalinde.

1. INTRODUCTION-PRELIMINARIES

The symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J. In what follows, "continuous" means "strongly continuous". The symbols $\partial D,\ \overline{D}$ denote the strong boundary and the strong closure of the set D, respectively. An operator $T : X \supset D(T) \to Y$, with Y another real Banach space, is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. For an operator $T : X \supset D(T) \to 2^Y$ and a set $G \subset D(T)$ we set $TG = \bigcup\{Tx \ :\ x \in G\}$. An operator $T : X \supset D(T) \to 2^X$ is "accretive" if for every $x,\ y \in D(T)$ there exists $j \in J(x-y)$ such that

$$\text{(A)} \qquad \langle u - v, j \rangle \geq 0 \text{ for every } u \in Tx,\ v \in Ty.$$

An accretive operator T is "strongly accretive" if 0 in the right-hand side of (A) is replaced by $\alpha\|x-y\|^2$, where $\alpha > 0$ is a fixed constant. An accretive operator T is called "m-accretive" if $R(T + \lambda I) = X$ for every $\lambda > 0$, where I denotes the identity operator on X.

We denote by $B_r(0)$ the open ball of X with center at zero and radius $r > 0$. For an m-accretive operator T, the "resolvents" $J_\lambda \ :\ X \to D(T)$ of T are defined

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by $J_\lambda = (I + \lambda T)^{-1}$ for all $\lambda \in (0, \infty)$. J_λ is a non-expansive mapping on X for all $\lambda > 0$. Also the operator $T_\lambda \equiv (1/\lambda)(I - J_\lambda)$ is a global Lipschitzian mapping with $T_\lambda x \in TJ_\lambda x$, for every $x \in X$. For facts involving accretive operators, and other related concepts, the reader is referred to Barbu [1], Browder [2], Cioranescu [4], Deimling [5], Lakshmikantham and Leela [18] and Petryshyn [22]. For a survey paper on compactness and accretivity, we cite the paper [14].

For a bounded set $\Omega \subset X$, the Kuratowski measure of noncompactness, $\gamma(\Omega)$, is defined by

$$\gamma(\Omega) = \inf\{\epsilon > 0 \ : \ \Omega \text{ can be covered by a finite family of sets of diameter } < \epsilon\}.$$

Let $k \in (0, \infty)$ be fixed. A continuous mapping $g : X \supset D(g) \to X$ is a "k-set-contraction" if $g(\Omega)$ is bounded and $\gamma(g(\Omega)) \le k\gamma(\Omega)$, for any bounded subset Ω of $D(g)$. It is called "condensing" if, for every non-empty, bounded and non-compact subset Ω of $D(g)$, $g(\Omega)$ is bounded and $\gamma(g(\Omega)) < \gamma(\Omega)$. A k-set-contraction with $k < 1$ is also called a "strict set-contraction". Hirano and Kalinde gave in [11] the following result.

Theorem A. *Let $T : X \supset D(T) \to 2^X$ be m-accretive with $(T+I)^{-1}$ compact. Let $C : X \supset D(T) \to X$ be bounded and such that $C(I+\lambda T)^{-1} : X \to X$ is condensing for some $\lambda \in (0,1)$. Let $p \in X$ and assume that there exists a constant $r > 0$ and $z \in D(T)$ such that $\|z\| < r$ and*

$$(*) \qquad \langle u + Cx - p, j\rangle \ge 0,$$

for all $x \in D(T)$ with $\|x\| \ge r$, all $u \in Tx$ and all $j \in J(x-z)$. Then $p \in R(T+C)$.

Our main purpose in this paper is to improve this result. Namely, we show that the above theorem is true with $(*)$ holding for just $x \in D(T) \cap \partial G$, where G is an open and bounded set in X containing the point z. We also assume the boundedness of the operator C only on the set $D(T) \cap \overline{G}$. We do this by applying a degree theory for mappings of the type $I - T$, where T is condensing and defined on the closure of a possibly unbounded open set. A good account of the degree theory for 1-set-contractions and condensing mappings can be found in the book of Lloyd [20, p. 92].

The results of this paper, as well as a good number of papers in the references, have applications in the control theory with preassigned responses (cf. Kaplan and the author [12]) and the construction of methods of lines (cf. [15]). For other recent results of this nature, we refer to the papers by Ding and the author [6], Guan [7], [8], Guan and the author [9], [10], and the papers [13] and [17].

2. Main results

Let G denote an open subset of X. In the next theorem we sketch the proof of how we can extend the concept of a degree function for mappings $I - T$, with $T : \overline{G} \to X$ condensing and G bounded, to such mappings with unbounded domains G and $T(\overline{G})$ bounded. In this connection, we make reference to the four degree axioms ((I)-(IV)) in Lloyd's book [20, p. 73]. We denote by co$G$ the convex hull of the set G.

Theorem 1. *Let G be a non-empty open subset of X and let $T : \overline{G} \to X$ be condensing. Assume that the set $T(\overline{G})$ is bounded. Let $p \in X$ be such that $p \notin (I-T)(\partial G)$. Then there exists a degree function $d(I-T, G, p)$ which satisfies the four axioms (I)-(IV).*

Proof. We follow the approach of Lloyd [20, pp. 95-101]. We let

$$\Sigma_k(\overline{G}) \equiv \{\phi : \overline{G} \to X \ : \ \phi \equiv I - T, \text{ where } T \text{ is a } k\text{-set-contraction} \\ \text{with } T(\overline{G}) \text{ bounded}\},$$

$$\Gamma(\overline{G}) \equiv \{\phi : \overline{G} \to X \ : \ \phi \equiv I - T, \text{ where } T \text{ is condensing with } T(\overline{G}) \text{ bounded}\}.$$

Obviously, $\Sigma_k(\overline{G}) \subset \Gamma(\overline{G})$, for $k < 1$. We define first a degree function for mappings in $\Sigma_k(\overline{G})$ provided that $k < 1$. To this end, we note that every function in the family $\Gamma(\overline{G})$ is bounded and that it is also proper (and thus closed) by a proof identical to that of Lemma 6.2.1 in Lloyd's book. In particular, every function in $\Sigma_k(\overline{G}),\ k < 1$, is also bounded and proper. Given a function $\phi \equiv I - T \in \Sigma_k(\overline{G}),\ k < 1$, we let $\Delta_1 \equiv \overline{\mathrm{co} T(\overline{G})}$ and, inductively,

$$\Delta_n \equiv \overline{\mathrm{co} T(\Delta_{n-1} \cap \overline{G})},\ n = 2, 3, \dots .$$

Since $T(\overline{G})$ is bounded, each set Δ_n is also bounded. Moreover, each set Δ_n is closed and convex, $\{\Delta_n\}$ is a decreasing sequence and the set

$$\Delta \equiv \bigcap_{n=1}^{\infty} \Delta_n$$

is non-empty, convex and compact. Since the set $\overline{G} \cap \Delta$ is closed, Dugundji's theorem says that the mapping T, restricted to the set $\overline{G} \cap \Delta$, has a continuous extension $\widetilde{T}$ to all of X. This function $\widetilde{T}$ has values in the set $\mathrm{co} T(\overline{G} \cap \Delta) \subset \mathrm{co}\Delta = \Delta$ and coincides with T on the set $\overline{G} \cap \Delta$. Since the values of the function $\widetilde{T}$ on the set $\overline{G} \cap \Delta$ lie in the compact set Δ, the Nagumo degree (cf. [21]) $d(I - \widetilde{T}, G, 0)$ is well defined, provided that $0 \notin (I - \widetilde{T})(\partial G)$. If $0 \notin \phi(\partial G)$ we define $d(\phi, G, 0) \equiv d(I - \widetilde{T}, G, 0)$. If $p \notin \phi(\partial G)$, we define $d(\phi, G, p) \equiv d(\phi - p, G, 0)$. A careful examination of the material on pages 97-100 of Lloyd's book shows that this degree function has the four properties (I)-(IV).

Given a function $\phi \equiv I - T \in \Gamma(\overline{G})$, we let

$$\delta \equiv (1/3)\rho(0, \phi(\partial G)),$$

where ρ denotes the distance function between two sets. We know that $\rho > 0$, for $0 \notin \phi(\partial G)$, because the set $\phi(\partial G)$ is closed. If $0 \notin \phi(\partial G)$, we define $d(\phi, G, 0) \equiv d(I - g, G, 0)$, where $g : \overline{G} \to X$ is any strict set-contraction such that $g(\overline{G})$ is bounded and

$$\sup_{x \in \overline{G}} \|T(x) - g(x)\| < \delta.$$

If $p \notin \phi(\partial G)$, we let $d(\phi, G, p) \equiv d(\phi - p, G, 0)$. Again, it is easy to see that this degree function is well defined and has the four basic degree properties (I)-(IV). $\square$

We are now ready for our main result.

Theorem 2. *Let $T : X \supset D(T) \to 2^X$ be m-accretive with $(T+I)^{-1}$ compact. Let $C : X \supset D(T) \to X$ be such that $C(I+\lambda T)^{-1} : X \to X$ is condensing for some $\lambda \in (0,1)$. Let $p \in X$ and assume that there exists a bounded open set $G \subset X$ and $z \in D(T) \cap G$ such that $C(D(T) \cap \overline{G})$ is bounded and*

$$(*) \qquad \langle u + Cx - p, j \rangle \geq 0,$$

for all $x \in D(T) \cap \partial G$, all $u \in Tx$ and all $j \in J(x-z)$. Then $p \in R(T+C)$. Actually, $p \in (T+C)(D(T) \cap \overline{G})$.

Proof. We show first that we may assume $z = 0 \in D(T) \cap G$ and $0 \in T(0)$. In fact, if this is not true, we consider the new operators $\widetilde{T}$, $\widetilde{C}$ defined by

$$\widetilde{T}x \equiv T(x+z) - v,\ \widetilde{C}x \equiv C(x+z) + v,\ x \in \widetilde{D}(T),$$

where v is a fixed point in $T(z)$, $D(\widetilde{T}) \equiv D(T) - z$. We also set $\widetilde{G} \equiv G - z$. It is easy to see the operator $\widetilde{T}$ is m-accretive on $D(\widetilde{T})$. To show the compactness of the resolvent $(I+\widetilde{T})^{-1}$, and hence any resolvent of $\widetilde{T}$ (by the resolvent identity), we note first that the m-accretivity of $\widetilde{T}$ implies the continuity of this resolvent. Let $\{y_n\}$ be a bounded sequence in X and let

$$x_n = (I+\widetilde{T})^{-1} y_n.$$

Then

$$y_n = x_n + \widetilde{v}_n = x_n + v_n - v,\ n = 1, 2, \dots ,$$

where $\widetilde{v}_n \in \widetilde{T}x_n$ and $v_n \in T(x_n + z)$. From

$$y_n = (x_n + z) + v_n - (v+z),$$

we obtain that

$$(x_n + z) + T(x_n + z) \ni y_n + (v+z),$$

or

$$x_n = (I+T)^{-1}[y_n + (v+z)] - z,$$

which, by the compactness of $(I+T)^{-1}$, implies the existence of a convergent subsequence of $\{x_n\}$. This finishes the proof of the compactness of $(I+\widetilde{T})^{-1}$.

We now show that the operator $\widetilde{C}(I+\lambda\widetilde{T})^{-1}$ is condensing. To this end, we seek to express this operator in terms of the operator $C(I+\lambda T)^{-1}$. In fact, letting

$$\widetilde{J}_\lambda \equiv (I+\lambda\widetilde{T})^{-1}$$

and

$$u \equiv \widetilde{J}_\lambda y = (I+\lambda\widetilde{T})^{-1} y \in D(\widetilde{T}),$$

for some $y \in X$, we have that there exists $w \in D(T)$ such that $u = w - z$ and

$$y \in u + \lambda\widetilde{T}u = u + \lambda(T(u+z) - v) = w - z + \lambda T(w) - \lambda v.$$

This says that

$$w = (I + \lambda T)^{-1}(y + z + \lambda v),$$

i.e., that

$$\widetilde{J}_\lambda y = (I + \lambda \widetilde{T})^{-1} y = (I + \lambda T)^{-1}(y + z + \lambda v) - z = J_\lambda(y + z + \lambda v) - z.$$

Finally,

$$\widetilde{C}\widetilde{J}_\lambda y = C(\widetilde{J}_\lambda y + z) + v = (CJ_\lambda)(y + z + \lambda v) + v.$$

Since $\gamma(B+q) = \gamma(B)$, for any bounded subset B of X and any $q \in X$, we have the condensity of the operator $\widetilde{C}\widetilde{J}_\lambda$. To see that $(*)$ is satisfied with $z = 0$, it suffices to observe that

$$\langle (w - v) + (C(u + z) + v) - p, j\rangle \geq 0,$$

for every $u \in D(\widetilde{T}) \cap \partial\widetilde{G}$, every $w \in T(u + z)$ and every $j \in J(u)$. It is also easy to see that the set $\widetilde{C}(D(\widetilde{T}) \cap \overline{\widetilde{G}})$ is bounded. Thus, it suffices to prove the theorem with $z = 0$ and $0 \in T(0)$.

As in Theorem 1 of [11], we are planning to solve the problem

$$T_\lambda y + CJ_\lambda y = p. \tag{1}$$

Since $T_\lambda y \in TJ_\lambda y,\ y \in X$, the solvability of (1) leads immediately to the solvability of $Tx + Cx \ni p$ if we let $x = J_\lambda y$. Also, Equation (1) can be rewritten as

$$(1/\lambda)(I - J_\lambda)y + CJ_\lambda y = p,$$

or $(I - S)y = 0$, where

$$Sy \equiv (I - \lambda C)J_\lambda y + \lambda p.$$

Unlike [11], we consider the homotopy

$$H(t, y) \equiv y - tSy,\ (t, y) \in [0, 1] \times \overline{U},$$

where

$$U = (I + \lambda T)\left(D(T) \cap G\right).$$

Since the operator J_λ is a continuous mapping on all of X, its inverse, $(I + \lambda T)$, is a set-valued mapping that maps relatively open (closed) sets of its domain $D(T)$ onto open (closed) sets of the space X. Because of this, the set U is open and the set

$$(I + \lambda T)\left(D(T) \cap \overline{G}\right)$$

is closed. Because of this, we have

$$\begin{aligned}
\overline{(I + \lambda T)\left(D(T) \cap \overline{G}\right)} &= (I + \lambda T)\left(D(T) \cap \overline{G}\right) \\
&= (I + \lambda T)\left(D(T) \cap G\right) \cup (I + \lambda T)\left(D(T) \cap \partial G\right) \\
&\supset \overline{(I + \lambda T)\left(D(T) \cap G\right)} \\
&= (I + \lambda T)\left(D(T) \cap G\right) \cup \partial\left((I + \lambda T)\left(D(T) \cap G\right)\right),
\end{aligned}$$

which implies that $(I+\lambda T)(D(T)\cap\partial G)\supset\partial((I+\lambda T)(D(T)\cap G))$. Since

$$\overline{(I+\lambda T)\,(D(T)\cap G)}\subset(I+\lambda T)\left(D(T)\cap\overline{G}\right),$$

the homotopy $H(t,y)$ is well-defined. It is also a condensing mapping in y, for every $t\in[0,1]$, because of the compactness of the operator J_λ, the fact that CJ_λ is condensing and $\lambda\in(0,1)$. Since J_λ maps the set $\overline{U}$ onto the bounded set $D(T)\cap\overline{G}$ and $C(D(T)\cap\overline{G})$ is bounded, the operator S has a bounded range and we may (and do) apply the degree function from Theorem 1 to the mapping $H(t,y)$. In order to obtain a solution $y\in\overline{U}$ to the equation $H(1,y)=0$, which will provide us with a solution of (1) lying in $D(T)\cap\overline{G}$, we need to show that $H(t,y)=0$ has no solution lying in the set ∂U, for any $t\in(0,1)$. Here, we are using this fact, $0\in U$, and the fact that if $0\notin(I-S)(\partial U)$, then we have

$$d(I-tS,U,0)=d(I,U,0)=1,\ \ t\in[0,1].$$

Assume that there is $t\in(0,1)$ and a point $y_t\in\partial U$ such that $H(t,y_t)=0$. Then we have

$$y_t=t[(I-\lambda C)J_\lambda y_t+\lambda p].$$

If we let $x_t=J_\lambda y_t$, then the above imply that $x_t\in D(T)\cap\partial G$ and the equation

$$x_t+\lambda w_t=t[(I-\lambda C)x_t+\lambda p]$$

holds, for some $w_t\in Tx_t$. Thus,

$$(1-t)x_t+\lambda[w_t+t(Cx_t-p)]=0. \tag{2}$$

Evaluating any functional $j_t\in J(x_t)$ on this equation, we get

$$(1-t)\|x_t\|^2+\lambda\langle w_t+t(Cx_t-p),j_t\rangle=0$$

and, since $t\in(0,1)$ and $x_t\neq0$,

$$\langle w_t+t(Cx_t-p),j_t\rangle<0.$$

Let us now pick $j_t\in J(x_t)$ such that

$$\langle w_t,j_t\rangle\geq0.$$

We can do this because $0\in T(0)$ and T is accretive. If

$$\langle Cx_t-p,j_t\rangle\geq0,$$

then we have a contradiction. Let us assume that

$$\langle Cx_t-p,j_t\rangle<0.$$

Then

$$\langle w_t,j_t\rangle<-t\langle Cx_t-p,j_t\rangle<-\langle Cx_t-p,j_t\rangle,$$

or

$$\langle w_t + Cx_t - p, j_t \rangle < 0,$$

which is a contradiction to our assumed boundary condition. The proof is complete. □

The compactness of the resolvents of an m-accretive operator T appears in a necessary and sufficient condition for the compactness of the semigroup generated by T. If T is time-dependent, then the compactness of its resolvents is often one of the necessary conditions for the compactness in x of the evolution operator $U(t,s)x,\ s<t$, generated by T. For such a compact evolution operator and related material, we cite the papers [14], [16] and several of the references therein.

In order to be able to apply the above proof to the case of a possibly unbounded operator C, we must impose a condition on the operator T so that whenever y_t, in the proof of Theorem 2, lies on the boundary of a certain open set B, we have that x_t satisfies the boundary condition $(*)$. Naturally, the degree function of Theorem 1 will be well defined if $(C(I+\lambda T)^{-1})(\overline{B})$ is a bounded set. The following result reflects this situation and is actually more general than Theorem 2. However, it is rather difficult to check whether its boundary condition is satisfied in such a degree of generality. We assume, for convenience, that $0 \in D(T)$ and $0 \in T(0)$.

Theorem 3. *Let $T : X \supset D(T) \to 2^X$ be m-accretive with $(T+I)^{-1}$ compact, $0 \in D(T)$ and $0 \in T(0)$. Let $C : X \supset D(T) \to X$ be such that $C(I+\lambda T)^{-1} : X \to X$ is condensing, for some $\lambda \in (0,1)$. Let $p \in X$ and assume that there exists an open set $B \subset X$ such that $0 \in B,\ 0 \notin (I+\lambda T)^{-1}(\partial B)$, the set $(C(I+\lambda T)^{-1})(\overline{B})$ is bounded and*

$$(*) \qquad \langle u + Cx - p, j \rangle \geq 0,$$

for all $x \in (I+\lambda T)^{-1}(\partial B),\ u \in Tx$ and $j \in Jx$. Then $p \in R(T+C)$.

Proof. The proof follows exactly as in Theorem 2. In fact, proceeding as in the proof of Theorem 2, we have that the homotopy $H(t,x)$ is now defined on $[0,1]\times\overline{B}$ and the equation (2) is impossible for $t \in (0,1),\ y_t \in \partial B$ and $x_t = J_\lambda y_t$. □

As a corollary to this theorem, we obtain Theorem 3 of Hirano and Kalinde [11]. Their proof was based on the proof of Theorem 1 in [11].

Corollary 1. *Let $T : X \supset D(T) \to 2^X$ be m-accretive with $(T+I)^{-1}$ compact. Let $C : X \supset D(T) \to X$ be such that $C(I+\lambda T)^{-1} : X \to X$ is condensing, for some $\lambda \in (0,1)$. Let $p \in X$ and assume that there exists a positive constant b and $z \in D(T)$ such that $\|z\| < b$, Tz is bounded and the inequality $(*)$ is satisfied for all $x \in D(T),\ u \in Tx$ with $\max\{\|x\|,\|u\|\} > b$ and all $j \in J(x-z)$. Then $p \in R(T+C)$.*

Proof. We note first that if $\widetilde{T},\ \widetilde{C}$ are as in Theorem 2, then, for every $x \in D(\widetilde{T}),\ w = u - v \in \widetilde{T}x = T(x+z) - v,\ j \in Jx$ such that

$$\max\{\|x+z\|,\|u\|\} > b,$$

we have

$$\langle w + \widetilde{C}x - p, j \rangle = \langle u - v + (C(x+z)+v) - p, j \rangle = \langle u + C(x+z) - p, j \rangle \geq 0.$$

We apply Theorem 3 and its proof by taking B to be the open ball $B_r(-z)$. Here, the radius r is chosen so that $r > b + 2\lambda M$, where

$$M = \max\{b, \sup\{\|u\| : u \in Tz\}\}.$$

In fact, $0 \in B_r(-z)$ and if $y_t \in \partial B_r(-z)$, then $\|y_t + z\| = r$ and $x_t = \widetilde{J}_\lambda y_t \in D(\widetilde{T})$ satisfies $(x_t + z) + \lambda(u - v) = y_t + z$, where $u \in T(x_t + z)$. We show that $x_t \neq 0$ and that either $\|x_t + z\| > b$ or $\|u\| > b$. The relation

$$\|(x_t + z) + \lambda(u - v)\| = r > b + 2\lambda M$$

implies that either $\|x_t + z\| > b$ or $\|u - v\| > 2M$. In the first case, we cannot have $x_t = 0$ because $\|z\| \leq b$. In the second case, if $x_t = 0$, then $u \in Tz$ and $2M \geq \|u - v\| > 2M$, i.e., a contradiction. Moreover, if the second case holds, then $\|u\| > 2M - \|v\| \geq 2M - M = M \geq b$. Thus, all the assumptions of Theorem 3 are satisfied and the proof is complete. □

The following result extends Theorem 7 of [11] to the effect that we only require a local boundary condition.

Theorem 4. *Let $T : X \supset D(T) \to 2^X$ be m-accretive and let $C : X \supset D(T) \to X$ be such that $C([(\lambda + n)/n]I + \lambda T)^{-1} : X \to X$ is compact, for some $\lambda \in (0, 1)$ and all integers $n \geq n_0$, where n_0 is a positive integer. Let $p \in X$ and assume that there exists a bounded open set $G \subset X$ such that $0 \in D(T) \cap G$, $C(D(T) \cap \overline{G})$ is bounded and*

$$(*) \qquad \langle u + Cx - p, j\rangle \geq 0,$$

for all $x \in D(T) \cap \partial G$, all $u \in Tx$ and all $j \in Jx$. Then $p \in \overline{R(T + C)}$. Actually, $p \in \overline{(T + C)(D(T) \cap \overline{G})}$.

Proof. We only consider $n \geq n_0$. As in [11], we observe that, for the mapping $V_n \equiv T + (1/n)I$, the resolvent $(I + \lambda V_n)^{-1}$ is Lipschitz continuous with Lipschitz constant $n/(n + \lambda)$. Thus, it is condensing. Also, the boundary condition $(*)$ holds true for the mapping V_n in place of T in Theorem 1. Using the proof of Theorem 1 with V_n in place of T (the homotopy $H(t, x)$ is still condensing in x), we obtain that the problem

$$Tx + Cx + (1/n)x \ni p$$

has a solution $x_n \in D(T) \cap \overline{G}$. Thus, $\{x_n\}$ is a bounded sequence. This says that $x_n/n \to 0$ and completes the proof of the theorem. □

3. Discussion-Example

It is now apparent that Theorems 2-4 can be extended to a variety of situations involving perturbations of m-accretive operators. For example, the "inner product" conditions in these theorems may be replaced by "norm" conditions as in the various examples in the survey article [14].

It is important to mention here that the degree theories of Chen (see the end of Chen's paper [3]) and Liu [19] concerning condensing, or 1-set-contractive, perturbations of m-accretive operators contain a basic flaw. In fact, both authors claim that, for an m-accretive operator T, the operator $Q_\lambda \equiv (T+\lambda I)^{-1}$ is non-expansive for small $\lambda > 0$. However, this is not true in general because

$$(T+\lambda I)^{-1}x = ((1/\lambda)T + I)^{-1}((1/\lambda)x),$$

which says that the operator Q_λ is a Lipschitzian mapping with Lipschitz constant $1/\lambda > 1$, for all small $\lambda > 0$. It seems that these two degree theories are therefore valid for operators of the type $(T+I)+C$ with T m-accretive.

Example 1. It is easy to see that we can improve upon the example of [11]. As in [11], we consider the problem

$$(BV) \qquad \begin{cases} -\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i\left(\dfrac{\partial u}{\partial x_i}\right) + g\left(x, \dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_N}, u\right) = p(x), & \text{on } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

With the notation of [11], we let $X = L^2(\Omega)$,

$$Tu \equiv -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(\frac{\partial u}{\partial x_i}\right), \ u \in D(T) \equiv W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

and $(Cu)(x) \equiv g(x, \nabla u(x), u(x)),\ u \in D(T),\ x \in \Omega$. We also let $p \in L^2(\Omega)$. The solvability of (BV) in $D(T)$ may now be achieved if we assume exactly what was assumed in [11], but with condition (g2) there replaced by

$$\int_\Omega (g(x, \nabla u(x), u(x)) - p(x))u(x)dx \geq 0,$$

for all $u \in D(A) \cap \partial G$, where G is a bounded open set in $L^2(\Omega)$ containing zero. In particular, we may take $G = B_r(0)$. $\square$

Maximal monotone versions of some of the results in this paper can be found in the author's paper [17].

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