

## UNIQUENESS FOR NON-HARMONIC TRIGONOMETRIC SERIES

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ABSTRACT. When  $\lambda_n > 0$ ,  $\lambda_n \uparrow \infty$  and

$$\frac{1}{2}|a_0| + \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty,$$

if

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty),$$

then

$$a_0 = a_1 = b_1 = \cdots = a_n = b_n = \cdots = 0.$$

More generalized results are given.

### 1. INTRODUCTION

Let  $\{\lambda_n\}_n$  be a strictly increasing sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} \lambda_n = \infty.$$

For example,  $\lambda_n = \log(n+1)$  for  $n = 1, 2, \dots$ . In this paper we shall discuss a uniqueness problem for non-harmonic trigonometric series:

$$(1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x).$$

Many mathematicians have discussed some uniqueness problems for harmonic trigonometric series (see [1] and [3]).

Zygmund [2] discussed the same problem for the integral case

$$\int_0^{\infty} (c_s \cos sx + d_s \sin sx) ds,$$

where  $c_s$  and  $d_s$  are continuous.

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It is easy to see that the series (1) is zero at  $x$  and  $-x$  if and only if

$$(1-a) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x = 0;$$

$$(1-b) \quad \sum_{n=1}^{\infty} b_n \sin \lambda_n x = 0.$$

Using the same argument as in the proof of Theorem 2 in Section 68 of Chapter 1 of [1] (see p. 190), we can prove that if (1-a) and (1-b) hold, then

$$(2-a) \quad \lim_{h \rightarrow \infty} \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 = 0;$$

$$(2-b) \quad \lim_{h \rightarrow \infty} \sum_{n=1}^{\infty} b_n \sin \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 = 0.$$

When (1-a) and (1-b) hold, the convergences of the two series

$$H_m^e(x) := \sum_{n=1}^{\infty} \frac{a_n \cos \lambda_n x}{\lambda_n^{2m}},$$

$$H_m^o(x) := \sum_{n=1}^{\infty} \frac{b_n \sin \lambda_n x}{\lambda_n^{2m}} \quad (m = 1, 2, \dots)$$

are certified by the following lemma.

**Lemma 1.** *When  $\{\theta_n\}_n$  is a strictly decreasing sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \theta_n = 0$ , if a series  $\sum_{n=0}^{\infty} \alpha_n$  converges, then the series  $\sum_{n=0}^{\infty} \alpha_n \theta_n$  converges.*

*Proof.* Put  $R_n := \sum_{k=n}^{\infty} \alpha_k$  for  $n = 1, 2, \dots$ . By the Abel transform, we have

$$\sum_{k=n}^{\infty} \alpha_k \theta_k = R_n \theta_n - \sum_{k=n}^{N-1} R_{k+1} (\theta_k - \theta_{k+1}) - R_{N+1} \theta_N.$$

Thus,

$$\left| \sum_{k=n}^N \alpha_k \theta_k \right| \leq |R_n \theta_n| + \sup_{n \leq k} |R_{k+1}| (\theta_n + \theta_N) + |R_{N+1}| \theta_N$$

and each term in the right-hand side tends to zero when  $n$  and  $N$  tend to infinity. Hence the sequence  $\{\sum_{k=1}^n \alpha_k \theta_k\}_n$  is a Cauchy sequence. The lemma is proved.  $\square$

In this paper, we shall give the following results:

**Theorem 2.** *If*

$$(3) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty),$$

and if for  $m = 1, 2, \dots$

$$(4-a) \quad H_m^e(x) \text{ and } H_m^o(x) \text{ are continuous;}$$

$$(4-b) \quad \lim_{x \rightarrow \infty} \frac{H_m^e(x)}{x^2} = \lim_{x \rightarrow \infty} \frac{H_m^o(x)}{x} = 0,$$

then

$$(5) \quad a_0 = a_1 = b_1 = \dots = a_n = b_n = \dots = 0.$$

**Corollary 3.** *When*

$$(6) \quad \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n^2} < \infty,$$

if (3) holds, then (5) is valid.

**Theorem 4.** *When  $E$  is an enumerable set (without loss of generality, we can assume that  $E$  satisfies  $-x \in E$  if  $x \in E$ ), if*

$$(7) \quad \frac{1}{2}a_n + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) = 0 \quad \text{everywhere } (-\infty, \infty) \text{ except } E,$$

and if

$$(8) \quad H_1^e(x) \text{ and } H_1^o \text{ are smooth in } E$$

and (4-a) and (4-b) hold for  $m = 1, 2, \dots$ , then (5) is valid.

**Corollary 5.** *When*

$$(9) \quad \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{\lambda_n} < \infty,$$

if (7) holds, then (5) is valid.

## 2. PROOF OF THEOREM 2

Put

$$F_1(x) := \frac{1}{4}a_0x^2 - H_1^e(x).$$

Thus we have

$$\frac{1}{4h^2} \{F_1(x+2h) - 2F_1(x) + F_1(x-2h)\} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left(\frac{\sin \lambda_n h}{\lambda_n h}\right)^2$$

and

$$\frac{1}{4h^2} \{H_1^o(x+2h) - 2H_1^o(x) + H_1^o(x-2h)\} = \sum_{n=1}^{\infty} b_n \sin \lambda_n x \left(\frac{\sin \lambda_n h}{\lambda_n h}\right)^2.$$

From (3), the two second symmetric derivatives satisfy

$$(10) \quad D^2 F_1(x) = D^2 H_1^o(x) = 0 \quad \text{everywhere } (-\infty, \infty).$$

From (4-a) and by Lemma (3.4) in Section 3 of Chapter IX of [3] (see p. 327),  $F_1(x)$  and  $H_1^o$  are linear, that is,

$$(11-a) \quad F_1(x) = A_1 x + \frac{1}{2} B_1;$$

$$(11-b) \quad H_1^o(x) = C_1 x + D_1 \quad \text{everywhere } (-\infty, \infty).$$

From (11-a),

$$A_1 x = \frac{1}{4} a_0 x^2 - \frac{1}{2} B_1 - H_1^e(x),$$

where the left-hand side is an odd function and the right-hand side is an even function. Thus,  $A_1 = 0$ . And from (4-b),  $a_0 = 0$ . Thus

$$(12-a) \quad \frac{1}{2} B_1 + H_1^e(x) = 0 \quad \text{everywhere } (-\infty, \infty).$$

Arguing analogously, for  $H_1^o(x)$ , we can prove that  $C_1 = D_1 = 0$  and

$$(12-b) \quad H_1^o = 0 \quad \text{everywhere } (-\infty, \infty).$$

Let us discuss similarly to the above for non-harmonic trigonometric series, (12-a) and (12-b). And we can prove that  $B_1 = 0$  and for some  $B_2$

$$\begin{aligned} \frac{1}{2} B_2 + H_2^e(x) &= 0; \\ H_2^o(x) &= 0 \quad \text{everywhere } (-\infty, \infty). \end{aligned}$$

Continuing this process, we have  $B_{m-1} = 0$  and for some  $B_m$

$$(13-a) \quad \frac{1}{2} B_m + H_m^e(x) = 0;$$

$$(13-b) \quad H_m^o(x) = 0 \quad \text{everywhere } (-\infty, \infty).$$

Obviously  $B_m = 0$  for all  $m$ ; then

$$(13-a') \quad H_m^e(x) = 0 \quad \text{everywhere } (-\infty, \infty).$$

The conclusion follows if the following lemma is proved.

**Lemma 6.** *Let  $\{\theta_n\}_n$  be a sequence satisfying the condition of Lemma 1 and  $\theta_1 < 1$ . If  $\sum_{n=1}^{\infty} \alpha_n$  converges and*

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \theta_n^m = 0 \quad \text{for all } m = 1, 2, \dots,$$

*then  $\alpha_0 = 0$ .*

*Proof of Lemma 6.* Put  $R_n = \sum_{k=n}^{\infty} \alpha_k$  for  $n = 1, 2, \dots$ . For each  $\varepsilon > 0$ , there exists  $N$  such that  $|R_n| < \varepsilon$  for  $n > N$ . Since

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \theta_n^m = \alpha_0 + \alpha_1 \theta_1^m + \dots + \alpha_N \theta_N^m - R_{N+1} \theta_{N+1}^m - \sum_{N+1}^{\infty} R_{k+1} (\theta_k^m - \theta_{k+1}^m),$$

$$|R_{N+1} \theta_{N+1}^m| < \varepsilon,$$

$$\begin{aligned} \left| \sum_{k=N+1}^{\infty} R_{k+1} (\theta_k^m - \theta_{k+1}^m) \right| &\leq \sum_{k=N+1}^{\infty} |R_{k+1}| |\theta_k^m - \theta_{k+1}^m| \\ &\leq \varepsilon \sum_{k=N+1}^{\infty} \theta_k^m - \theta_{k+1}^m = \varepsilon \theta_{N+1}^m < \varepsilon \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} \sum_{k=1}^N \alpha_k \theta_k^m = 0,$$

we have

$$|\alpha_0| = \left| \sum_{k=1}^{\infty} \alpha_k \theta_k^m \right| \leq \left| \sum_{k=1}^N \alpha_k \theta_k^m \right| + \left| \sum_{k=N+1}^{\infty} \alpha_k \theta_k^m \right| \leq \left| \sum_{k=1}^N \alpha_k \theta_k^m \right| + 2\varepsilon.$$

Thus

$$|\alpha_0| \leq \lim_{m \rightarrow \infty} \left| \sum_{k=1}^N \alpha_k \theta_k^m \right| + 2\varepsilon = 2\varepsilon.$$

Consequently  $\alpha_0 = 0$ . Lemma 6 is proved.  $\square$

Now put  $\theta_n = \frac{\lambda_{n+1}}{\lambda_1}$  for  $n = 1, 2, \dots$ . Thus  $\{\theta_n\}_n$  satisfies the condition of Lemma 6. And put  $\alpha_n = a_{n+1} \cos \lambda_{n+1} x$ . Then from (13-a')

$$a_1 \cos \lambda_1 x = 0 \quad \text{everywhere } (-\infty, \infty).$$

And analogously from (13-b),

$$b_1 \sin \lambda_1 x = 0 \quad \text{everywhere } (-\infty, \infty).$$

Continuing this process we can easily prove

$$a_n \cos \lambda_n x = b_n \sin \lambda_n x = 0 \quad \text{everywhere for all } n.$$

Consequently

$$a_n = b_n = 0 \quad \text{for } n = 1, 2, \dots$$

We have proved Theorem 2.

## 3. PROOFS OF THEOREM 4 AND COROLLARIES

By Lemma (3.20) in Section 3 of Chapter IX of [3] (see p.328) and from (7), (8) and (4-a),  $F_1(x)$  and  $H_1^o(x)$  are linear. Then using the same argument as in the proof of Theorem 2, we can easily prove Theorem 4.

Obviously condition (6) in Corollary 3 is stronger than (4-a) and (4-b) in Theorem 2, and (9) in Corollary 5 is stronger than (4-a), (4-b) and (8) in Theorem 4.

*Remark 1.* Under the conditions (4-a) and (4-b),  $H_m^e(x)$  and  $H_m^o(x)$  are continuous if and only if

$$\lim_{h \rightarrow 0} \sum_{n=1}^{\infty} \frac{a_n \sin \lambda_n x}{\lambda_n^{2m}} \sin \lambda_n h = \lim_{h \rightarrow 0} \sum_{n=1}^{\infty} \frac{b_n \cos \lambda_n x}{\lambda_n^{2m}} \sin \lambda_n h = 0$$

everywhere  $(-\infty, \infty)$ .

*Remark 2.*  $H_1^e(x)$  and  $H_1^o(x)$  are smooth at  $x$  if and only if

$$\lim_{h \rightarrow 0} h \left( \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 \right) = \lim_{h \rightarrow 0} h \left( \sum_{n=1}^{\infty} b_n \sin \lambda_n x \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^2 \right) = 0.$$

(See p. 43 (3.1) in Chapter II and p. 328 (3.21) in Chapter IX of [3].)

## REFERENCES

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