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ON CONTRAVARIANT FINITENESS OF SUBCATEGORIES OF MODULES OF PROJECTIVE DIMENSION $\leq I$

BANGMING DENG

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ABSTRACT. Let $\wedge$ be an artin algebra. This paper presents a sufficient condition for the subcategory $\mathcal{P}^i(\wedge)$ of mod $\wedge$ to be contravariantly finite in mod $\wedge$, where $\mathcal{P}^i(\wedge)$ is the subcategory of mod $\wedge$ consisting of $\wedge$–modules of projective dimension less than or equal to i. As an application of this condition it is shown that $\mathcal{P}^i(\wedge)$ is contravariantly finite in mod $\wedge$ for each i when $\wedge$ is stably equivalent to a hereditary algebra.

INTRODUCTION AND PRELIMINARIES

Throughout this paper, all algebras are artin algebras, all modules are finitely generated left modules, and all subcategories are full subcategories. For an artin algebra $\wedge$, we denote by mod $\wedge$ the category of all finitely generated $\wedge$–modules.

The notions of contravariantly and covariantly finite subcategories of mod $\wedge$ were first introduced and studied by Auslander and Smalϕ in connection with the study of the existence of almost split sequences in a subcategory of mod $\wedge$ (see [2] and [3]). We now recall these notions from [2].

Definition. A full subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to be

(i) contravariantly finite in $\mathcal{B}$ if for each object X in $\mathcal{B}$, the representable functor $\hom_{\mathcal{B}}(\ ,X)$ restricted to $\mathcal{A}$ is finitely generated as a functor on $\mathcal{A}$,

(ii) covariantly finite in $\mathcal{B}$ if for each object Y in $\mathcal{B}$, the representable functor $\hom_{\mathcal{B}}(Y,\ )$ restricted to $\mathcal{A}$ is finitely generated, and

(iii) functorially finite if $\mathcal{A}$ is both contravariantly and covariantly finite in $\mathcal{B}$.

For an artin algebra $\wedge$, an interesting class of subcategories of mod $\wedge$ is the subcategory $\mathcal{P}^i(\wedge)$ which consists of all $\wedge$–modules of projective dimension $\leq i$ for $i \geq 0$, as well as the subcategory $\mathcal{P}^\infty(\wedge)$ consisting of all $\wedge$–modules of finite projective dimension. The contravariant and covariant finiteness of $\mathcal{P}^i(\wedge)$ and $\mathcal{P}^\infty(\wedge)$ is studied by many authors; see, for example, [1], [5], and [6].

The aim of this paper is to present a condition which is sufficient for the subcategory $\mathcal{P}^i(\wedge)$ to be contravariantly finite in mod $\wedge$. As an application of this condition we show in section 2 that $\mathcal{P}^i(\wedge)$ is contravariantly finite in mod $\wedge$ for each i when $\wedge$ is stably equivalent to a hereditary algebra. The main idea of the

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proofs is from [1]. Note that in [1] Auslander and Reiten gave a sufficient condition for $\mathcal{P}^\infty(\wedge)$ to be contravariantly finite in mod $\wedge$ and proved that $\mathcal{P}^\infty(\wedge)$ is contravariantly finite in mod $\wedge$ when $\wedge$ is stably equivalent to a hereditary algebra.

The terminology used throughout this article is taken from [1].

1. A sufficient condition for $\mathcal{P}^i(\wedge)$ to be contravariantly finite

The proofs in this section are analogous to those in sect.4 in [1], but for the completeness of the article, we give here the proofs. Before presenting the sufficient condition, we first give the following more general result.

Proposition 1.1. *Suppose that $\underline{a}$ is an ideal in $\wedge$ with $\mathrm{pd}_\wedge \wedge/\underline{a} \le i$ such that if M is a $\wedge$–module with $\mathrm{pd}_\wedge M \le i$, then $M/\underline{a}M$ is a projective $\wedge/\underline{a}$–module. Let C be a $\wedge/\underline{a}$–module. Then we have the following.*

- (a) *A map $B \to C$ in mod $\wedge/\underline{a}$ is a right $\mathcal{P}^0(\wedge/\underline{a})$–approximation of C if and only if it is a right $\mathcal{P}^i(\wedge)$–approximation of C.*
- (b) *If $A \to C$ is a right $\mathcal{P}^i(\wedge)$–approximation of C, then $A/\underline{a}A \to C$ is a right $\mathcal{P}^0(\wedge/\underline{a})$–approximation of C.*

Proof. (a) Suppose $f : B \to C$ is a right $\mathcal{P}^0(\wedge/\underline{a})$–approximation of C. Since B is a projective $\wedge/\underline{a}$–module and $\mathrm{pd}_\wedge \wedge/\underline{a} \le i$, as a $\wedge$–module B is in $\mathcal{P}^i(\wedge)$. Let $g : X \to C$ be a morphism in mod $\wedge$ with X in $\mathcal{P}^i(\wedge)$. Then g is the composition of the canonical projection $\pi : X \to X/\underline{a}X$ and the induced map $g_1 : X/\underline{a}X \to C$. Since $X/\underline{a}X$ is a projective $\wedge/\underline{a}$–module, the morphism g_1 can be lifted to B, so g can be lifted to B, that is, $f : B \to C$ is a right $\mathcal{P}^i(\wedge)$–approximation of C.

Conversely, suppose that a morphism $f : B \to C$ in mod $\wedge/\underline{a}$ is a right $\mathcal{P}^i(\wedge)$–approximation of C. Since $\mathrm{pd}_\wedge B \le i$, as a $\wedge/\underline{a}$–module $B = B/\underline{a}B$ is projective. Hence $f : B \to C$ is a right $\mathcal{P}^0(\wedge/\underline{a})$–approximation of C.

(b) This is trivial.

Corollary 1.2. *Let $\underline{a}$ be an ideal in $\wedge$ satisfying the hypothesis of Proposition* 1.1. *If $\underline{a}S = 0$ for each simple $\wedge$–module S with $\mathrm{pd}_\wedge S > i$, then $\mathcal{P}^i(\wedge)$ is contravariantly finite in* mod $\wedge$.

Proof. First note that $\mathcal{P}^i(\wedge)$ is a resolving subcategory (i.e. $\mathcal{P}^i(\wedge)$ satisfying the following three conditions: (a) closed under extension, (b) closed under kernels of surjections, and (c) contains all projective $\wedge$–modules). Then by [1, Proposition 3.7], $\mathcal{P}^i(\wedge)$ is contravariantly finite in mod $\wedge$ if and only if each simple $\wedge$–module has a right $\mathcal{P}^i(\wedge)$–approximation. Suppose S is a simple $\wedge$–module. If $\mathrm{pd}_\wedge S \le i$, we are done. Suppose now $\mathrm{pd}_\wedge S > i$; then S is a $\wedge/\underline{a}$–module. Hence there is a right $\mathcal{P}^0(\wedge/\underline{a})$–approximation $A \to S$ of S since $\mathcal{P}^0(\wedge/\underline{a})$ is contravariantly finite in mod $\wedge/\underline{a}$. By (a) of Proposition 1.1, $A \to S$ is also a right $\mathcal{P}^i(\wedge)$–approximation of S. Therefore, $\mathcal{P}^i(\wedge)$ is contravariantly finite.

For an artin algebra $\wedge$, we denote by $\Omega(\text{mod } \wedge)$ the subcategory consisting of the syzygy modules $\Omega(C)$ of all C in mod $\wedge$. Further, we denote by $\tau_{\mathcal{A}}(M)$ the trace of a category $\mathcal{A}$ in the module M, that is, the submodule of M generated by the images of all maps $A \to M$ with A in $\mathcal{A}$. Finally, for each $i \ge 0$, we denote by $\underline{a}_i$ the trace of $\Omega(\text{mod } \wedge) \cap \mathcal{P}^i(\wedge)$ in $\underline{r}$, i.e. $\underline{a}_i = \tau_{\Omega(\text{mod } \wedge)\cap\mathcal{P}^i(\wedge)}(\underline{r})$, where $\underline{r}$ denotes the radical of $\wedge$. It is obvious that $\underline{a}_i \subset \underline{r}$ is an ideal in $\wedge$.

Proposition 1.3. *If* $\mathrm{pd}_\wedge \underline{a}_i \leq i$, *then* $\mathcal{P}^{i+1}(\wedge)$ *is contravariantly finite in* mod $\wedge$.

Proof. We first observe that $\underline{a}_i P = \tau_{\Omega(\mathrm{mod}\ \wedge)\cap\mathcal{P}^i(\wedge)}(\underline{r}P)$ if P is a projective module, where $\underline{r}P$ is the radical of P.

Let M be a $\wedge$–module with projective dimension $\leq i+1$. We then consider the following exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

with f a projective cover of M. Since $\mathrm{pd}_\wedge \Omega(M) \leq i$, one gets that $\Omega(M)$ is in $\Omega(\mathrm{mod}\ \wedge) \cap \mathcal{P}^i(\wedge)$. Therefore, it holds that $\Omega(M) \subset \tau_{\Omega(\mathrm{mod}\ \wedge)\cap\mathcal{P}^i(\wedge)}(\underline{r}P) = \underline{a}_i P$ and that $P/\underline{a}_i P \cong M/\underline{a}_i M$, that is, $M/\underline{a}_i M$ is a projective $\wedge/\underline{a}_i$–module. The condition $\mathrm{pd}_\wedge \underline{a}_i \leq i$ implies that $\mathrm{pd}_\wedge \wedge/\underline{a}_i \leq i+1$. Because each simple $\wedge$–module is annihilated by $\underline{a}_i$, one has by Corollary 1.2 that $\mathcal{P}^{i+1}(\wedge)$ is contravariantly finite.

Corollary 1.4. *Suppose* $\mathrm{pd}_\wedge \underline{a}_i \leq i$. *Then the* $\wedge$*–modules of projective dimension* $\leq i+1$ *are the summands of modules* M *which have filtrations* $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ *such that each subquotient* M_l/M_{l+1} *is an indecomposable projective* $\wedge/\underline{a}_i$*–module.*

Proof. Note that the minimal right $\mathcal{P}^0(\wedge/\underline{a})$–approximation of a $\wedge/\underline{a}$–module is its projective cover. It then follows that the $\wedge/\underline{a}$–projective covers of the simple $\wedge$–modules are just the minimal right $\mathcal{P}^i(\wedge)$–approximations of the simple $\wedge$–modules. The corollary then follows directly from [1, Proposition 3.7].

Combining the above results with the results in [1] and [3], we get the following corollary.

Corollary 1.5. *Suppose* $\underline{a}_0$ *is projective. Then* $\mathcal{P}^1(\wedge)$ *has almost split sequences.*

Proof. From [1] one knows that $\mathcal{P}^1(\wedge)$ is covariantly finite in mod $\wedge$ for all artin algebras. By Proposition 1.3, one has that $\mathcal{P}^1(\wedge)$ is also contravariantly finite in mod $\wedge$, so $\mathcal{P}^1(\wedge)$ is functorially finite. Then from [3, Theorem 2.4] it follows that $\mathcal{P}^1(\wedge)$ has almost split sequences.

As seen the example in [5], one knows that $\mathcal{P}^i(\wedge)$ is not always contravariantly finite. In the following we construct an algebra with finite global dimension $n \geq 2$ which satisfies that $\mathcal{P}^i(\wedge)$ are not contravariantly finite for $1 \leq i \leq n-1$.

Example. Let k be an algebraically closed field. For each $n \geq 2$, let $\wedge_n$ be given as the path algebra of the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{n-2}} n \xrightarrow{\gamma_{n-1}} n+1$$

modulo the ideal generated by the paths $\gamma_1\alpha$ and $\gamma_{i+1}\gamma_i$, $1 \leq i \leq n-2$. It is easy to calculate that $\wedge_n$ has global dimension n and that the simple module S_1 corresponding to the vertex 1 has projective dimension n. On one hand, the module S_1 is a preinjective module (see [2]) and each indecomposable preinjective module has projective dimension n. On the other hand, there is a family of indecomposable $\wedge$–modules $\{Y_\lambda | 0 \neq \lambda \in k\}$ in $\mathcal{P}^1(\wedge)$ of dimension 2 with support containing vertices 1 and 2 only (i.e. the Y_λ can be considered as modules over double arrows; see [4, sect. 8]). Moreover, it holds that $\hom_\wedge(Y_\lambda, Y_\mu) = 0$ for $\lambda \neq \mu$ and there is a nonzero morphism from each Y_λ to S_1. Suppose $f : X \to S_1$ is a right $\mathcal{P}^1(\wedge)$–approximation. One can easily show that each Y_λ should appear as a summand of X, but this is impossible. Hence S_1 has no right $\mathcal{P}^1(\wedge)$–approximation, that is, $\mathcal{P}^1(\wedge)$ is not contravariantly finite in mod $\wedge$. Similarly, one can prove that $\mathcal{P}^i(\wedge)$ is not contravariantly finite in mod $\wedge$ for $2 \leq i \leq n-1$.

An easy observation shows that $\mathrm{pd}_\wedge \underline{a}_i = n-1$ for $0 \leq i \leq n-2$.

2. An application

As an application of the sufficient condition given in section 1, in this section we prove that $\mathcal{P}^i(\wedge)$ is contravariantly finite in mod $\wedge$ when $\wedge$ is stably equivalent to a hereditary algebra. The proof is based on the following two lemmas.

Lemma 2.1 [1, Proposition 4.12]**.** *For an artin algebra, the following are equivalent.*

(a) $\wedge$ *satisfies the conditions*:

  (i) *A simple module S is a torsion module* (*i.e.* $S^* = \hom_\wedge(S, \wedge) = 0$) *if it is a composition factor of $\underline{r}P/\operatorname{soc} P$ for some indecomposable projective module P.*

  (ii) *Every indecomposable torsionless module* (*i.e. a submodule of a free* $\wedge$–*module*) *is simple or projective.*

(b) $\wedge$ *is stably equivalent to a hereditary algebra.*

Lemma 2.2. *Suppose that $\wedge$ is stably equivalent to a hereditary algebra. Then it holds that* $\mathrm{pd}_\wedge \underline{a}_i \leq i$ *for each $i \geq 0$, where* $\underline{a}_i = \tau_{\Omega(\mathrm{mod}\ \wedge)\cap\mathcal{P}^i(\wedge)}(\underline{r})$.

Proof. By Lemma 2.1 one gets that

$$\underline{a}_i = P \oplus S_1 \oplus \cdots \oplus S_t,$$

where P is projective and the S_j are simple modules.

Suppose now that $\mathrm{pd}_\wedge \underline{a}_i > i$. Then there is a simple module $S := S_j$ satisfying that $\mathrm{pd}_\wedge S > i$. By the construction of $\underline{a}_i$, there is an epimorphism $f : L \to S$ with L indecomposable in $\Omega(\mathrm{mod}\ \wedge) \cap \mathcal{P}^i(\wedge)$. We claim that $f(\mathrm{soc}(L)) = 0$. Otherwise, f would be a splittable epimorphism, so $\mathrm{pd}_\wedge S \leq i$, but this is impossible. Thus S is a composition factor of $L/\mathrm{soc}L$. Since L is in $\Omega(\mathrm{mod}\ \wedge)$, it follows that $L \subset \underline{r}Q$ for some projective module Q. Then $L/\mathrm{soc}L \subset \underline{r}Q/\mathrm{soc}Q$; that is, S is a composition factor of $\underline{r}Q/\mathrm{soc}Q$. This implies by Lemma 2.1 (ii) that S is a torsion module. This contradicts that $S \subset \underline{a}_i$. Therefore, it holds that $\mathrm{pd}_\wedge \underline{a}_i \leq i$.

Proposition 2.3. *Suppose that $\wedge$ is stably equivalent to a hereditary algebra. Then $\mathcal{P}^i(\wedge)$ is contravariantly finite in* mod $\wedge$ *for each $i \geq 0$.*

Proof. By Lemma 2.2 one knows that $\mathrm{pd}_\wedge \underline{a}_i \leq i$ for each i. Then by applying Proposition 1.3 one gets that $\mathcal{P}^{i+1}(\wedge)$ is contravariantly finite in mod $\wedge$. The proposition then follows from the fact that the subcategory $\mathcal{P}^0(\wedge)$ is always contravariantly finite in mod $\wedge$.

Remark. In [1] it is proved that $\mathcal{P}^\infty(\wedge)$ is contravariantly finite in mod $\wedge$ when $\wedge$ is stably equivalent to a hereditary algebra. Combining this result with Proposition 2.3, one can see that if $\wedge$ is stably equivalent to a hereditary algebra, then $\mathcal{P}^i(\wedge)$ is contravariantly finite in mod $\wedge$ for all $i \in \mathbb{N} \cup \{\infty\}$.

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Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China

E-mail address: dengbm@bnu.ihep.ac.cn