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S^1-QUOTIENTS OF QUATERNION-KÄHLER MANIFOLDS

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ABSTRACT. The notion of symplectic reduction has been generalized to manifolds endowed with other structures, in particular to quaternion-Kähler manifolds, namely Riemannian manifolds with holonomy in $Sp(n)Sp(1)$. In this work we prove that the only complete quaternion-Kähler manifold with positive scalar curvature obtainable as a quaternion-Kähler quotient by a circle action is the complex Grassmannian $Gr_2(\mathbb{C}^n)$.

0. INTRODUCTION

A Riemannian manifold can be endowed with various kinds of geometric structures which are reflected in a corresponding reduction of the holonomy group. A quaternion-Kähler manifold is a connected oriented Riemannian manifold with holonomy in $Sp(n)Sp(1)$, $n \geq 2$, a group which appears in the list, given by Berger's classification theorem [4], of possible holonomy groups of non-symmetric, non-reducible Riemannian manifolds .

Quaternion-Kähler geometry therefore arises naturally from the classification of possible holonomies. The first significant fact is that quaternion-Kähler manifolds are always Einstein. The sign of the scalar curvature s provides a coarse classification of this class of manifolds: when $s = 0$ the holonomy reduces to $Sp(n)$ and we are in the case of locally hyperkähler manifolds; the case of scalar curvature strictly different from zero is the one that fully characterizes quaternion-Kähler geometry, and by quaternion-Kähler we shall intend from now on that $s \neq 0$. This implies in particular that the manifolds under consideration are Einstein but not Ricci flat.

Examples with positive scalar curvature are given by the compact symmetric spaces classified by Wolf [27] and Alekseevskii [1], each corresponding to a compact simple Lie group:

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}, Gr_2(\mathbb{C}^n) = \frac{SU(n)}{S(U(n-2)\times U(2))}, \widetilde{Gr}_4(\mathbb{R}^n) = \frac{SO(n)}{SO(n-4)\times SO(4)},$$

plus the exceptional cases. Their non-compact duals provide symmetric examples of quaternion-Kähler manifolds with negative scalar curvature. Many other examples, non-symmetric, complete and non-compact exist for $s < 0$ [2, 6, 8, 17], whereas for $s > 0$ the only known complete (and thus, by Myers' theorem, compact) examples are the Wolf spaces. The problem of the existence of complete, non-symmetric positive examples still remains unsolved.

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Twistorial methods apply to the study of quaternion-Kähler manifolds, translating quaternion-Kähler geometry into holomorphic geometry. Our aim is to explore a "symplectic approach" to the study of quaternion-Kähler geometry, investigating the extent to which techniques introduced in symplectic geometry, starting from the notion of moment map and symplectic reduction, can contribute to our knowledge of quaternion-Kähler manifolds.

The first step in this direction is the quotient construction introduced by Galicki and Lawson in [7, 9]. When a Lie group G acts on a quaternion-Kähler manifold preserving its structure, a moment map with respect to the G-action is constructed. It is a uniquely defined section of a certain bundle over M, and its construction differs from the Kähler and hyperkähler ones, which are directly derived from the symplectic moment map construction [20, 11]. However a process of reduction is defined also in this case: if G acts freely on the inverse image of the zero section, then the quotient manifold inherits a quaternion-Kähler structure from the original one. Only two of the Wolf spaces are known to be global quaternion-Kähler quotients; they are the complex and real Grassmannians, quotients of the quaternionic projective space by the natural actions of $U(1)$ and $Sp(1)$ respectively, both diagonally immersed in $Sp(n+1)$ [7].

In this work, we specialize to circle actions and prove the following

0.1. Theorem. *The complex Grassmannian $Gr_2(\mathbb{C}^n)$ is the only positive complete quaternion-Kähler manifold that can be obtained as a quaternion-Kähler quotient by a circle action.*

The proof of the theorem brings together various aspects of quaternion-Kähler theory:

a) properties of the curvature tensor, which are consequences of the holonomy reduction to $Sp(n)Sp(1)$ (see [1, 2] and [5]);

b) deeper topological constraints which hold for compact quaternionic-Kähler manifolds with positive scalar curvature (see [16]);

c) the geometry of $\mu^{-1}(0)$ as a fibre bundle over the quotient, and its relationship with self-duality (see [21]).

1. Preliminaries

Notation. We shall use throughout the notation of [23]. The complexified cotangent space at a point $x \in M$ will be

$$T_x^* M_{\mathbb{C}} = E \otimes_{\mathbb{C}} H$$

and we often identify it with the tangent space $T_x M_{\mathbb{C}}$ using the metric. The inclusion of the holonomy Lie algebra $\mathfrak{sp}(n) + \mathfrak{sp}(1)$ in $\mathfrak{so}(4n)$ corresponds to the immersion of the bundle $S^2E \oplus S^2H$ into the bundle of 2-forms $\Lambda^2 T^* M_{\mathbb{C}}$ via

$$S^2H \oplus S^2E \hookrightarrow (S^2H \otimes \omega_H) \oplus (S^2E \otimes \omega_E) \hookrightarrow \Lambda^2(T^*M_{\mathbb{C}}) \tag{1.1}$$

where the skew-forms ω_H and ω_E are induced from the quaternionic structures of H and E respectively.

By duality, S^2E and S^2H can also be regarded as subbundles of the endomorphism bundle $\mathrm{End}(TM)$. Sections of S^2E and S^2H will often be treated both as 2-forms and endomorphisms of the (co-)tangent bundle. For example, if ϕ is a section of S^2E, we may write $\phi(Y,Z) = g(\phi(Y), Z)$, where g is the Riemannian metric and Y, Z are tangent vectors.

We recall that the bundle S^2H has local bases consisting of three almost complex structures I_1, I_2, I_3, behaving like quaternions. We shall call these bases *quaternionic bases.*

a) *Curvature properties.* As recalled in the introduction, a first remarkable consequence of the holonomy being in $Sp(n)Sp(1)$ is that a quaternion-Kähler manifold is Einstein. Moreover the curvature tensor R has the characteristic structure

$$R = tR_0 + R_1, \tag{1.2}$$

where t is a fixed positive multiple of the scalar curvature s, R_0 is the curvature tensor of the projective quaternionic space $\mathbb{H}P^n$, and R_1 is a section of S^4E [1, 23]. The decomposition (1.2) translates into very useful and explicit formulae on the curvature [5]. We shall need the following one: let g be the Riemannian metric on M and R the curvature operator of the Levi-Civita connection. Fix a quaternionic basis; then, if Y is orthogonal to $\langle X, I_1X, I_2X, I_3X\rangle$, we have

$$\frac{s}{4n(n+2)}\|X\|^2\|Y\|^2 = \sum_{k=0}^{3} g(R(X, I_kY)X, I_kY) \tag{1.3}$$

where I_0 =Identity.

b) *Topological aspects.* We already observed that there are no known examples of compact, non-symmetric, positive quaternion-Kähler manifolds. There are several results on the geometry and topology of these manifolds supporting the conjecture that no non-symmetric examples exist [16, 18] as happens in the 8-dimensional case [22]. In particular we need the following theorem.

1.1. Theorem (LeBrun [16])**.** *Let M be a complete quaternion-Kähler manifold with positive scalar curvature. If $b_2(M) > 0$, then M is homothetic to $Gr_2(\mathbb{C}^n)$ with its symmetric space metric.*

c) *Self-duality.* The decomposition (1.1) allows one to generalize the notion of self-duality to quaternion-Kähler manifolds: a 2-form is self-dual if it is a section of S^2E [24]. The manifold $\mu^{-1}(0)$ is a self-dual principal bundle over the quotient [21, 10]; this follows naturally in the course of the proof of Theorem 0.1 for the case of circle actions (see (2.9)) and it is an essential point in our proof.

Quaternion-Kähler moment map: a brief review. Since we are aiming to study circle actions, it is convenient to present the quaternion-Kähler moment map and quotient only for that case. In this way definitions will be simplified and tailored to our purposes.

Assume that an isometric $U(1)$-action is defined on the quaternion-Kähler manifold M. We shall call

i) X the vector field generated by $1 \in \mathbb{R} \cong \mathfrak{u}(1)$;
ii) $\overline{X} = g(X, \cdot)$ the 1-form dual to X with respect to the Riemannian metric;
iii) ∇ the Levi-Civita connection on M.

Remark. A connected compact group of isometries of M automatically preserves its quaternion-Kähler structure, because by Kostant [15], since M is irreducible, the covariant derivative ∇ of any Killing field lies in the holonomy algebra $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$.

In general, overlined vector fields will indicate the corresponding dual forms. Let I_1, I_2, I_3 be a local quaternionic basis. By [23, Lemma 6.4,6.5] the globally defined equation

$$\nabla\mu = \sum_{i=1}^{3} \overline{I_i X} \otimes I_i, \quad \mu \in \Gamma(S^2 H), \tag{1.4}$$

admits a unique solution, given explicitly by

$$\mu = \tfrac{1}{t}(\nabla\overline{X})^{S^2H} \in \Gamma(S^2H) \tag{1.5}$$

[9, 26]. Equation (1.4) is the so-called twistor equation and it can be found in this form in [12].

The section $\mu \in \Gamma(S^2H)$ is the *moment map* with respect to the circle action, as defined by Galicki in [7]. It is not necessary to postulate that this is equivariant: unlike for the symplectic moment map, the equivariance of μ is guaranteed by the uniqueness of the solution of the differential equation.

The next theorem generalizes to quaternion-Kähler manifolds the notion of symplectic reduction introduced by Marsden-Weinstein.

1.2. Theorem (Galicki-Lawson [9]). *Let M be a quaternion-Kähler manifold acted on isometrically by $U(1)$, and let $\mu^{-1}(0)$ be the inverse image of the zero section. If $U(1)$ acts freely on $\mu^{-1}(0)$, then the quotient manifold $\mu^{-1}(0)/U(1)$ inherits a quaternion-Kähler structure from that of M.*

Remark. The moment map μ on M can be characterized, as in the symplectic case, in terms of forms [25]: let Ω be the 4-form defined on M by $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$, where $\omega_1, \omega_2, \omega_3$ are the locally defined 2-forms corresponding to a quaternionic basis I_1, I_2, I_3. Ω is globally defined and the stabilizer of Ω in $SO(4n)$ is exactly $Sp(n)Sp(1), \quad n \geq 2$. Therefore $M^{4n}, \quad n \geq 2$, is quaternion-Kähler if and only if $\nabla\Omega = 0$. It follows from (1.4) that the moment map satisfies the equation $d\mu = i_X\Omega$.

Before moving to the next section it is convenient to mention explicitly a basic case whose detailed study underlies the whole work.

Example. It is well known that the complex Grassmannian $Gr_2(\mathbb{C}^{n+1})$ of 2-dimensional subspaces in $\mathbb{C}^{n+1}$, i.e. the Wolf space $SU(n+1)/S(U(n-1)\times U(2))$, can be obtained as a quaternion-Kähler quotient by a circle action on $\mathbb{H}P^n$. The latter is given by the diagonal immersion of $U(1)$ in $Sp(n+1)$, which acts naturally on the quaternionic projective space. Let S be the subset of $\mathbb{H}^{n+1}$ given by the points $p = a + bj$ such that $a, b \in \mathbb{C}^{n+1}$ are linearly dependent. In addition to the description arising from Theorem 1.2 it can be proved that:

a) the subset $\underline{S} = S/\mathbb{H}^*$ of $\mathbb{H}P^n$ is the fixed point set of the $U(1)$-action on $\mathbb{H}P^n$ and it is isomorphic to $\mathbb{C}P^n$;

b) there is an action of $GL(2,\mathbb{C})$ on $\mathbb{H}^{n+1}$ and the quotient $(\mathbb{H}^{n+1} \setminus S)/GL(2,\mathbb{C})$ is $Gr_2(\mathbb{C}^{n+1})$;

c) S^1 acts freely on the open subset $\mathbb{H}P^n \backslash \underline{S}$ of $\mathbb{H}P^n$. By b) the former is a fibre bundle over $Gr_2(\mathbb{C}^{n+1})$ and its fibres are quaternionic submanifolds of $\mathbb{H}P^n \backslash \underline{S}$ isomorphic to $\mathbb{H}P^1 \backslash \mathbb{C}P^1$.

2. Main result

In this section we shall prove Theorem 0.1, restated as follows.

2.1. Theorem. *Let M^{4n} be a quaternion-Kähler manifold with positive scalar curvature, endowed with an isometric circle action. Suppose that the inverse image of the zero section by the moment map μ is compact and that the circle action is free on it. Then the quotient $\mu^{-1}(0)/S^1$ is homothetic to the complex Grassmannian $Gr_2(\mathbb{C}^n)$ of 2-dimensional subspaces in $\mathbb{C}^n$.*

Proof. As a first step we need to analyse closely the 2-form $d\overline{X}$. Let M^{S^1} be the fixed point set of the $U(1)$-action. On $M \setminus M^{S^1}$ we define

$$\alpha = \frac{\overline{X}}{\|X\|^2}, \tag{2.1}$$

Since $\overline{X}$ is a Killing 1-form, we can essentially identify $\nabla\overline{X}$ with $d\overline{X}$ [14, Prop. 3.2, Chap. VI]; moreover, since M has non-zero scalar curvature, M is irreducible and has non-zero Ricci tensor. By [19] we have

$$\nabla\overline{X} \in \mathfrak{sp}(1) + \mathfrak{sp}(n) \cong S^2H + S^2E \subset \Lambda^2T^*M; \tag{2.2}$$

thus, according to (2.2) and (1.5)

$$d\overline{X} = \nabla\overline{X} = t\,\mu + \sigma \tag{2.3}$$

splits into two components. □

We consider $\mu^{-1}(0)$ as a principal fibre bundle over the quotient $\hat{M}$. The Riemannian metric on M determines a natural connection, the horizontal space at each point being the orthogonal complement to the one-dimensional space tangent to the $U(1)$-orbit.

The tangent space T_xM at a point $x \in M \setminus M^{S^1}$ decomposes orthogonally as

$$T_xM = \langle X\rangle \oplus \langle IX, JX, KX\rangle \oplus \mathcal{H},$$

where $\mathcal{H}$ is the orthogonal complement of the quaternionic span of X. In particular, if $x \in \mu^{-1}(0)$, by (1.4) we obtain $T_x\mu^{-1}(0) = \langle IX, JX, KX\rangle^{\perp}$, and the horizontal and vertical spaces of the $U(1)$-connection defined on the principal bundle $\mu^{-1}(0)$ are $\mathcal{H}$ and $\langle X\rangle$ respectively.

Let $i : \mu^{-1}(0) \hookrightarrow M$ be the immersion; the 1-form $i^*\alpha$ is the connection form of the connection on $\mu^{-1}(0)$ defined above. If

$$\beta = d\alpha,$$

then the corresponding curvature form is given by $d(i^*\alpha) = i^*\beta$.

From (2.1) and (2.3) we obtain, at every point $x \in \mu^{-1}(0)$,

$$\sigma = \|X\|^2\beta + d(\|X\|^2) \wedge \alpha. \tag{2.4}$$

To gain information on the behaviour of σ at least at a chosen point, consider on M the real function $\|X\|^2$. Its exterior derivative is given by

$$d(\|X\|^2) = 2g(\nabla X, X) = -2g(\nabla_X X, \cdot),$$

and on $\mu^{-1}(0)$ becomes

$$d(\|X\|^2) = -2\sigma(X,\,\cdot\,). \tag{2.5}$$

Moreover, as X is a Killing field, $\|X\|$ is constant on the $U(1)$-orbits, in particular on the fibres of the $U(1)$-bundle $\mu^{-1}(0)$; hence it defines a real function over the quotient $\mu^{-1}(0)/U(1)$. Let $x_0 \in \mu^{-1}(0)$ be in a critical orbit; then (2.5) gives

$$\sigma_{x_0}(X, Z) = 0 \quad \text{for any } Z \in \mathcal{H}_{x_0}. \tag{2.6}$$

A stronger property can be obtained using the fact that $\sigma \in \Gamma(S^2E)$: with x_0 and Z as before, $\sigma_{x_0}(I_iX, Z) = \langle \sigma_{x_0}(I_iX), Z\rangle$, $\quad i = 1, 2, 3$. But $I_k \in \Gamma(S^2H)$ and σ commute, therefore $\sigma_{x_0}(I_iX, Z) = -\langle \sigma_{x_0}(X), I_iZ\rangle = 0$, $\quad i = 1, 2, 3$, by (2.6). This, with (2.6), implies

$$(\nabla_Z X)_{x_0} = \sigma_{x_0}(Z) \in \mathcal{H}_{x_0} \quad \text{for any } Z \in \mathcal{H}_{x_0}. \tag{2.7}$$

In order to make use of curvature property (1.3), we need to prove the following

2.2. Lemma. *For any $Z \in \mathcal{H}_{x_0}$ the Hessian of the function $\|X\|^2_{|_{\mu^{-1}(0)}}$ satisfies the identity*

$$\begin{aligned} \tfrac{1}{2} Hess(\|X\|^2)_{x_0}(Z, Z) &= \langle R(X, Z)X, Z\rangle_{x_0} \\ &\quad + \|\sigma_{x_0}(Z)\|^2 - \tfrac{1}{\|X\|^2} \textstyle\sum_{i=1}^3 \sigma_{x_0}(X, I_iX)\sigma_{x_0}(Z, I_iZ). \end{aligned}$$

Proof. Let Z be a vector field on $\mu^{-1}(0)$ extending the given $Z \in \mathcal{H}_{x_0}$ in such a way that Z is the local horizontal lift of a vector field on the quotient. Recall that the bundle S^2H is preserved by ∇, thus $\nabla I_i = \sum_{j=1}^3 \gamma_{ij} I_j$, with γ_{ij} locally defined 1-forms. Then on $\mu^{-1}(0)$

$$\begin{aligned} \langle \nabla_Z Z, I_iX\rangle &= Z\langle Z, I_iX\rangle - \langle Z, \nabla_Z(I_iX)\rangle \\ &= 0 - \langle Z, \nabla_Z(I_i)X\rangle + \langle I_iZ, \nabla_Z X\rangle \\ &= -\langle Z, \textstyle\sum_1^3 \gamma_{ij}(Z)I_jX\rangle + \sigma(Z, I_iZ) \\ &= \sigma(Z, I_iZ) \end{aligned} \tag{2.8}$$

This implies also $\langle \nabla_Z Z, X\rangle = 0$.

Consider now the Hessian

$$\begin{aligned} \mathrm{Hess}(\|X\|^2)_{x_0}(Z, Z) &= ZZ(\|X\|^2) \\ &= -2Z\langle \nabla_X X, Z\rangle \\ &= -2Z(\sigma(X, Z)) \\ &= -2[(\nabla_Z\sigma)_{x_0}(X, Z) + \sigma_{x_0}(\nabla_Z X, Z) + \sigma_{x_0}(X, \nabla_Z Z)]; \end{aligned}$$

by (2.6) the third term of the summand is equal to $\sum_{i=1}^3 \frac{\langle \nabla_Z Z, I_iX\rangle}{\|X\|^2}\sigma(X, I_iX)$ which, by (2.8), is $\frac{1}{\|X\|^2}\sum_{i=1}^3 \sigma(Z, I_iZ)\sigma(X, I_iX)$.

The second term is $\langle \sigma(\sigma(Z)), Z\rangle = -\|\sigma(Z)\|^2$, and finally the first term is equal to $-2(\nabla_Z\nabla X)(X, Z) = 2\langle R(X, Z)X, Z\rangle$ by [14, Prop. 2.5, Chap. VI]. □

Since α is zero on horizontal vectors, (2.4) implies that the curvature of the $U(1)$-bundle $\mu^{-1}(0)$, viewed as a global form on the base manifold $\hat{M}$, is the closed 2-form $\hat{\sigma}/\|X\|^2$, where $\hat{\sigma}$ is the form induced by σ on the quotient $\hat{M}$. Theorem 1.2 gives

$$\frac{\hat{\sigma}}{\|X\|^2} \in \Gamma(S^2\hat{E}), \tag{2.9}$$

since $\sigma \in \Gamma(S^2E)$; here $S^2\hat{E}$ is the $\mathfrak{sp}(n)$-bundle on $\hat{M}$. This implies that $\hat{\sigma}/\|X\|^2$ is not only closed, but also co-closed. In order to prove that, recall that $S^2\hat{E}$ is associated to an irreducible representation of $Sp(n-1)$ and this gives $*(\hat{\sigma}/\|X\|^2) =$

$c\,\hat{\Omega}^{n-2} \wedge (\hat{\sigma}/\|X\|^2)$ for some constant c, where $\hat{\Omega}$ is the invariant $Sp(n-1)Sp(1)$ 4-form on the $(4n-4)$-dimensional quotient. Hence $\hat{\sigma}/\|X\|^2$ is harmonic on $\hat{M}$.

Suppose now that the quotient manifold $\hat{M}$ is not homothetic to the Grassmannian; then by Theorem 1.1 its second Betti number must be zero:

$$b_2(\hat{M}) = 0.$$

We want to show that this is in contradiction with $\hat{M}$ being a quaternion-Kähler quotient.

Since $b_2(\hat{M}) = 0$, we have $\hat{\sigma} \equiv 0$ on $\hat{M}$. Therefore at points in $\mu^{-1}(0)$

$$\sigma(Z_1, Z_2) = 0 = \beta(Z_1, Z_2), \quad \text{for all } Z_1, Z_2 \in \mathcal{H}.$$

This, together with (2.7), implies $\sigma_{x_0}(Z) = 0$. And the identity in Lemma 2.2 becomes

$$\tfrac{1}{2}\mathrm{Hess}(\|X\|^2)_{x_0}(Z, Z) = \langle R(X, Z)X, Z\rangle_{x_0}.$$

Let $Z_0 \in \mathcal{H}_{x_0}$ with $\|Z_0\| = 1$; applying (1.3) we have

$$\frac{s}{4n(n+2)}\|X\|^2 = \frac{1}{2}\sum_{k=0}^{3} \mathrm{Hess}(\|X\|^2)_{x_0}(I_k Z_0, I_k Z_0).$$

In order to get a contradiction recall that by hypothesis $\hat{M}$ is compact; we can therefore choose x_0 to be in the $U(1)$-orbit on which $\|X\|_{|_{\mu^{-1}(0)}}$ reaches its maximum value. This concludes the proof as $s > 0$. □

Concluding remarks. A further step in the application of symplectic techniques to the case of quaternion-Kähler manifolds is the study of $f = \|\mu\|^2$ as an equivariant Morse function, as suggested by analogy with Kirwan's work on symplectic and algebraic manifolds [13]. Some results analogous to the symplectic case can be obtained in this framework and aspects of the general situation mimic the basic example reported in §1.

One goal for this theory is to obtain some positive results in the study of quaternion-Kähler manifolds with positive scalar curvature. This has been the essential motivation for the present work, and will be developed by the author in a subsequent paper.

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