

BOUNDED HARMONIC MAPS ON A CLASS OF MANIFOLDS

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ABSTRACT. Without imposing any curvature assumptions, we show that bounded harmonic maps with image contained in a regular geodesic ball share similar behaviour at infinity with the bounded harmonic functions on the domain manifold.

§0

In this note, among other things we will prove a Liouville type theorem on harmonic maps. It was first proved by Yau [Y] that any positive harmonic function on a complete noncompact manifold with nonnegative Ricci curvature must be constant. Later, Cheng [Cg] proved that any harmonic map with bounded image from a complete noncompact manifold with nonnegative Ricci curvature into a Cartan-Hadamard manifold must be a constant map. Using a similar method, Choi [Ch] was able to generalize Cheng's result. He showed that any harmonic map from a complete noncompact manifold with nonnegative Ricci curvature into a complete manifold with sectional curvature bounded from above by $K > 0$ is a constant map, provided that the image of the harmonic map lies inside a regular geodesic ball (see Definition 1.3). It was proved by Kendall [Ke] that the result is still true by only assuming the domain manifold supports no nonconstant bounded harmonic functions. If we relax the condition that the manifold has nonnegative Ricci curvature outside a compact set, then the theorem of Yau [Y] is no longer true. In fact, the behavior of bounded and positive harmonic functions on a complete noncompact manifold with nonnegative sectional curvature outside a compact set has been studied thoroughly in [L-T 1]. One of the results in [L-T 1] is that if M has nonnegative sectional curvature outside a compact set, then M has finitely many ends (see §1), and a bounded harmonic function defined near the infinity on an end will be asymptotically constant. There are many kinds of manifolds which satisfy the same property. See the examples in §1. One of the main results we obtain in this note is that if every bounded harmonic function defined near infinity of an end of a manifold M with respect to some compact set is asymptotically constant, then every bounded harmonic map from M into a regular ball of another manifold is also asymptotically constant at the infinity of each end. There is no curvature assumption on M . Moreover, such a map will have finite total energy. We will also

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discuss some properties of bounded harmonic maps from such a manifold into a complete noncompact manifold with nonnegative sectional curvature. We will also prove some existence and uniqueness results on harmonic maps. All manifolds in this paper are assumed to be connected.

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§1

Let us begin with some definitions.

Definition 1.1. An end E of a manifold M is an unbounded component of the complement of some compact, smooth domain D of M . In this case, we say that E is an end corresponding to D .

For each D , $M \setminus \overline{D}$ has only finitely many unbounded components since M is a connected manifold. Also the boundary of each component is smooth; in particular, each end has smooth boundary. It is also clear that if $D_1 \subset D_2$ are compact, smooth domains of M , then the number of ends corresponding to D_1 is less than or equal to the number of ends corresponding to D_2 . Hence, we say that M has finitely many ends if there exists $b < \infty$ such that the number of ends corresponding to D is less than or equal to b for any compact, smooth domain $D \subset M$. One also observes that in this case there exist an integer $b_0 < \infty$ and a compact, smooth domain $D_0 \subset M$ such that the number of ends corresponding to D is b_0 for each compact, smooth domain D containing D_0 . We say that b_0 is the number of ends of M . In the following, when M has finitely many ends and we say E is an end of M , it is understood that E is an end corresponding to D_0 . If there is no such b_0 , then we say M has infinitely many ends.

Definition 1.2. A Riemannian manifold M with boundary ∂M (∂M may be empty) is said to be parabolic if it does not admit any positive Green's function satisfying Neumann boundary condition on ∂M . Otherwise, it is said to be non-parabolic. In particular, an end E of M is parabolic if it does not admit any positive Green's function satisfying the Neumann boundary condition on ∂E , and is nonparabolic otherwise.

We want to prove the following lemma:

Lemma 1.1. *Let M be a complete noncompact manifold without boundary. Let D be a bounded domain in M with smooth boundary, and let E be an end of M with respect to D . Suppose every bounded harmonic function defined at the infinity of E is asymptotically constant near infinity. Let g_1, \dots, g_k be any finite set of bounded superharmonic functions defined on E , with $\liminf_{x \rightarrow \infty, x \in E} g_j(x) = 0$, for all $1 \leq j \leq k$, and let $g = \max\{g_1, \dots, g_k\}$. Then*

$$\liminf_{\substack{x \rightarrow \infty \\ x \in E}} g(x) = 0.$$

Proof. Since $\liminf_{x \rightarrow \infty, x \in E} g_j(x) = 0$ for all j , so $\liminf_{x \rightarrow \infty, x \in E} g(x) \geq 0$. Suppose the lemma is not true. Then there is $\epsilon > 0$ such that

$$(1.1) \quad \liminf_{\substack{x \rightarrow \infty \\ x \in E}} g(x) \geq 2\epsilon.$$

Let o be a fixed point in M . Then there is R_0 such that for all $1 \leq j \leq k$, for all $x \in E$ with $r(x) \geq R_0$, where $r(x)$ is the geodesic distance from x to o , we have

$$(1.2) \quad g_j(x) \geq -\frac{\epsilon}{2k},$$

and

$$(1.3) \quad g(x) \geq \epsilon.$$

By (1.2), (1.3) and the definition of g , if $x \in E$ with $r(x) \geq R_0$, we have

$$(1.4) \quad \begin{aligned} \sum_{j=1}^k g_j(x) &\geq \epsilon - (k-1) \cdot \frac{\epsilon}{2k} \\ &> \frac{\epsilon}{2}. \end{aligned}$$

For each $R > R_0$, and for each j , let $h_{R,j}$ be the harmonic function defined on $E \cap (B_o(R) \setminus B_o(R_0))$, such that $h_{R,j} = g_j$ on $E \cap \partial B_o(R)$ and $E \cap \partial B_o(R_0)$. Since g_j is bounded, there exists $R_i \rightarrow \infty$ such that $h_j = \lim_{i \rightarrow \infty} h_{R_i,j}$ exists, for $1 \leq j \leq k$. Since g_j is superharmonic and h_j is harmonic, $h_j \leq g_j$. In particular

$$\begin{aligned} \liminf_{\substack{x \rightarrow \infty \\ x \in E}} h_j(x) &\leq \liminf_{\substack{x \rightarrow \infty \\ x \in E}} g_j(x) \\ &= 0. \end{aligned}$$

By the assumptions of the lemma, h_j is asymptotically constant at infinity of E , so

$$(1.5) \quad \lim_{\substack{x \rightarrow \infty \\ x \in E}} h_j(x) = a_j$$

for some nonpositive constant a_j . However, by (1.4) and the definition of $h_{R,j}$,

$$\begin{aligned} \sum_{j=1}^k h_{R_i,j} &= \sum_{j=1}^k g_j \\ &\geq \frac{\epsilon}{2} \end{aligned}$$

on $E \cap \partial B_o(R)$ and $E \cap \partial B_o(R_0)$, for all i . Hence $\sum_{j=1}^k h_{R_i,j} \geq \epsilon/2$ for all i . Let $i \rightarrow \infty$; then we have $\sum_{j=1}^k h_j \geq \epsilon/2$. This contradicts (1.5). The proof of the lemma is completed.

Using a similar argument, one can obtain the following:

Lemma 1.2. *Let M be a complete noncompact manifold without boundary, such that M supports no nonconstant bounded harmonic functions. Let g_1, \dots, g_k be a finite set of bounded superharmonic functions such that $\inf_M g_j = 0$ for all $1 \leq j \leq k$. Let $g = \max\{g_1, \dots, g_k\}$; then $\inf_M g = 0$.*

Lemma 1.2 is basically due to Kendall [Ke], but the method is different. In Lemma 1.2, the assumption is weaker in the sense that there may be bounded harmonic functions defined at the infinity of M which is not asymptotically constant at infinity. However, we have the following:

Proposition 1.3. *Let M be a complete noncompact manifold without boundary, and let D be a bounded domain of $M \setminus \overline{D}$. Suppose there are at least two nonparabolic ends of M with respect to D . Let E be one of the nonparabolic ends. Then every bounded harmonic function on M is asymptotically constant at the infinity of E if and only if every bounded harmonic function defined near the infinity of E is asymptotically constant at the infinity of E .*

Proof. Suppose every bounded harmonic function on M is asymptotically constant at infinity of E . Since M has at least two nonparabolic ends, by [L-T 4], there is a nonconstant bounded harmonic function f on M such that $0 < f < 1$ and $\limsup_{x \rightarrow \infty, x \in E} f(x) = 1$. By the assumption, we have $\lim_{x \rightarrow \infty, x \in E} f(x) = 1$. Hence $1 - f$ is a positive harmonic function on M such that $(1 - f) \rightarrow 0$ at the infinity of E . Let g be a harmonic function defined near the infinity of E . We can extend g to be a bounded function which is harmonic outside a compact set. For $R > 0$, let h_R be the harmonic function defined on $B_o(R)$, such that $h_R = g$ on $\partial B_o(R)$, where o is a fixed point. There exists $R_i \rightarrow \infty$ such that h_{R_i} converges to a bounded harmonic function h on M . Since h_{R_i} are uniformly bounded on compact sets, $g - h_{R_i}$ is harmonic in $B_o(R_i) \setminus B_o(R_0)$ for some R_0 , and $g - h_{R_i} = 0$ on $\partial B_o(R)$, hence there is a constant C independent of i such that $C(1 - f) \geq |g - h_{R_i}|$ for i large enough. Hence $C(1 - f) \geq |g - h|$. Since $(1 - f) \rightarrow 0$ at the infinity of E and h is asymptotically constant at the infinity of E , so g is also asymptotically constant at the infinity of E . The converse is obvious.

Using the above results, we generalize Liouville theorems on harmonic maps of Cheng [Cg], Choi [Ch] and Kendall [Ke]. Let us first introduce the following definition.

Definition 1.3. Let N be a complete manifold with sectional curvature bounded from above by $k^2 \geq 0$. Let $B_p(R)$ be a geodesic ball in N of radius R and with center p , and denote by $C(p)$ the cut locus of its center. We call the ball $B_p(R)$ a regular ball in N if $kR < \pi/2$ and $C(p) \cap B_p(R)$ is empty.

Clearly, if N is simply connected with nonpositive sectional curvature, then each geodesic ball in N is regular by the Hadamard's theorem.

Theorem 1.4. *Let M be a complete noncompact manifold and N be a complete manifold with sectional curvature bounded above by $k^2 > 0$.*

- (a) *Suppose M has no nonconstant bounded harmonic function; then every harmonic map from M into a regular ball of N must be constant.*
- (b) *Let D be a bounded domain in M . Suppose E is an end of M such that every bounded harmonic function defined near the infinity of E is asymptotically constant. Then every harmonic map from E into a regular geodesic ball of N is asymptotically constant at infinity of E .*

Proof. (a) was proved in [Ke] by using Lemma 1.2. To prove (b), let u be a harmonic map from E into a regular geodesic ball $B_p(R)$ of N . Let $K = \overline{u(E)}$. Then by [J-K, A-C-M, Ke], for all $q \in K$ the function

$$\Psi_q(s) = \frac{1 - \cos(kd_N(q, s))}{k \cos(kd_N(p, s))}$$

is convex in $B_p(R)$. Therefore $\Psi_q(u(x))$ is a bounded subharmonic function in E by [Go]. Let $c_q = \limsup_{x \rightarrow \infty, x \in E} \Psi_q(u(x))$ and let $g_q(x) = c_q - \Psi_q(u(x))$. So g_q

is bounded and superharmonic, and $\liminf_{x \rightarrow \infty, x \in E} g_q(x) = 0$. By Lemma 1.1, it is easy to see that for all $\epsilon > 0$ and for any finitely many points q_1, \dots, q_k ,

$$(1.6) \quad \bigcap_{i=1}^k \{x \in E \mid g_{q_i}(x) \leq \epsilon\} \neq \emptyset.$$

Consider the family \mathcal{F} of subsets $A(p, \epsilon)$ of K , for $q \in K$ and $\epsilon > 0$, where

$$A(q, \epsilon) = \{s \in K \mid c_q - \Psi_q(s) \leq \epsilon\}.$$

By (1.6) and the definitions of K and g_q , we see that the intersection of finitely many $A(p, \epsilon)$ is nonempty. By the compactness of K , there is $q_0 \in A(q, \epsilon)$ for all $q \in K$ and for all $\epsilon > 0$. In particular, we have $c_{q_0} - \Psi_{q_0}(q_0) \leq 0$. Hence $c_{q_0} \leq 0$. But c_{q_0} is nonnegative, so $c_{q_0} = 0$. By the definition of c_{q_0} , we see that $\limsup_{x \in E, x \rightarrow \infty} \Psi_{q_0}(u(x)) = 0$. Hence $\lim_{x \in E, x \rightarrow \infty} u(x) = q_0$. This completes the proof of the theorem.

By Proposition 1.3, we have the following:

Corollary 1.5. *Let M be a complete noncompact manifold and let D be a bounded domain in M . Suppose any bounded harmonic function on M is asymptotically constant at each nonparabolic end. Then any harmonic map from M into a regular geodesic ball of a manifold N is also asymptotically constant at each nonparabolic end.*

Note that if M has only one nonparabolic end, and M satisfies the condition of the corollary, then M has no nonconstant bounded harmonic function. See [L-T 2].

The method of proof in Theorem 1.4 can be applied to harmonic maps into a complete manifold with nonnegative sectional curvature. Let N be a complete noncompact manifold with nonnegative sectional curvature. Let $p \in N$, and let γ be a ray emanating from p parametrized by arclength. Let β_γ be the Busemann function of γ . That is, $\beta_\gamma(x) = \lim_{t \rightarrow \infty} (t - d(x, \gamma(t)))$. Let $\beta_p = \sup_\gamma \beta_\gamma$, where the supremum is taken on the set of all rays γ from p . It is well known that β_p is a convex function which is Lipschitz with Lipschitz constant equal to 1; see [C-G]. Moreover, $\beta_p(p) = 0$.

Proposition 1.6. *Let M be a complete noncompact manifold and let D be a bounded domain. Assume that every bounded harmonic function defined on an end with respect to D is asymptotically constant. Let u be a bounded harmonic map from M into a complete noncompact manifold with nonnegative sectional curvature. Then for each end E there is a point $p \in N$ such that $\limsup_{x \rightarrow \infty, x \in E} \beta_p(u(x)) = 0$.*

Proof. The proof is similar to the proof of Theorem 1.4, by considering $\beta_p \circ u$. We omit the details.

Examples. (1) Let M be a complete noncompact manifold with asymptotically nonnegative sectional curvature. Namely, there is a monotonic nonincreasing continuous function $k : [0, \infty) \rightarrow [0, \infty)$, such that $\int_0^\infty tk(t)dt < \infty$, and the sectional curvature of M at x satisfies $K_M(x) \geq -k(r(x))$ for all x , where $r(x)$ is the distance of x from a fixed point o . Then M has finitely many ends and each end satisfies the condition in Theorem 1.4 (b). See [L-T 1] and [K 1, K 2]. This is also true for \tilde{M} which is quasi-isometric to M . See [S].

(2) Let M be a complete noncompact manifold which is quasi-isometric to a complete manifold with nonnegative Ricci curvature outside a compact set; then M has finitely many ends. If in addition, M has finite first Betti number, then each end satisfies the condition in Theorem 1.4 (b). See [L-T 5], [Cai], and [Liu]. We may relax the condition on the Ricci curvature, provided the manifold satisfies certain conditions on volume comparison. If the condition on the first Betti number is removed, then in some cases, M still satisfies the condition in Proposition 1.3. See [L-T 5] for more details.

§2

In this section, we will prove an existence theorem on harmonic maps and will give more information on bounded harmonic maps considered in §1.

Theorem 2.1. *Let E_1, \dots, E_l be the nonparabolic ends of M with respect to some compact set. Suppose that every bounded harmonic function on M is asymptotically constant at infinity of each E_i . Let $B_p(R)$ be a regular geodesic ball in a manifold N . Then for any p_1, \dots, p_l in $B_p(R)$ there exists a harmonic map $u : M \rightarrow N$ with finite total energy such that $u(M) \subset B_p(R)$ and $\lim_{x \rightarrow \infty, x \in E_i} u(x) = p_i$ for all $i = 1, \dots, l$.*

Proof. The theorem can be proved as in [A-C-M] using the result of [H-K-W]. To be more precise, let n be the dimension of N and let (y_1, \dots, y_n) be the normal coordinates of $B_p(R)$ with center at p . In this coordinate system, $p_i = (a_{1i}, \dots, a_{ni})$, $1 \leq i \leq l$. As in the proof of Proposition 1.3, by [L-T 4], and using the assumptions, for each i there is a bounded harmonic function f_i such that $\lim_{x \rightarrow \infty, x \in E_i} f_i = 1$ and $\lim_{x \rightarrow \infty, x \in E_j} f_i = 0$ for $j \neq i$. Let $h = (h_1, \dots, h_n)$ be the vector-valued harmonic function with $h_k = \sum_{i=1}^l a_{ki} f_i$, and let v be the harmonic function such that $v = \sum_{i=1}^l (\sum_{k=1}^n (a_{ki})^2) f_i$. As in [A-C-M], one can find a harmonic map u from M into $B_p(R)$ and a constant C such that

$$d_N(u, h) \leq C \left(v - \sum_{k=1}^n (h_k)^2 \right).$$

By the definition of h , it is easy to see that $\lim_{x \rightarrow \infty, x \in E_i} u(x) = p_i$ for all $i = 1, \dots, l$. Moreover, h has finite Dirichlet integral and hence has finite total energy; see for example [L-T 5]. By the construction of u , it is easy to see u has finite total energy.

We also have the following uniqueness result.

Proposition 2.2. *Let u, v be two harmonic maps from M into N with sectional curvature bounded from above by $k^2 > 0$, such that their images are contained in a regular ball $B_p(R)$ of N . Suppose that on each nonparabolic end E of M ,*

$$\lim_{x \rightarrow \infty, x \in E} u(x) = \lim_{x \rightarrow \infty, x \in E} v(x).$$

Then $u \equiv v$ on M .

Proof. According to [Ke] there exists a continuous nonnegative bounded convex function Ψ defined on $B_o(R) \times B_o(R)$ and $\Psi(x, y) = 0$ if and only if $x = y$. Consider

$f(x) = \Psi(u(x), v(x))$. Since both u and v are harmonic maps and Ψ is convex, we conclude $\Delta f \geq 0$ on M . Clearly, f is bounded. Hence, f attains its maximum at the infinity of M , that is, at the infinity of some end. But the maximum cannot be attained at the infinity of a parabolic end e unless it is a constant. Otherwise $\sup_M f - f$ will be a positive superharmonic function on e attaining its infimum at the infinity of e , which implies that the end e is nonparabolic. At the infinity of each nonparabolic end, from our assumption, $d_N(u(x), v(x)) \rightarrow 0$ as $x \rightarrow \infty$. Thus, $\max_M f = 0$ and $f \equiv 0$ as $f \geq 0$. Therefore, $u \equiv v$.

Corollary 2.3. *Let M be a complete manifold, and let D be a bounded domain in M . Assume every bounded harmonic function on M is asymptotically constant near infinity of each nonparabolic end. Then every bounded harmonic map from M into a regular geodesic ball of another manifold N has finite total energy.*

Proof. The corollary follows from Corollary 1.5, Theorem 2.1 and Proposition 2.2.

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