

## DIFFERENCES OF VECTOR-VALUED FUNCTIONS ON TOPOLOGICAL GROUPS

BOLIS BASIT AND A. J. PRYDE

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let  $G$  be a locally compact group equipped with right Haar measure. The right differences  $\Delta_h\varphi$  of functions  $\varphi$  on  $G$  are defined by  $\Delta_h\varphi(t) = \varphi(th) - \varphi(t)$  for  $h, t \in G$ . Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h\varphi \in L^p(G)$  for some  $1 \leq p < \infty$  and all  $h \in G$ . We prove that  $\|\Delta_h\varphi\|_p$  is a right uniformly continuous function of  $h$ . If  $G$  is abelian and the Beurling spectrum  $sp(\varphi)$  does not contain the unit of the dual group  $\hat{G}$ , then we show  $\varphi \in L^p(G)$ . These results have analogues for functions  $\varphi : G \rightarrow X$ , where  $X$  is a separable or reflexive Banach space. Finally, we apply our methods to vector-valued right uniformly continuous differences and to absolutely continuous elements of left Banach  $G$ -modules.

### §1. INTRODUCTION

Let  $\xi \in L^p(\mathbb{R})$  for some  $1 \leq p \leq \infty$ . Consider the indefinite integral  $\varphi(t) = P\xi(t) = \int_0^t \xi(x)dx$ . Now  $\Delta_h\varphi(t) = \int_t^{t+h} \xi(x)dx = \chi_h * \varphi(t)$  where  $\chi_h$  is the characteristic function of  $[-h, 0]$ . It follows that  $\Delta_h\varphi \in L^p(\mathbb{R})$  and moreover that  $\varphi$  is continuous. We seek conditions under which there exists a constant function  $c$  such that  $\varphi + c \in L^p(\mathbb{R})$ . In short we write  $\varphi \in L^p(\mathbb{R}) + \mathbb{C}$ .

More generally, let  $\varphi \in L^\infty(G)$  where  $G$  is a locally compact group equipped with right Haar measure and suppose  $\Delta_h\varphi \in L^p(G)$  for some  $1 \leq p < \infty$  and all  $h \in G$ . What additional conditions ensure  $\varphi \in L^p(G)$ ?

To answer this question, we study the function  $\psi(h) = \Delta_h\varphi$  and develop a new method for investigating difference problems.

Firstly, let  $X$  be a Banach space. The *right* and *left differences* of a function  $\varphi : G \rightarrow X$  are defined by  $\Delta_h\varphi(t) = \varphi(th) - \varphi(t)$  and  $\Delta^h\varphi(t) = \varphi(ht) - \varphi(t)$  respectively. Let  $e$  be the unit in  $G$ . We say that  $\varphi$  is *right uniformly continuous* if  $\lim_{v \rightarrow e} \sup_{t \in G} \|\Delta_v\varphi(t)\| = 0$ , and let  $C_{rub}(G, X)$  be the space of all right uniformly continuous bounded functions  $\varphi : G \rightarrow X$ . For functions  $f, g : G \rightarrow \mathbb{C}$  we will use the *involution* given by  $f^*(t) = f(t^{-1})$  and the *right convolution*  $f * g(t) = \int_G f(th^{-1})g(h)dh$ . The space of compactly supported continuous functions  $\varphi : G \rightarrow X$  will be denoted by  $C_c(G, X)$  or  $C_c(G)$  if  $X = \mathbb{C}$ .

In section 2 we prove that the function  $\psi$  defined above is right uniformly continuous. This allows us in section 3 to construct a continuous weight function  $w$  on

---

Received by the editors September 21, 1994 and, in revised form, January 4, 1995.

1991 *Mathematics Subject Classification*. Primary 43A15; Secondary 28B05, 39A05.

*Key words and phrases*. Differences, weight functions, spectrum, right uniform continuity,  $G$ -modules, weak continuity, absolutely continuous elements.

$G$  which dominates  $\psi$ . The corresponding Beurling algebra  $L_w^1(G)$  is a Wiener algebra (see [10, pages 22, 83, 142]). Under the assumption that  $G$  is abelian and the spectrum  $sp(\varphi)$  does not contain the unit  $\hat{e}$  of the dual group  $\widehat{G}$ , we use a Bochner-Haar integral (see [11, page 132]) to show that  $\varphi \in L^p(G)$ . For the definition of spectrum see (3.1) below ([10, page 139] and [2]). As a consequence, we show that if  $\xi \in L^p(\mathbb{R})$  for some  $1 \leq p \leq \infty$  and if  $0 \notin sp(\xi)$ , then there exists a constant function  $c$  such that  $P\xi + c \in L^p(\mathbb{R})$ . We also show that these results remain valid for  $X$ -valued functions where  $X$  is a separable or reflexive Banach space.

In section 4 we use some of these techniques to prove that vector-valued bounded functions with right uniformly continuous right differences are right uniformly continuous. The abelian case was obtained in [4] and [6]. As a consequence, we obtain in section 5 a characterization of absolutely continuous elements of left Banach  $G$ -modules.

## §2. TECHNICAL LEMMAS

**Lemma 2.1.** *Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h\varphi \in L^p(G)$  for some  $1 \leq p \leq \infty$  and all  $h \in G$ . Then the function  $\psi : G \rightarrow L^p(G)$ ,  $\psi(h) = \Delta_h\varphi$ , is right uniformly continuous if and only if it is continuous at one point  $h_0 \in G$ .*

*Proof.* For arbitrary  $h, v \in G$  we have  $\|\psi(hv) - \psi(h)\|_p = \|\psi(v) - \psi(e)\|_p = \|\psi(h_0v) - \psi(h_0)\|_p$  and the lemma follows.

**Lemma 2.2.** *Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h\varphi \in L^p(G)$  for some  $1 < p < \infty$  and all  $h \in G$ . Let  $g \in L^q(G)$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then the function  $\psi_g : G \rightarrow \mathbb{C}$ ,  $\psi_g(h) = \int_G \Delta_h\varphi(t)g(t)dt$ , is continuous.*

*Proof.* Firstly let  $g \in C_c(G)$ . Then for  $h, v \in G$  we have

$$\psi_g^*(h) = \int_G \varphi(t)\Delta_h g(t) dt \quad \text{and} \quad \Delta_v\psi_g^*(h) = \int_G \varphi(th^{-1})\Delta_v g(t) dt.$$

Hence  $\psi_g^*$  is right uniformly continuous. In particular,  $\psi_g$  is continuous.

Secondly, take an arbitrary  $g \in L^q(G)$ . There exists a sequence  $\{g_n\}$  in  $C_c(G)$  converging to  $g$  in the  $L^q$ -norm. This implies  $|\psi_{g_n}(h) - \psi_g(h)| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $h \in G$ . By the Baire category theorem [11, page 12],  $\psi_g$  is continuous on a set  $D$  of the second category. Since  $G$  is locally compact,  $D \neq \emptyset$ . Now we show that continuity of  $\psi_g$  at one point  $h_0$  implies its continuity on  $G$ . Indeed, note that for  $h, k \in G$  we have  $\Delta_k\psi_g(h) = \psi_g(hk) - \psi_g(h) = \int_G [\varphi(thk) - \varphi(th)]g(t)dt = (\Delta_k\varphi)^* * g(h^{-1})$ . By [7, 20.32 (e)],  $\Delta_k\psi_g \in C_0(G)$ . From the identity  $\Delta^v\psi_g(h) = \Delta^v\psi_g(h_0) + \Delta^v\Delta_{h_0^{-1}h}\psi_g(h_0)$ , the continuity of  $\psi_g$  at  $h_0$  and the continuity of  $\Delta_{h_0^{-1}h}\psi_g$  at  $h_0$  we conclude that  $\psi_g$  is continuous.

**Theorem 2.3.** *Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h\varphi \in L^p(G)$  for some  $1 < p < \infty$  and all  $h \in G$ . Then  $\psi : G \rightarrow L^p(G)$ ,  $\psi(h) = \Delta_h\varphi$ , is right uniformly continuous.*

*Proof.* By Lemma 2.2,  $\psi$  is weakly continuous. That is,  $\langle \psi(h), g \rangle = \psi_g(h)$  is a continuous function of  $h$  for all  $g \in L^q(G)$ . By a Theorem of Namioka [9, Theorem 4.1],  $\psi$  is continuous on a dense  $G_\delta$  subset of  $G$ . By Lemma 2.1,  $\psi$  is right uniformly continuous.

We need the following proposition.

**Proposition 2.4.** *Let  $X$  be a Banach space. Let  $\varphi : G \rightarrow X$  be bounded on an open subset  $U$  of  $G$ . Suppose  $\Delta_h\varphi$  is continuous for each  $h \in G$ . Then  $\varphi$  is continuous.*

*Proof.* For  $g \in X^*$ , the dual of  $X$ , set  $\varphi_g = g \circ \varphi$ . Then  $\varphi_g$  is bounded on  $U$  and the differences  $\Delta_h\varphi_g = g \circ \Delta_h\varphi$  are all continuous. By [1, Theorem 2.1]  $\varphi_g$  is continuous at each  $h_o \in U$ . From the identity  $\Delta^v\varphi_g(h) = \Delta^v\Delta_{h_o^{-1}h}\varphi_g(h_o) + \Delta^v\varphi_g(h_o)$  we conclude that  $\varphi_g$  is continuous on  $G$ . By [9, Theorem 4.1],  $\varphi$  is continuous on a dense  $G_\delta$  subset of  $G$ . The identity  $\Delta^v\varphi(h) = \Delta^v\Delta_{h_1^{-1}h}\varphi(h_1) + \Delta^v\varphi(h_1)$  shows that  $\varphi$  is continuous on  $G$ .

**Corollary 2.5.** *Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h\varphi \in L^1(G)$  for all  $h \in G$ . Then  $\psi : G \rightarrow L^1(G)$ ,  $\psi(h) = \Delta_h\varphi$ , is right uniformly continuous.*

*Proof.* Since  $\Delta_h\varphi \in L^1(G) \cap L^\infty(G)$ , we conclude  $\Delta_h\varphi \in L^p(G)$  for all  $1 \leq p \leq \infty$ . By Theorem 2.3,  $\|\Delta_h\varphi\|_{1+\frac{1}{n}}$  is a continuous function of  $h$  for each  $n \in \mathbb{N}$ , the natural numbers. Moreover,  $\lim_{n \rightarrow \infty} \|\Delta_h\varphi\|_{1+\frac{1}{n}} = \|\Delta_h\varphi\|_1$ . By the Baire category theorem,  $\|\Delta_h\varphi\|_1$  is a continuous function of  $h$  except on a subset of  $G$  of the first category. So it is continuous at some  $h_o \in G$ . Hence there exists a neighbourhood  $V$  of the unit  $e$  in  $G$  such that  $\|\psi(h_o v)\|_1 = \|\Delta_{h_o v}\varphi\|_1 \leq 1 + \|\Delta_{h_o}\varphi\|_1$  for all  $v \in V$ . Consider the differences  $\Delta_k\psi$  for  $k \in G$ . We have  $\|\Delta_v\Delta_k\psi(h)\|_1 = \|\Delta_v\Delta_k\varphi\|_1 \rightarrow 0$  as  $v \rightarrow e$ , by [7, Theorem 20.4], since  $\Delta_k\varphi \in L^1(G)$  for each  $k \in G$ . Hence  $\Delta_k\psi : G \rightarrow L^1(G)$  is continuous. By Proposition 2.4,  $\psi$  is continuous, and by Lemma 2.1,  $\psi$  is right uniformly continuous.

*Remark 2.6.* Proposition 2.4 also holds true for the more general case of  $\sigma$ -well  $\alpha$ -favorable topological groups as defined in [3]. In this case we use [3, Theorem 1] instead of [9, Theorem 4.1].

*Remark 2.7.* Let  $X$  be a Banach space and  $1 \leq p \leq \infty$ . Then  $L^p(G, X)$  denotes the Banach space of strongly measurable functions  $\varphi : G \rightarrow X$  for which  $\|\varphi(\cdot)\|_X \in L^p(G)$ . If  $1 \leq p < \infty$ , then  $C_c(G, X)$  is dense in  $L^p(G, X)$ . Moreover, if  $1 < p < \infty$  and  $X$  is separable or reflexive, then the dual of  $L^p(G, X)$  is  $L^q(G, X^*)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . For this, see [5, 8.20.3 and 8.20.5]. It follows that the results of this section remain valid with  $L^p(G)$  replaced by  $L^p(G, X)$  for  $1 \leq p \leq \infty$  whenever  $X$  is a separable or reflexive Banach space.

*Remark 2.8.* If  $X$  is a Banach space not containing a subspace isomorphic to  $c_o$  (the Banach space of convergent to zero complex sequences), then  $L^1(G, X)$  is also a Banach space not containing a subspace isomorphic to  $c_o$  (see [8]). It follows that in the proof of Corollary 2.5 with  $X = \mathbb{C}$  we can avoid application of Proposition 2.4 and conclude instead from [1, Theorem 2.1] that  $\psi$  is continuous at some point  $h_o$  and hence is right uniformly continuous. This simplification is not available for Banach spaces  $X$  containing subspaces isomorphic to  $c_o$ .

### §3. BOUNDED FUNCTIONS WITH DIFFERENCES IN $L^p(G)$

In this section  $G$  is a locally compact abelian group. Let  $\varphi \in L^p(G)$  for some  $1 \leq p \leq \infty$ . Then [7, Corollary 20.14],  $f * \varphi \in L^p(G)$  for each  $f \in L^1(G)$ . It follows that

$$I(\varphi) = \{f \in L^1(G) : f * \varphi = 0\}$$

is a closed ideal of  $L^1(G)$ . We define

$$(3.1) \quad sp(\varphi) = \text{hull } I(\varphi) = \{\gamma \in \widehat{G} : \hat{f}(\gamma) = 0 \text{ for all } f \in I(\varphi)\}$$

where  $\widehat{G}$  is the dual group of  $G$  and  $\hat{f}$  is the Fourier transform of  $f$ . For the following, see [2, Proposition 1.1].

**Proposition 3.1.** *Let  $\varphi \in L^p(G)$  for some  $1 \leq p \leq \infty$ . Then*

- (i)  $sp(\varphi * f) \subset sp(\varphi) \cap \text{supp}(\hat{f})$  for all  $f \in L^1(G)$ ; and
- (ii)  $sp(\varphi) = \emptyset$  if and only if  $\varphi = 0$ .

Now let  $w$  be a weight function on  $G$  satisfying the Beurling-Domar condition. See [10, page 132]. Then the Beurling algebra  $L_w^1(G) = \{f \in L^1(G) : wf \in L^1(G)\}$  is a Wiener algebra [10, 6.3.1]. The dual of  $L_w^1(G)$  is  $L_w^\infty(G) = \{\varphi : \frac{\varphi}{w} \in L^\infty(G)\}$ . If  $\varphi \in L_w^\infty(G)$ , then its spectrum with respect to  $L_w^1(G)$  will be denoted by  $sp_w(\varphi) = \text{hull } I_w(\varphi)$ , where  $I_w(\varphi) = \{f \in L_w^1(G) : f * \varphi = 0\}$  (see [10, page 142]).

We show

**Theorem 3.2.** *Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h \varphi \in L^p(G)$  for some  $1 \leq p < \infty$  and all  $h \in G$ . If  $\hat{e} \notin sp(\varphi)$ , then  $\varphi \in L^p(G)$ .*

*Proof.* By Theorem 2.3 and Corollary 2.5,  $\psi : G \rightarrow L^p(G)$ ,  $\psi(h) = \Delta_h \varphi$ , is uniformly continuous. Define  $w : G \rightarrow \mathbb{R}$  by  $w(h) = 1 + \|\psi(h)\|_p + \|\psi(h^{-1})\|_p$ . Then

- (i)  $w(hk) \leq w(h)w(k)$  for all  $h, k \in G$ ;
- (ii)  $w$  is uniformly continuous; and
- (iii)  $w(h^n) \leq nw(h)$  for all  $h \in G, n \in \mathbb{N}$ .

It follows that  $w$  is a weight function on  $G$  satisfying the Beurling-Domar condition. Hence  $L_w^1(G)$  is a Wiener algebra. Since  $\hat{e} \notin sp(\varphi)$ , by [10, 2.1.3, Remark] there exist a neighbourhood  $V$  of  $\hat{e}$  and a function  $f \in L_w^1(G)$  such that  $\text{supp}(\hat{f}) \subset V$ ,  $\hat{f}(\hat{e}) = 1$  and  $V \cap sp(\varphi) = \emptyset$ . By Proposition 3.1,  $\varphi * f = 0$ . Hence  $\varphi(t) = \int_G [\varphi(t) - \varphi(ts^{-1})]f(s)ds = - \int_G \Delta_{s^{-1}}\varphi(t)f(s)ds$ . The integrand as a function of  $s$  from  $G$  to  $L^p(G)$  is weakly Borel measurable. Moreover, we claim that it is almost separably-valued with respect to Haar measure. Indeed  $f \in L^1(G)$ , so its (essential) support is  $\sigma$ -compact. The function  $\Delta_{s^{-1}}\varphi$  of  $s$  is continuous and, therefore, restricted to the support of  $f$  it has a range which is  $\sigma$ -compact in  $L^p(G)$  and hence separable. The claim follows. By Pettis's theorem [11, page 131] the integrand is strongly Borel measurable. Moreover, since  $\|\Delta_{s^{-1}}\varphi\|_p \leq w(s)$ , by Bochner's theorem [11, page 133] the Bochner-Haar integral  $\int_G \Delta_{s^{-1}}\varphi f(s)ds$  exists and belongs to  $L^p(G)$ . That is,  $\varphi \in L^p(G)$ .

Letting  $w$  denote the weight function defined above, one can show by the same method as used in the proof of Theorem 3.2.

**Theorem 3.3.** *Let  $\varphi \in L^\infty(G)$  and suppose  $1 \leq p < \infty$ . Then  $\varphi \in L^p(G)$  if and only if*

- (i)  $\Delta_h \varphi \in L^p(G)$  for all  $h \in G$ , and
- (ii) there exists  $f \in L_w^1(G), f \neq 0$ , such that  $f * \varphi \in L^p(G)$ .

Similarly, we have

**Theorem 3.4.** Let  $\varphi \in L^\infty(G)$  and suppose  $\Delta_h \varphi \in L^p(G)$  for some  $1 \leq p < \infty$  and all  $h \in G$ . If  $f \in L_w^1(G)$  and  $\hat{f}(\hat{e}) = 1$ , then  $\varphi - f * \varphi \in L^p(G)$ .

To study indefinite integrals, we use the weight

$$(3.2) \quad v(x) = 1 + |x|, \quad x \in \mathbb{R}.$$

It is readily seen that  $v$  is a symmetric weight function satisfying Beurling conditions (see [10, page 17]). It follows that  $L_v^1(\mathbb{R})$  is a Wiener algebra. We denote by  $C_u(\mathbb{R})$  ( $C_{ub}(\mathbb{R})$ ) the set of all complex-valued uniformly continuous (uniformly continuous bounded) functions defined on  $\mathbb{R}$ .

**Proposition 3.5.** If  $\xi \in L^p(\mathbb{R})$  where  $1 \leq p \leq \infty$  and  $v$  is given by (3.2), then  $P\xi \in C_u(\mathbb{R}) \cap L_w^\infty(\mathbb{R})$ .

*Proof.* If  $p = 1$ , it is well-known that  $P\xi$  is absolutely continuous and hence uniformly continuous. For arbitrary  $p$ , and  $x, h \in \mathbb{R}$ ,  $|P\xi(x+h) - P\xi(x)| = |\int_0^h \xi(x+t) dt| \leq |h|^{1-1/p} \|\xi\|_p$  showing that  $P\xi \in C_u(\mathbb{R})$ . Moreover,  $|P\xi(x)| = |\int_0^x \xi(t) dt| \leq |x|^{1-1/p} \|\xi\|_p$ , showing that  $P\xi \in L_w^\infty(\mathbb{R})$ .

**Theorem 3.6.** If  $\xi \in L^p(\mathbb{R})$  where  $1 \leq p \leq \infty$  and  $0 \notin sp(\xi)$ , then  $P\xi \in C_{ub}(\mathbb{R})$ . Moreover,  $P\xi \in L^p(\mathbb{R}) + \mathbb{C}$ .

*Proof.* Since  $0 \notin sp(\xi)$ , there exists a neighbourhood  $V = [-\delta, \delta]$  such that  $sp(\xi) \cap V = \emptyset$ . Since  $L_v^1(\mathbb{R})$  is a Wiener algebra, there is a function  $h \in L_v^1(\mathbb{R})$  such that  $\hat{h} = 1$  for  $|\lambda| \leq \delta/4$  and  $\hat{h} = 0$  for  $|\lambda| \geq \delta/3$ . By Proposition 3.1,  $h * \varphi = 0$ . Similarly, by [10, page 140-141] and [2, Proposition 1.1],  $sp_v(h * P\xi) \subset \text{supp}(\hat{h}) \cap sp_v(P\xi) \subset \{0\}$ . Since  $\frac{d(h * P\xi)}{dx} = h * \xi = 0$ , we conclude that  $h * P\xi = c$ , a constant. If  $\eta = P\xi - c$ , then  $0 \notin sp_w(\eta)$ . Indeed,  $h * \eta = h * P\xi - h * c = c - c = 0$ . Thus  $h \in I_v(\eta)$  and  $\hat{h}(0) = 1$ , showing  $0 \notin sp_w(\eta)$ . By Proposition 3.5,  $\eta \in C_u(\mathbb{R})$  and so by [2, Theorem 4.2],  $\eta$  is bounded and so is  $P\xi$ . This proves that  $P\xi \in C_{ub}(\mathbb{R})$ . The function  $\eta$  satisfies all the conditions of Theorem 3.2, therefore  $\eta \in L^p(\mathbb{R})$ . Hence  $P\xi \in L^p(\mathbb{R}) + \mathbb{C}$ .

*Remark 3.7.* The results of section 3 remain true for  $X$ -valued functions provided that  $X$  is separable or reflexive. The spectrum of  $\varphi \in L^p(G, X)$  is defined again by (3.1).

#### §4. RIGHT UNIFORMLY CONTINUOUS DIFFERENCES

In this section and the next, we again take a locally compact group  $G$  and a Banach space  $X$ . The following theorem, for the case of abelian groups, was proved by Datry and Muraz [4, Théorème 4, Corollaire]. Their proof was indirect, using deep results for Banach  $G$ -modules. We give a different direct proof, using the techniques of the previous sections, and then deduce results for  $G$ -modules in section 5.

**Theorem 4.1.** Let  $\varphi : G \rightarrow X$  be a bounded function and suppose  $\Delta_h \varphi \in C_{rub}(G, X)$  for all  $h \in G$ . Then  $\varphi \in C_{rub}(G, X)$ .

*Proof.* Define  $\psi : G \rightarrow C_{rub}(G, X)$  by  $\psi(h) = \Delta_h \varphi$ . Then for  $h, k \in G$  we have

$$\|\Delta_v \Delta_k \psi(h)\|_\infty = \|\Delta_v \Delta_k \varphi\|_\infty \rightarrow 0 \quad \text{as } v \rightarrow e.$$

So  $\Delta_k\psi : G \rightarrow C_{rub}(G, X)$  is continuous. By Proposition 2.4,  $\psi$  is continuous. Finally, continuity of  $\psi$  at  $e$  implies that  $\varphi$  is right uniformly continuous.

*Remark 4.2.* In view of Remark 2.6, Theorem 4.1 holds true for the more general case of  $\sigma$ -well  $\alpha$ -favorable topological groups.

### §5. APPLICATION TO LEFT $G$ -MODULES

A Banach space  $X$  together with a family of bounded linear operators  $A_h : X \rightarrow X$ , for  $h \in G$ , is called a *left Banach  $G$ -module* if

- (i)  $A_e(x) = x$  for all  $x \in X$ ;
- (ii)  $A_{hk}(x) = A_h(A_k(x))$  for all  $h, k \in G$  and all  $x \in X$ ;
- (iii)  $\|A_h(x)\| \leq \kappa\|x\|$  for all  $h \in G$ , all  $x \in X$ , and some  $\kappa > 0$ .

The space  $X_{abs} = \{x \in X : \|A_v x - x\| \rightarrow 0 \text{ as } v \rightarrow e\}$  is a closed submodule of  $X$ . Its elements are called *absolutely continuous*. See Datry and Muraz [4], where Theorem 5.2 below is obtained in the case  $G$  is abelian using a different proof.

For a fixed vector  $x \in X$  we study the function  $\psi : G \rightarrow X$  given by  $\psi(h) = A_h x$ . So  $x$  is absolutely continuous if and only if  $\psi$  is continuous at  $e$ . In fact the following is true.

**Theorem 5.1.** *For an element  $x$  of a left Banach  $G$ -module  $X$ , the following are equivalent.*

- (a)  $x \in X_{abs}$ ;
- (b)  $\psi$  is weakly continuous at some point in  $G$ ;
- (c)  $\psi$  is right uniformly continuous.

*Proof.* If  $x \in X_{abs}$ , then  $\psi$  is continuous, and therefore weakly continuous, at  $e$ . So (a) implies (b). Next suppose  $\psi$  is weakly continuous at  $h_0$ . From the identity  $\langle \Delta_v \psi(h), x^* \rangle = \langle \Delta_v \psi(h_0), A_{hh_0^{-1}}^*(x^*) \rangle$  for  $h, v \in G$  and  $x^* \in X^*$ , it follows that  $\psi$  is weakly continuous on  $G$ . By [9, Theorem 4.1]  $\psi$  is continuous at some point  $h_1$ . From the identity  $\Delta_v \psi(h) = A_{hh_1^{-1}}(\Delta_v \psi(h_1))$  it follows that  $\psi$  is right uniformly continuous. Hence (b) implies (c). That (c) implies (a) is obvious.

**Theorem 5.2.** *Let  $x$  be an element of a Banach  $G$ -module  $X$ . If  $A_h x - x \in X_{abs}$  for all  $h \in G$ , then  $x \in X_{abs}$ .*

*Proof.* For each  $k \in G$ ,  $\Delta_k \psi(h) = A_h(A_k x - x)$  which by Theorem 5.1 defines a right uniformly continuous function  $\Delta_k \psi : G \rightarrow X$ . As  $\psi$  is bounded, Theorem 4.1 shows that  $\psi$  is right uniformly continuous. Hence  $x \in X_{abs}$ .

*Remark 5.3.* In view of Remark 2.6, the results of this section hold true for the more general case of  $\sigma$ -well  $\alpha$ -favorable topological groups.

### REFERENCES

1. B. Basit and M. Emam, *Differences of functions in locally convex spaces and applications to almost periodic and almost automorphic functions*, Annales Polonici Math. **XLI** (1983), 193–201. MR **85d**:43005
2. B. Basit and A.J. Pryde, *Polynomials and functions with finite spectra on locally compact abelian groups*, Bull. Austral. Math. Soc. **51** (1994), 33–42. CMP 95:07
3. J.P.R. Christensen, *Joint continuity of separately continuous functions*, Proc. Amer. Math. Soc. **82** (1981), 455–461. MR **82h**:54012
4. C. Datry and G. Muraz, *Analyse harmonique dans les modules de Banach I: propriétés générales*, Bull. Science Mathématique **119** (1995), 299–337.

5. R.E. Edwards, *Functional Analysis—Theory and Applications*, Holt, Rinehart and Winston Inc., New York, 1965. MR **36**:4308
6. F. Galvin, G. Muraz et P. Szeptycki, *Fonction aux différence  $f(x) - f(a + x)$  continues*, C.R.Acad.Sci. Paris, série I **315** (1991), 397–400. MR **94b**:39035
7. E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis, Part I*, Springer-Verlag, 1979. MR **81k**:43001
8. S. Kwapien, *On Banach spaces containing  $c_0$* , Studia Math. **52** (1974), 187–188. MR **50**:8627
9. I. Namioka, *Separate continuity and joint continuity*, Pacific Journal of Math. **51** (1974), 515–531. MR **51**:6693
10. H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford Math. Monographs, Oxford Univ., 1968. MR **46**:5933
11. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1966. MR **50**:2851

DEPARTMENT OF MATHEMATICS, MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA  
E-mail address: [bbasit\(ajpryde\)vaxc.cc.monash.edu.au](mailto:bbasit(ajpryde)vaxc.cc.monash.edu.au)