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DIFFERENCES OF VECTOR-VALUED FUNCTIONS ON TOPOLOGICAL GROUPS

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ABSTRACT. Let G be a locally compact group equipped with right Haar measure. The right differences $\triangle_h\varphi$ of functions φ on G are defined by $\triangle_h\varphi(t) = \varphi(th) - \varphi(t)$ for $h, t \in G$. Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 \le p < \infty$ and all $h \in G$. We prove that $\|\triangle_h\varphi\|_p$ is a right uniformly continuous function of h. If G is abelian and the Beurling spectrum $sp(\varphi)$ does not contain the unit of the dual group $\hat{G}$, then we show $\varphi \in L^p(G)$. These results have analogues for functions $\varphi : G \to X$, where X is a separable or reflexive Banach space. Finally, we apply our methods to vector-valued right uniformly continuous differences and to absolutely continuous elements of left Banach G-modules.

§1. INTRODUCTION

Let $\xi \in L^p(\mathbb{R})$ for some $1 \le p \le \infty$. Consider the indefinite integral $\varphi(t) = P\xi(t) = \int_0^t \xi(x)dx$. Now $\triangle_h\varphi(t) = \int_t^{t+h} \xi(x)\,dx = \chi_h * \varphi(t)$ where χ_h is the characteristic function of $[-h, 0]$. It follows that $\triangle_h\varphi \in L^p(\mathbb{R})$ and moreover that φ is continuous. We seek conditions under which there exists a constant function c such that $\varphi + c \in L^p(\mathbb{R})$. In short we write $\varphi \in L^p(\mathbb{R}) + \mathbb{C}$.

More generally, let $\varphi \in L^\infty(G)$ where G is a locally compact group equipped with right Haar measure and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 \le p < \infty$ and all $h \in G$. What additional conditions ensure $\varphi \in L^p(G)$?

To answer this question, we study the function $\psi(h) = \triangle_h\varphi$ and develop a new method for investigating difference problems.

Firstly, let X be a Banach space. The *right* and *left differences* of a function $\varphi : G \to X$ are defined by $\Delta_h\varphi(t) = \varphi(th) - \varphi(t)$ and $\Delta^h\varphi(t) = \varphi(ht) - \varphi(t)$ respectively. Let e be the unit in G. We say that φ is *right uniformly continuous* if $\lim_{v\to e} \sup_{t\in G} ||\Delta_v\varphi(t)|| = 0$, and let $C_{rub}(G, X)$ be the space of all right uniformly continuous bounded functions $\varphi : G \to X$. For functions $f, g : G \to \mathbb{C}$ we will use the *involution* given by $f^*(t) = f(t^{-1})$ and the *right convolution* $f * g(t) = \int_G f(th^{-1})g(h)dh$. The space of compactly supported continuous functions $\varphi : G \to X$ will be denoted by $C_c(G, X)$ or $C_c(G)$ if $X = \mathbb{C}$.

In section 2 we prove that the function ψ defined above is right uniformly continuous. This allows us in section 3 to construct a continuous weight function w on

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G which dominates ψ. The corresponding Beurling algebra $L^1_w(G)$ is a Wiener algebra (see [10, pages 22, 83, 142]). Under the assumption that G is abelian and the spectrum $sp(\varphi)$ does not contain the unit $\hat{e}$ of the dual group $\widehat{G}$, we use a Bochner-Haar integral (see [11, page 132]) to show that $\varphi \in L^p(G)$. For the definition of spectrum see (3.1) below ([10, page 139] and [2]). As a consequence, we show that if $\xi \in L^p(\mathbb{R})$ for some $1 \le p \le \infty$ and if $0 \notin sp(\xi)$, then there exists a constant function c such that $P\xi + c \in L^p(\mathbb{R})$. We also show that these results remain valid for X-valued functions where X is a separable or reflexive Banach space.

In section 4 we use some of these techniques to prove that vector-valued bounded functions with right uniformly continuous right differences are right uniformly continuous. The abelian case was obtained in [4] and [6]. As a consequence, we obtain in section 5 a characterization of absolutely continuous elements of left Banach G-modules.

§2. Technical lemmas

Lemma 2.1. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 \le p \le \infty$ and all $h \in G$. Then the function $\psi : G \to L^p(G)$, $\psi(h) = \triangle_h\varphi$, is right uniformly continuous if and only if it is continuous at one point $h_0 \in G$.*

Proof. For arbitrary $h, v \in G$ we have $\|\psi(hv) - \psi(h)\|_p = \|\psi(v) - \psi(e)\|_p = \|\psi(h_0v) - \psi(h_0)\|_p$ and the lemma follows.

Lemma 2.2. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 < p < \infty$ and all $h \in G$. Let $g \in L^q(G)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then the function $\psi_g : G \to \mathbb{C}$, $\psi_g(h) = \int_G \triangle_h\varphi(t)g(t)dt$, is continuous.*

Proof. Firstly let $g \in C_c(G)$. Then for $h, v \in G$ we have

$$\psi^*_g(h) = \int_G \varphi(t)\triangle_h g(t)\, dt \quad \text{and} \quad \Delta_v\psi^*_g(h) = \int_G \varphi(th^{-1})\Delta_v g(t)\, dt.$$

Hence ψ^*_g is right uniformly continuous . In particular, ψ_g is continuous.

Secondly, take an arbitrary $g \in L^q(G)$. There exists a sequence $\{g_n\}$ in $C_c(G)$ converging to g in the L^q-norm. This implies $|\psi_{g_n}(h) - \psi_g(h)| \to 0$ as $n \to \infty$ for all $h \in G$. By the Baire category theorem [11, page 12], ψ_g is continuous on a set D of the second category. Since G is locally compact, $D \ne \emptyset$. Now we show that continuity of ψ_g at one point h_0 implies its continuity on G. Indeed, note that for $h, k \in G$ we have $\triangle_k\psi_g(h) = \psi_g(hk) - \psi_g(h) = \int_G[\varphi(thk) - \varphi(th)]g(t)dt = (\triangle_k\varphi)^* * g(h^{-1})$. By [7, 20.32 (e)], $\triangle_k\psi_g \in C_0(G)$. From the identity $\triangle^v\psi_g(h) = \triangle^v\psi_g(h_0) + \triangle^v\triangle_{h_0^{-1}h}\psi_g(h_0)$, the continuity of ψ_g at h_0 and the continuity of $\Delta_{h_0^{-1}h}\psi_g$ at h_0 we conclude that ψ_g is continuous.

Theorem 2.3. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 < p < \infty$ and all $h \in G$. Then $\psi : G \to L^p(G)$, $\psi(h) = \triangle_h\varphi$, is right uniformly continuous.*

Proof. By Lemma 2.2, ψ is weakly continuous. That is, $\langle\psi(h), g\rangle = \psi_g(h)$ is a continuous function of h for all $g \in L^q(G)$. By a Theorem of Namioka [9, Theorem 4.1], ψ is continuous on a dense G_δ subset of G. By Lemma 2.1, ψ is right uniformly continuous.

We need the following proposition.

Proposition 2.4. *Let X be a Banach space. Let $\varphi : G \to X$ be bounded on an open subset U of G. Suppose $\triangle_h\varphi$ is continuous for each $h \in G$. Then φ is continuous.*

Proof. For $g \in X^*$, the dual of X, set $\varphi_g = g \circ \varphi$. Then φ_g is bounded on U and the differences $\Delta_h\varphi_g = g \circ \Delta_h\varphi$ are all continuous. By [1, Theorem 2.1] φ_g is continuous at each $h_o \in U$. From the identity $\Delta^v\varphi_g(h) = \Delta^v\Delta_{h_o^{-1}h}\varphi_g(h_o) + \Delta^v\varphi_g(h_o)$ we conclude that φ_g is continuous on G. By [9, Theorem 4.1], φ is continuous on a dense G_δ subset of G. The identity $\Delta^v\varphi(h) = \Delta^v\Delta_{h_1^{-1}h}\varphi(h_1) + \Delta^v\varphi(h_1)$ shows that φ is continuous on G.

Corollary 2.5. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^1(G)$ for all $h \in G$. Then $\psi : G \to L^1(G)$, $\psi(h) = \triangle_h\varphi$, is right uniformly continuous.*

Proof. Since $\triangle_h\varphi \in L^1(G) \cap L^\infty(G)$, we conclude $\triangle_h\varphi \in L^p(G)$ for all $1 \le p \le \infty$. By Theorem 2.3, $\|\triangle_h\varphi\|_{1+\frac{1}{n}}$ is a continuous function of h for each $n \in \mathbb{N}$, the natural numbers. Moreover, $\lim_{n\to\infty}\|\triangle_h\varphi\|_{1+\frac{1}{n}} = \|\triangle_h\varphi\|_1$. By the Baire category theorem, $\|\triangle_h\varphi\|_1$ is a continuous function of h except on a subset of G of the first category. So it is continuous at some $h_0 \in G$. Hence there exists a neighbourhood V of the unit e in G such that $\|\psi(h_0v)\|_1 = \|\triangle_{h_0v}\varphi\|_1 \le 1 + \|\triangle_{h_0}\varphi\|_1$ for all $v \in V$. Consider the differences $\triangle_k\psi$ for $k \in G$. We have $||\Delta_v\Delta_k\psi(h)||_1 = ||\Delta_v\Delta_k\varphi||_1 \to 0$ as $v \to e$, by [7, Theorem 20.4], since $\triangle_k\varphi \in L^1(G)$ for each $k \in G$. Hence $\triangle_k\psi : G \to L^1(G)$ is continuous. By Proposition 2.4, ψ is continuous, and by Lemma 2.1, ψ is right uniformly continuous.

Remark 2.6. Proposition 2.4 also holds true for the more general case of σ-well α-favorable topological groups as defined in [3]. In this case we use [3, Theorem 1] instead of [9, Theorem 4.1].

Remark 2.7. Let X be a Banach space and $1 \le p \le \infty$. Then $L^p(G, X)$ denotes the Banach space of strongly measurable functions $\varphi : G \to X$ for which $||\varphi(.)||_X \in L^p(G)$. If $1 \le p < \infty$, then $C_c(G, X)$ is dense in $L^p(G, X)$. Moreover, if $1 < p < \infty$ and X is separable or reflexive, then the dual of $L^p(G, X)$ is $L^q(G, X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. For this, see [5, 8.20.3 and 8.20.5]. It follows that the results of this section remain valid with $L^p(G)$ replaced by $L^p(G, X)$ for $1 \le p \le \infty$ whenever X is a separable or reflexive Banach space.

Remark 2.8. If X is a Banach space not containing a subspace isomorphic to $c_\circ$ (the Banach space of convergent to zero complex sequences), then $L^1(G, X)$ is also a Banach space not containing a subspace isomorphic to $c_\circ$ (see [8]). It follows that in the proof of Corollary 2.5 with $X = \mathbb{C}$ we can avoid application of Proposition 2.4 and conclude instead from [1, Theorem 2.1] that ψ is continuous at some point h_o and hence is right uniformly continuous. This simplification is not available for Banach spaces X containing subspaces isomorphic to $c_\circ$.

§3. Bounded functions with differences in $L^p(G)$

In this section G is a locally compact abelian group. Let $\varphi \in L^p(G)$ for some $1 \le p \le \infty$. Then [7, Corollary 20.14], $f * \varphi \in L^p(G)$ for each $f \in L^1(G)$. It follows that

$$I(\varphi) = \{f \in L^1(G) : f * \varphi = 0\}$$

is a closed ideal of $L^1(G)$. We define

$$sp(\varphi) = \text{hull } I(\varphi) = \{\gamma \in \widehat{G} : \hat{f}(\gamma) = 0 \text{ for all } f \in I(\varphi)\} \tag{3.1}$$

where $\widehat{G}$ is the dual group of G and $\hat{f}$ is the Fourier transform of f. For the following, see [2, Proposition 1.1].

Proposition 3.1. *Let $\varphi \in L^p(G)$ for some $1 \leq p \leq \infty$. Then*
(i) *$sp(\varphi * f) \subset sp(\varphi) \cap$ supp $(\hat{f})$ for all $f \in L^1(G)$; and*
(ii) *$sp(\varphi) = \emptyset$ if and only if $\varphi = 0$.*

Now let w be a weight function on G satisfying the Beurling-Domar condition. See [10, page 132]. Then the Beurling algebra $L^1_w(G) = \{f \in L^1(G) : wf \in L^1(G)\}$ is a Wiener algebra [10, 6.3.1]. The dual of $L^1_w(G)$ is $L^\infty_w(G) = \{\varphi : \frac{\varphi}{w} \in L^\infty(G)\}$. If $\varphi \in L^\infty_w(G)$, then its spectrum with respect to $L^1_w(G)$ will be denoted by $sp_w(\varphi) =$ hull $I_w(\varphi)$, where $I_w(\varphi) = \{f \in L^1_w(G) : f * \varphi = 0\}$ (see [10, page 142]).

We show

Theorem 3.2. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h \varphi \in L^p(G)$ for some $1 \leq p < \infty$ and all $h \in G$. If $\hat{e} \notin sp(\varphi)$, then $\varphi \in L^p(G)$.*

Proof. By Theorem 2.3 and Corollary 2.5, $\psi : G \to L^p(G)$, $\psi(h) = \triangle_h \varphi$, is uniformly continuous. Define $w : G \to \mathbb{R}$ by $w(h) = 1 + \|\psi(h)\|_p + \|\psi(h^{-1})\|_p$. Then
(i) $w(hk) \leq w(h)w(k)$ for all $h, k \in G$;
(ii) w is uniformly continuous; and
(iii) $w(h^n) \leq nw(h)$ for all $h \in G, n \in \mathbb{N}$.

It follows that w is a weight function on G satisfying the Beurling-Domar condition. Hence $L^1_w(G)$ is a Wiener algebra. Since $\hat{e} \notin sp(\varphi)$, by [10, 2.1.3, Remark] there exist a neighbourhood V of $\hat{e}$ and a function $f \in L^1_w(G)$ such that $\text{supp}(\hat{f}) \subset V$, $\hat{f}(\hat{e}) = 1$ and $V \cap sp(\varphi) = \emptyset$. By Proposition 3.1, $\varphi * f = 0$. Hence $\varphi(t) = \int_G [\varphi(t) - \varphi(ts^{-1})] f(s) ds = -\int_G \triangle_{s^{-1}} \varphi(t) f(s) ds$. The integrand as a function of s from G to $L^p(G)$ is weakly Borel measurable. Moreover, we claim that it is almost separably-valued with respect to Haar measure. Indeed $f \in L^1(G)$, so its (essential) support is σ-compact. The function $\triangle_{s^{-1}} \varphi$ of s is continuous and, therefore, restricted to the support of f it has a range which is σ-compact in $L^p(G)$ and hence separable. The claim follows. By Pettis's theorem [11, page 131] the integrand is strongly Borel measurable. Moreover, since $\|\triangle_{s^{-1}} \varphi\|_p \leq w(s)$, by Bochner's theorem [11, page 133] the Bochner-Haar integral $\int_G \triangle_{s^{-1}} \varphi f(s) ds$ exists and belongs to $L^p(G)$. That is, $\varphi \in L^p(G)$.

Letting w denote the weight function defined above, one can show by the same method as used in the proof of Theorem 3.2.

Theorem 3.3. *Let $\varphi \in L^\infty(G)$ and suppose $1 \leq p < \infty$. Then $\varphi \in L^p(G)$ if and only if*
(i) *$\triangle_h \varphi \in L^p(G)$ for all $h \in G$, and*
(ii) *there exists $f \in L^1_w(G), f \neq 0$, such that $f * \varphi \in L^p(G)$.*

Similarly, we have

Theorem 3.4. *Let $\varphi \in L^\infty(G)$ and suppose $\triangle_h\varphi \in L^p(G)$ for some $1 \le p < \infty$ and all $h \in G$. If $f \in L^1_w(G)$ and $\hat{f}(\hat{e}) = 1$, then $\varphi - f * \varphi \in L^p(G)$.*

To study indefinite integrals, we use the weight

$$v(x) = 1 + |x|,\ x \in \mathbb{R}. \tag{3.2}$$

It is readily seen that v is a symmetric weight function satisfying Beurling conditions (see [10, page 17]). It follows that $L^1_v(\mathbb{R})$ is a Wiener algebra. We denote by $C_u(\mathbb{R})$ ($C_{ub}(\mathbb{R})$) the set of all complex-valued uniformly continuous (uniformly continuous bounded) functions defined on $\mathbb{R}$.

Proposition 3.5. *If $\xi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ and v is given by (3.2), then $P\xi \in C_u(\mathbb{R}) \cap L^\infty_w(\mathbb{R})$.*

Proof. If $p = 1$, it is well-known that $P\xi$ is absolutely continuous and hence uniformly continuous. For arbitrary p, and $x, h \in \mathbb{R}$, $|P\xi(x+h) - P\xi(x)| = |\int_0^h \xi(x+t)\,dt| \le |h|^{1-1/p}\|\xi\|_p$ showing that $P\xi \in C_u(\mathbb{R})$. Moreover, $|P\xi(x)| = |\int_0^x \xi(t)\,dt| \le |x|^{1-1/p}\|\xi\|_p$, showing that $P\xi \in L^\infty_w(\mathbb{R})$.

Theorem 3.6. *If $\xi \in L^p(\mathbb{R})$ where $1 \le p \le \infty$ and $0 \notin sp(\xi)$, then $P\xi \in C_{ub}(\mathbb{R})$. Moreover, $P\xi \in L^p(\mathbb{R}) + \mathbb{C}$.*

Proof. Since $0 \notin sp(\xi)$, there exists a neighbourhood $V = [-\delta, \delta]$ such that $sp(\xi) \cap V = \emptyset$. Since $L^1_v(\mathbb{R})$ is a Wiener algebra, there is a function $h \in L^1_v(\mathbb{R})$ such that $\hat{h} = 1$ for $|\lambda| \le \delta/4$ and $\hat{h} = 0$ for $|\lambda| \ge \delta/3$. By Proposition 3.1, $h * \varphi = 0$. Similarly, by [10, page 140-141] and [2, Proposition 1.1], $sp_v(h * P\xi) \subset \text{supp }(\hat{h}) \cap sp_v(P\xi) \subset \{0\}$. Since $\frac{d(h*P\xi)}{dx} = h * \xi = 0$, we conclude that $h * P\xi$=c, a constant. If $\eta = P\xi - c$, then $0 \notin sp_w(\eta)$. Indeed, $h * \eta = h * P\xi - h * c = c - c = 0$. Thus $h \in I_v(\eta)$ and $\hat{h}(0) = 1$, showing $0 \notin sp_w(\eta)$. By Proposition 3.5, $\eta \in C_u(\mathbb{R})$ and so by [2, Theorem 4.2], η is bounded and so is $P\xi$. This proves that $P\xi \in C_{ub}(\mathbb{R})$. The function η satisfies all the conditions of Theorem 3.2, therefore $\eta \in L^p(\mathbb{R})$. Hence $P\xi \in L^p(\mathbb{R}) + \mathbb{C}$.

Remark 3.7. The results of section 3 remain true for X-valued functions provided that X is separable or reflexive. The spectrum of $\varphi \in L^p(G, X)$ is defined again by (3.1).

§4. Right uniformly continuous differences

In this section and the next, we again take a locally compact group G and a Banach space X. The following theorem, for the case of abelian groups, was proved by Datry and Muraz [4, Théorème 4, Corollaire]. Their proof was indirect, using deep results for Banach G-modules. We give a different direct proof, using the techniques of the previous sections, and then deduce results for G-modules in section 5.

Theorem 4.1. *Let $\varphi : G \to X$ be a bounded function and suppose $\triangle_h\varphi \in C_{rub}(G, X)$ for all $h \in G$. Then $\varphi \in C_{rub}(G, X)$.*

Proof. Define $\psi : G \to C_{rub}(G, X)$ by $\psi(h) = \triangle_h\varphi$. Then for $h, k \in G$ we have

$$||\Delta_v\Delta_k\psi(h)||_\infty = ||\Delta_v\Delta_k\varphi||_\infty \to 0 \quad \text{as } v \to e.$$

So $\Delta_k\psi : G \to C_{rub}(G, X)$ is continuous. By Proposition 2.4, ψ is continuous. Finally, continuity of ψ at e implies that φ is right uniformly continuous.

Remark 4.2. In view of Remark 2.6, Theorem 4.1 holds true for the more general case of σ-well α-favorable topological groups.

§5. Application to left G-modules

A Banach space X together with a family of bounded linear operators $A_h : X \to X$, for $h \in G$, is called a *left Banach G-module* if

(i) $A_e(x) = x$ for all $x \in X$;

(ii) $A_{hk}(x) = A_h(A_k(x))$ for all $h, k \in G$ and all $x \in X$;

(iii) $\|A_h(x)\| \leq \kappa\|x\|$ for all $h \in G$, all $x \in X$, and some $\kappa > 0$.

The space $X_{abs} = \{x \in X : \|A_v x - x\| \to 0 \text{ as } v \to e\}$ is a closed submodule of X. Its elements are called *absolutely continuous*. See Datry and Muraz [4], where Theorem 5.2 below is obtained in the case G is abelian using a different proof.

For a fixed vector $x \in X$ we study the function $\psi : G \to X$ given by $\psi(h) = A_h x$. So x is absolutely continuous if and only if ψ is continuous at e. In fact the following is true.

Theorem 5.1. *For an element x of a left Banach G-module X, the following are equivalent.*

(a) *$x \in X_{abs}$;*

(b) *ψ is weakly continuous at some point in G;*

(c) *ψ is right uniformly continuous.*

Proof. If $x \in X_{abs}$, then ψ is continuous, and therefore weakly continuous, at e. So (a) implies (b). Next suppose ψ is weakly continuous at h_0. From the identity $\langle \triangle_v \psi(h), x^* \rangle = \langle \triangle_v \psi(h_0), A^*_{hh_0^{-1}}(x^*) \rangle$ for $h, v \in G$ and $x^* \in X^*$, it follows that ψ is weakly continuous on G. By [9, Theorem 4.1] ψ is continuous at some point h_1. From the identity $\triangle_v \psi(h) = A_{hh_1^{-1}}(\triangle_v \psi(h_1))$ it follows that ψ is right uniformly continuous. Hence (b) implies (c). That (c) implies (a) is obvious.

Theorem 5.2. *Let x be an element of a Banach G-module X. If $A_h x - x \in X_{abs}$ for all $h \in G$, then $x \in X_{abs}$.*

Proof. For each $k \in G$, $\triangle_k \psi(h) = A_h(A_k x - x)$ which by Theorem 5.1 defines a right uniformly continuous function $\triangle_k \psi : G \to X$. As ψ is bounded, Theorem 4.1 shows that ψ is right uniformly continuous. Hence $x \in X_{abs}$.

Remark 5.3. In view of Remark 2.6, the results of this section hold true for the more general case of σ-well α-favorable topological groups.

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