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THE LU QI-KENG CONJECTURE FAILS GENERICALLY

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Abstract. The bounded domains of holomorphy in $\mathbf{C}^n$ whose Bergman kernel functions are zero-free form a nowhere dense subset (with respect to a variant of the Hausdorff distance) of all bounded domains of holomorphy.

A domain in $\mathbf{C}^n$ is called a *Lu Qi-Keng domain* if its Bergman kernel function has no zeroes. Lu Qi-Keng [11] raised the question of which domains, besides the ball and the polydisc, have this property. A motivation for the question is that the vanishing of the Bergman kernel function obstructs the global definition of Bergman representative coordinates. Over the years since Lu Qi-Keng's paper appeared, various versions of a *Lu Qi-Keng conjecture* have been mooted to the effect that all domains, or most domains, or all domains satisfying some geometrical hypothesis, are Lu Qi-Keng domains.

In the complex plane $\mathbf{C}^1$, a bounded domain with smooth boundary is a Lu Qi-Keng domain if and only if it is simply connected [16] (and thus biholomorphically equivalent to the disc). I have given a counterexample [1] showing that no analogous topological characterization of Lu Qi-Keng domains can hold in higher dimensions: there exists (in $\mathbf{C}^2$, and similarly in $\mathbf{C}^n$ for $n > 2$) a bounded, strongly pseudoconvex, contractible domain with C^∞ regular boundary whose Bergman kernel function does have zeroes.

In this note, I show that the Lu Qi-Keng domains of holomorphy may be viewed as exceptional: they form a nowhere dense set with respect to a suitable topology. Thus, contrary to former expectations, it is the normal situation for the Bergman kernel function of a domain to have zeroes.

To formulate the result precisely, I need a metric on bounded open sets. Since I impose no restriction on the regularity of the boundaries of the sets, some variant of the Hausdorff metric will be appropriate. The Hausdorff distance $\mathcal{H}$ is normally defined for nonempty, bounded, closed sets by the property that $\mathcal{H}(A, B) < \epsilon$ if and only if each point of A has Euclidean distance less than ϵ from some point of B, and vice versa.

After the seminal paper of Ramadanov [12], it is clear in the context of the Bergman kernel function that if a sequence of open sets $\{\Omega_j\}$ is going to be said to converge to an open set Ω, then every compact subset of Ω should eventually be contained in Ω_j. It is less clear what requirement should be imposed if the Ω_j contain points outside of Ω. The example [15, p. 39], [13, p. 280] of decreasing

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concentric disks in the complex plane converging to a disk with a slit removed shows that it is inadequate to require merely that for every open neighborhood of the closure $\overline{\Omega}$, eventually Ω_j is contained in the neighborhood.

I shall consider two different notions of convergence of open sets in $\mathbf{C}^n$. Both have the property that if $\Omega_j \to \Omega$, then the Ω_j eventually swallow every compact subset of Ω. However, they differ in what they require about the sets $\Omega_j \setminus \Omega$.

First I define a metric ρ_1 on bounded, nonempty, open sets via $\rho_1(U, V) = \mathcal{H}(\overline{U}, \overline{V}) + \mathcal{H}(\partial U, \partial V)$. If $\rho_1(\Omega_j, \Omega) \to 0$, then the Ω_j eventually swallow every compact subset of Ω and are eventually swallowed by every open neighborhood of $\overline{\Omega}$. The converse holds when Ω equals the interior of its closure, but not in general. By requiring that both the closures and the boundaries converge, convergence in the metric ρ_1 eliminates examples like the one above involving slits or punctures in the limit domain.

The metric ρ_1 can also be thought of in terms of functions. Define the distance function d_U of an open set U via $d_U(z) = \operatorname{dist}(z, \mathbf{C}^n \setminus U)$, where dist denotes the Euclidean distance. Then $\Omega_j \to \Omega$ according to the metric ρ_1 if and only if the continuous functions d_{Ω_j} converge uniformly on $\mathbf{C}^n$ to d_Ω and the $d_{\mathbf{C}^n \setminus \overline{\Omega_j}}$ converge uniformly to $d_{\mathbf{C}^n \setminus \overline{\Omega}}$.

In some contexts—one will appear below—it is useful to relax the hypothesis on how the sets $\Omega_j \setminus \Omega$ behave. They could be required to shrink in volume (Lebesgue $2n$-dimensional measure), but not necessarily in terms of Euclidean distance from Ω. I therefore introduce a second metric ρ_2 on bounded open sets via $\rho_2(U, V) = \operatorname{vol}(U \setminus V) + \operatorname{vol}(V \setminus U) + \sup_{z \in \mathbf{C}^n} |d_U(z) - d_V(z)|$. Convergence of Ω_j to Ω in this metric allows Ω_j to have a long thin tail whose width shrinks to zero but whose length does not shrink.

So far I have not assumed that the open sets in question are connected. It is easy to see that the Bergman kernel function $K(w, z)$ for a disconnected open set is identically equal to zero if w and z are in different connected components, while if the points are in the same connected component, then $K(w, z)$ equals the Bergman kernel function of that component. I will say that a (possibly) disconnected, bounded, nonempty, open set is a *Lu Qi-Keng open set* if its Bergman kernel function has no zeroes when the two variables are in the same connected component.

Theorem. *The Lu Qi-Keng open sets are nowhere dense in each of the following metric spaces, where the metric is ρ_1:*

(1) *the bounded pseudoconvex open sets;*
(2) *the bounded connected pseudoconvex open sets (domains of holomorphy);*
(3) *the bounded strongly pseudoconvex open sets;*
(4) *the bounded connected strongly pseudoconvex open sets.*

If one considers only open sets of Euclidean diameter less than some fixed constant M, then the same assertion holds when the metric is taken to be ρ_2.

I will base the proof of the theorem on the following two folklore lemmas. The ideas of the proofs are all in the literature, but since I do not know a reference for precisely these formulations, I will indicate proofs of the lemmas after the proof of the theorem.

Stability lemma for the Bergman kernel function. *Let $\{\Omega_j\}$ be a sequence of bounded pseudoconvex open sets that converges, in the sense of either ρ_1 or ρ_2, to a nonempty bounded open set Ω; in the case of ρ_2, assume also that the Ω_j have* *uniformly bounded diameters (this is automatic in the case of ρ_1). Suppose U is a connected component of Ω that has C^∞ regular boundary and that is separated from the rest of Ω (that is, the closure of U is disjoint from the closure of $\Omega \setminus U$). Then the Bergman kernel functions of the Ω_j converge to the Bergman kernel function of U uniformly on compact subsets of $U \times U$.*

In the statement of the stability lemma, pseudoconvexity of the limit set Ω is automatic: the limit function $-\log d_\Omega$ inherits plurisubharmonicity from the functions $-\log d_{\Omega_j}$, which converge uniformly on compact subsets of Ω.

A special case of considerable interest is when the Ω_j and Ω are all bounded, connected, pseudoconvex domains with C^∞ regular boundaries. The lemma then says that if the Ω_j converge to Ω, in the sense that Ω_j eventually swallows every compact subset of Ω and the volume of $\Omega_j \setminus \Omega$ tends to zero, then the Bergman kernel functions of the Ω_j converge uniformly on compact subsets of $\Omega \times \Omega$ to the Bergman kernel function of Ω.

The C^∞ regularity hypothesis in the lemma can be reduced to C^2 regularity, but I shall not prove this here.

I take the name of the second lemma from [10, Chap. 5, Exercise 21].

Barbell lemma. *Suppose G_1 and G_2 are bounded, connected, strongly pseudoconvex domains in $\mathbf{C}^n$ with C^∞ regular boundaries and with disjoint closures. Let γ be a smooth curve (that is, a C^∞ embedding of $[0,1]$ into $\mathbf{C}^n$) joining a boundary point of G_1 to a boundary point of G_2, and otherwise disjoint from the closures of G_1 and G_2, and let V be an arbitrary neighborhood in $\mathbf{C}^n$ of the curve γ. Then there exists a bounded, connected, strongly pseudoconvex domain Ω with C^∞ regular boundary such that Ω is contained in $G_1 \cup G_2 \cup V$, and Ω coincides with $G_1 \cup G_2$ outside V.*

When G_1 and G_2 are balls of equal size, and γ is the shortest line segment joining them, then the domain Ω is a "barbell", or dumbbell.

The C^∞ regularity can be changed everywhere in the statement of the lemma to C^k regularity, where k is any integer greater than or equal to 2.

Proof of the theorem. I have not claimed that the bounded pseudoconvex open sets which fail to be Lu Qi-Keng form an open set in either of the metrics ρ_1 or ρ_2, and I do not know whether or not this is the case for sets with irregular boundaries. However, if Ω is, for example, a bounded strongly pseudoconvex open set with C^∞ regular boundary, and the Bergman kernel function of Ω has zeroes on some connected component Ω_0, then there is a ρ_1 neighborhood of Ω containing no pseudoconvex Lu Qi-Keng open set. Indeed, if a sequence of pseudoconvex open sets converges to Ω in the metric ρ_1, then the corresponding Bergman kernel functions converge on Ω_0 to the Bergman kernel function of Ω_0 by the stability lemma, and by Hurwitz's theorem these approximating Bergman kernel functions cannot all be zero-free on Ω_0. The analogous statement holds for the metric ρ_2 if one restricts attention to sets of uniformly bounded Euclidean diameter.

Accordingly, to prove the theorem it will suffice to construct, arbitrarily close (according to either ρ_1 or ρ_2) to a given bounded pseudoconvex open set G, a bounded strongly pseudoconvex open set Ω with C^∞ regular boundary whose Bergman kernel function does have zeroes on some connected component; if G is connected, then Ω should be connected too.

It is standard that the pseudoconvex open set G can be exhausted from inside by strongly pseudoconvex open sets with C^∞ regular boundaries: namely, by sublevel sets of a smooth, strictly plurisubharmonic exhaustion function. It is evident that these interior approximating sets converge to G in both of the metrics ρ_1 and ρ_2. Consequently, there is no loss of generality in supposing from the start that G is a bounded strongly pseudoconvex open set with C^∞ regular boundary.

Place close to G a strongly pseudoconvex domain D with C^∞ regular boundary and small diameter, the Bergman kernel function of D having zeroes. (In $\mathbf{C}^1$, the domain D could be an annulus; in higher dimensions, D could be a small homothetic copy of the counterexample domain that I constructed in [1].) Then $G \cup D$ will be a disconnected strongly pseudoconvex open set that is close to G in both of the metrics ρ_1 and ρ_2. This open set $G \cup D$ will serve as the required Ω to prove parts (1) and (3) of the theorem.

To prove parts (2) and (4) of the theorem, I need to produce a connected Ω when G is connected. To do this, join G to D with a closed line segment L, and use the barbell lemma to construct a sequence of bounded, connected, strongly pseudoconvex open sets Ω_k with C^∞ regular boundaries, the Ω_k being contained in $G \cup D \cup V_k$, where the V_k are shrinking neighborhoods of the line segment L. The Ω_k converge to $G \cup D$ in the metric ρ_2, so the stability lemma and Hurwitz's theorem imply that the Bergman kernel function of Ω_k has zeroes (on D) when k is sufficiently large. Since the Euclidean distance of $D \cup V_k$ from G is small, Ω_k is close to G in the metric ρ_1 as well as in the metric ρ_2. Thus one of the Ω_k serves as the required Ω. $\square$

Proof of the stability lemma. The main point is to prove an L^2 approximation theorem for holomorphic functions. I claim that if f is a square-integrable holomorphic function on U, and if a positive ϵ is prescribed, then for all sufficiently large j there exists a square-integrable holomorphic function g_j on Ω_j such that $\|f - g_j\|_{L^2(\Omega_j \cap U)} < \epsilon$ and $\|g_j\|_{L^2(\Omega_j \setminus U)} < \epsilon$.

I first need to show that the holomorphic functions in the Sobolev space $W^1(U)$ of square-integrable functions with square-integrable first derivatives are dense in the space of square-integrable holomorphic functions on U. This is a consequence of Kohn's global regularity theorem [9] for the $\overline{\partial}$-Neumann problem with weights. Namely, for a suitably large positive number t, the weighted $\overline{\partial}$-Neumann operator N_t for U is a bounded operator on the Sobolev space $W^2(U)$. Consequently, the corresponding weighted Bergman projection operator P_t, which satisfies the relation $P_t = \operatorname{Id} - \overline{\partial}_t^* N_t \overline{\partial}$, maps $W^3(U)$ into the holomorphic subspace of $W^1(U)$. Now if f is a square-integrable holomorphic function in U, take a sequence $\{v_j\}$ of C^∞ functions converging to f in $L^2(U)$, and project these functions by P_t. The functions $P_t v_j$ are holomorphic functions in $W^1(U)$ that converge to f in $L^2(U)$.

Therefore, there is no loss of generality in assuming from the start that the holomorphic function f lies in $W^1(U)$. Consequently, f is the restriction to U of a function $F \in W^1(\mathbf{C}^n)$.

It follows from the hypothesis of the lemma that there is an open neighborhood V of the closure of U such that the $2n$-dimensional Lebesgue measure of $V \cap (\Omega_j \setminus U)$ tends to zero as $j \to \infty$. There is no harm in cutting off the function F so that its support lies inside V.

The one-form $\overline{\partial} F$ is then defined on all of $\mathbf{C}^n$, zero on U, zero outside V, and square-integrable. Since the measure of $V \cap (\Omega_j \setminus U)$ shrinks to zero, the $L^2(\Omega_j)$ norm of $\overline{\partial}F$ tends to zero as $j \to \infty$. Use Hörmander's L^2 theory [7] to solve the equation $\overline{\partial}u_j = \overline{\partial}F$ on Ω_j for a square-integrable function u_j whose $L^2(\Omega_j)$ norm is bounded by a constant (depending only on the uniform bound on the diameters of the Ω_j) times the $L^2(\Omega_j)$ norm of $\overline{\partial}F$. Thus the norm of u_j on Ω_j tends to zero as $j \to \infty$. Consequently, the function $g_j := F - u_j$, which is holomorphic and square-integrable on Ω_j, has norm on $\Omega_j \cap U$ close to the norm of f when j is large. Also, the norm of g_j on $\Omega_j \setminus U$ tends to zero with the measure of $V \cap (\Omega_j \setminus U)$. This confirms the claimed approximation property.

The remainder of the proof of the stability lemma follows standard lines. However, I mention that I am dispensing with the hypothesis of monotonicity of the domains that is typically assumed [8, pp. 180–182], [15, pp. 36–39].

Fix a point z in U. The Bergman kernel function $K(\cdot, z)$ (for U, or equivalently for Ω when the free variable is in U) is the unique square-integrable holomorphic function f on U that maximizes $f(z)$ subject to the nonlinear constraint $f(z) \geq \|f\|^2_{L^2(U)}$. Let f_j denote the corresponding extremal function for the approximating domain Ω_j. By the mean-value property of holomorphic functions, $f_j(z)$ is bounded by a constant times $\|f_j\|_{L^2(\Omega_j)}$ times an inverse power of the distance from z to the boundary of Ω_j; the extremal property of f_j then implies that $\|f_j\|_{L^2(\Omega_j)}$ too is bounded by a constant times an inverse power of the distance from z to the boundary of Ω_j. Therefore the $\|f_j\|_{L^2(\Omega_j)}$ are uniformly bounded, and so the f_j form a normal family on U. Consequently, the f_j have a subsequence that converges uniformly on compact subsets of U to a holomorphic limit function f_∞. (Once I show that the limit f_∞ actually is f, it will follow that the original sequence $\{f_j\}$, not just a subsequence, converges to f.)

By Fatou's lemma, it follows that the limit function f_∞ satisfies $f_\infty(z) \geq \|f_\infty\|^2_{L^2(U)}$. By the approximation property proved above, there exists a square-integrable holomorphic function g_j on Ω_j such that $g_j(z) \geq \|g_j\|^2_{L^2(\Omega_j)}$, and $g_j(z) \geq (1-\delta_j)f(z)$, where the positive numbers δ_j tend to zero as $j \to \infty$. The extremal function f_j therefore has the property that $f_j(z) \geq (1-\delta_j)f(z)$. Consequently, $f_\infty(z) \geq f(z)$. The uniqueness of the extremal function implies that $f_\infty = f$. This proves that the Bergman kernel functions $K_j(w, z)$ for the Ω_j converge pointwise to $K(w, z)$ on $U \times U$.

Since $|K_j(w, z)|^2 \leq K_j(w, w)K_j(z, z)$ by the Cauchy-Schwarz inequality, and the right-hand side is bounded by a constant depending only on the distances of z and w from the boundary of Ω_j, the functions $K_j(\cdot\,, \cdot)$ form a normal family in $U \times U$. From the normality and the pointwise convergence just proved, it is immediate that the convergence is uniform on compact subsets of $U \times U$. □

Proof of the barbell lemma. In the complex plane $\mathbf{C}^1$, there is nothing to prove, for every planar domain is strongly pseudoconvex. In higher dimensions, there is no loss of generality in supposing that the curve γ meets the boundaries of G_1 and G_2 transversely, since the barbell Ω is not prescribed inside the neighborhood V. By [4, Theorem 4] (a result that the authors attribute to [5]), the set $\overline{G}_1 \cup \overline{G}_2 \cup \gamma$ has a basis of Stein neighborhoods, so there exists a connected, strongly pseudoconvex domain with C^∞ regular boundary that outside V is a small perturbation of $G_1 \cup G_2$. This conclusion is already enough for the application to the proof of the main theorem.

The stronger statement that one can find a barbell that actually matches $G_1 \cup G_2$ outside a neighborhood of the curve γ was demonstrated by Shcherbina for the case when G_1 and G_2 are balls [14, Lemma 1.2 and its Corollary]. The general case follows from this special one because any strongly pseudoconvex domain can be perturbed in an arbitrarily small neighborhood of a boundary point to obtain a new strongly pseudoconvex domain whose boundary near that point is a piece of the boundary of a ball. This can be seen from the patching lemma for strictly plurisubharmonic functions in [3, Lemma 3.2.2] by taking the totally real set there to be a single point. $\square$

Open questions

(1) In the stability lemma, the C^∞ regularity hypothesis can be reduced to C^2 regularity by inspecting Kohn's proof [9] to see that C^{k+1} boundary regularity suffices for W^k regularity of the weighted $\overline{\partial}$-Neumann operator; one also needs techniques as in [2] to see that the weighted Bergman projection has the same regularity as the weighted $\overline{\partial}$-Neumann operator. Can the hypothesis in the stability lemma be reduced to C^1 boundary regularity?

(2) The conclusion of the theorem—that most pseudoconvex domains are not Lu Qi-Keng domains—changes if the topology on domains is changed. For example, any small C^∞ perturbation of the unit ball is a Lu Qi-Keng domain [6]. Does the set of bounded pseudoconvex Lu Qi-Keng domains with C^1 regular boundary have nonempty interior in the C^1 topology on pseudoconvex domains? This is the case for domains in the complex plane $\mathbf{C}^1$.

(3) My proof of the stability lemma for the Bergman kernel function uses pseudoconvexity. Can the word "pseudoconvex" be removed from the statement of the main theorem?

(4) Is every bounded *convex* domain a Lu Qi-Keng domain?

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