PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 7, July 1996

DIFFEOMORPHISMS WITH PERSISTENCY

KAZUHIRO SAKAI

(Communicated by Mary Rees)

ABSTRACT. The C^1 interior of the set of all diffeomorphisms satisfying Lewowicz's persistency is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

In [5], Lewowicz introduced a notion of persistency for a homeomorphism of a compact metric space X, and it is remarked that persistence is a weaker property than topological stability when X is a manifold. It is also proved there that every pseudo-Anosov map (on a surface) is persistent. The purpose of this paper is to analyze the dynamics of diffeomorphisms having persistency. More precisely we shall prove the following theorem.

Let M be a C^∞ closed manifold and let $\mathrm{Diff}(M)$ be the space of C^1 diffeomorphisms of M endowed with C^1 topology. We denote by $\mathcal{P}(M)$ the set of all $f \in \mathrm{Diff}(M)$ having persistency.

Theorem. *The C^1 interior of $\mathcal{P}(M)$ in $\mathrm{Diff}(M)$, $int\mathcal{P}(M)$, is characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.*

It was proved in [7] and [9] respectively that the C^1 interior of the set of all $f \in \mathrm{Diff}(M)$ having topological stability and the C^1 interior of the set of all $f \in \mathrm{Diff}(M)$ having the pseudo-orbit tracing property were characterized as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition. Therefore, if the theorem is established, then these two open sets and $int\mathcal{P}(M)$ are equal.

Let $\mathcal{E}(M)$ be the set of all expansive diffeomorphisms of M. Gerber and Katok [3] proved that if f is a pseudo-Anosov map on a surface M and if $\mathcal{N}^0(f)$ is a C^0 neighborhood of f, then there exists a smooth diffeomorphism $g \in \mathcal{N}^0(f)$ conjugating to f. Thus it can be checked that $g \in \mathcal{E}(M) \cap \mathcal{P}(M)$, and more precisely, the following corollary implies that g belongs to $\mathcal{E}(M) \cap \partial\mathcal{P}(M)$. Here $\partial\mathcal{P}(M)$ denotes the boundary of $\mathcal{P}(M)$ in $\mathrm{Diff}(M)$.

Corollary. *$\mathcal{E}(M) \cap int\mathcal{P}(M)$ is characterized as the set of all Anosov diffeomorphisms.*

The corollary is an easy consequence of our theorem. Indeed, since every $f \in \mathrm{Diff}(M)$ satisfying Axiom A and the strong transversality condition is structurally

Received by the editors October 12, 1994 and, in revised form, January 6, 1995.
1991 *Mathematics Subject Classification.* Primary 54H20, 58F10, 58F15.
Key words and phrases. Persistence, expansive, pseudo-Anosov map, Axiom A, Anosov diffeomorphism.

stable, if $f \in \mathcal{E}(M)$, then it is Anosov (by [6]). Conversely, if $f \in \mathrm{Diff}(M)$ is Anosov, then f is persistent since f is topologically stable.

Let d be a metric on M which is induced from a Riemannian metric $\|\cdot\|$ on TM, and let us denote by $\mathcal{H}(M)$ the set of all homeomorphisms of M. We say that $f \in \mathcal{H}(M)$ is *persistent* if for each $\varepsilon > 0$, there is $\delta > 0$ such that for every $x \in M$ and $g \in \mathcal{H}(M)$ with $d(f, g) < \delta$, there is $y \in M$ satisfying $d(f^n(x), g^n(y)) < \varepsilon$ ($\forall n \in \mathbf{Z}$). The notion is independent of a metric for M and is conjugacy invariant.

Let $\Lambda(f)$ be a hyperbolic set of $f \in \mathrm{Diff}(M)$. For any $\varepsilon > 0$ and $x \in \Lambda(f)$, the *local stable manifold* and the *local unstable manifold* are denoted by $W^s_\varepsilon(x, f)$ and $W^u_\varepsilon(x, f)$ respectively. The *stable manifold* $W^s(x, f)$ and the *unstable manifold* $W^u(x, f)$ of $x \in \Lambda(f)$ are defined by a usual way. Let $f \in \mathrm{Diff}(M)$ satisfy Axiom A. Then the *non-wandering set* of f, $\Omega(f)$, is a disjoint union of basic sets $\Lambda_1(f) \cup \cdots \cup \Lambda_\ell(f)$. Recall that the periodic points of $f_{|\Lambda_i(f)}$ are dense in $\Lambda_i(f)$ $(1 \le i \le \ell)$ and that for every $x \in M$ there are $p \in \Lambda_i(f)$ and $q \in \Lambda_j(f)$ $(1 \le i \ne j \le \ell)$ such that $x \in W^s(p, f) \cap W^u(q, f)$. We say that f satisfies the *strong transversality condition* if for every $x \in M$, $T_xW^s(p, f) + T_xW^u(q, f) = T_xM$ for some $p, q \in \Omega(f)$.

Let $P(f)$ denote the set of all periodic points of $f \in \mathrm{Diff}(M)$, and let $\mathcal{F}(M)$ be the set of all $f \in \mathrm{Diff}(M)$ having a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}(M)$ such that every $p \in P(g)$ $(\forall g \in \mathcal{U}(f))$ is hyperbolic. Then such a set was characterized as the set of all diffeomorphisms satisfying Axiom A with no-cycles (see [1, 4]). It is well known that every $f \in \mathrm{Diff}(M)$ satisfying Axiom A and the strong transversality condition is persistent (because f is topologically stable (see [8])). Therefore our theorem follows from the following two propositions.

Proposition A. *The C^1 interior of $\mathcal{P}(M)$, $int\mathcal{P}(M)$, is a subset of $\mathcal{F}(M)$.*

Proposition B. *Let $f \in \mathrm{Diff}(M)$ satisfy Axiom A with no-cycles. If $f \in int\mathcal{P}(M)$, then f satisfies the strong transversality condition.*

1. Proof of Proposition A

Let $f \in int\mathcal{P}(M)$. To get the conclusion, it is enough to show that every $p \in P(f)$ is hyperbolic. Indeed, if this is established, then for every C^1 neighborhood $\mathcal{V}(f) \subset int\mathcal{P}(M)$ of f, every $q \in P(g)$ $(\forall g \in \mathcal{V}(f))$ is hyperbolic because $g \in int\mathcal{P}(M)$. Thus $f \in \mathcal{F}(M)$ is obtained.

Fix a neighborhood $\mathcal{U}(f) \subset int\mathcal{P}(M)$ of f, and by assuming that there is a non-hyperbolic periodic point $p = f^n(p)$, we shall derive a contradiction. Here $n > 0$ is the prime period of p. The tangent space T_pM splits into the direct sum $T_pM = E^u_p \oplus E^s_p \oplus E^c_p$ where E^u_p, E^s_p and E^c_p are D_pf^n-invariant subspaces corresponding to the absolute values of the eigenvalues of D_pf^n greater than one, less than one and equal to one, and suppose $E^c_p \ne 0$. Then, for every $\varepsilon > 0$ there exists a linear automorphism $\mathcal{O} : T_pM \to T_pM$ such that

$$\begin{cases} \|\mathcal{O} - I\| \le \varepsilon, \\ \mathcal{O}(E^\sigma_p) = E^\sigma_p \text{ for } \sigma = s, u \text{ and } c, \\ \text{all eigenvalues of } \mathcal{O} \circ D_pf^n_{|E^c_p} \text{ are of a root of unity,} \end{cases}$$

where $I : T_pM \to T_pM$ is an identity map. By making use of Franks's lemma (see [2, Lemma 1.1]), we can find $\delta_0 > 0$ and $g \in \mathcal{U}(f)$ such that

(i) $B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(f^j(p)) = \phi$ for $0 \le i \ne j \le n - 1$,

(ii) $g(x) = f(x)$ for $x \in \{p, f(p), \cdots, f^{n-1}(p)\} \cup \{M \setminus \cup_{i=0}^{n-1} B_{4\delta_0}(f^i(p))\}$,

(iii) $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^i(p))$ $(0 \le i \le n-2)$,

(iv) $g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{n-1}(p)} f \circ \exp_{f^{n-1}(p)}^{-1}(x)$ for $x \in B_{\delta_0}(f^{n-1}(p))$,

where $B_\varepsilon(x) = \{y \in M | d(x,y) \le \varepsilon\}$ for $\varepsilon > 0$.

Define $G = \mathcal{O} \circ D_p f^n$. Then there exists $m > 0$ such that $G^m_{|E^c_p}$ is an identity map. For a sufficently small $0 < \delta_1 < \delta_0$, we have

$$g^{mn}_{|\exp_p T_pM(\delta_1)} = \exp_p \circ G^m \circ \exp_p^{-1}$$

where $T_pM(\delta_1) = \{v \in T_pM | \|v\| \le \delta_1\}$. Put $E^c_p(\delta_1) = E^c_p \cap T_pM(\delta_1)$. Then it is clear that

$$g^{mn}_{|\exp_p E^c_p(\delta_1)} = id_{|\exp_p E^c_p(\delta_1)}.$$

Let $v = (v_1, v_2, \cdots, v_r)$ $(r = \dim E^c_p)$ be the representation by components with respect to the fundamental vectors of $\mathbf{R}^r = E^c_p$. Put $\hat{\varepsilon} = \delta_1/8$ and fix any $0 < \delta \le \hat{\varepsilon}$. Let $\varphi : \mathbf{R}^r \to \mathbf{R}^r$ be the time-one map given by the vector field

$$\dot{v_i} = \delta' \chi(v_1) \cdots \chi(v_r) v_i$$

for $1 \le i \le r$. Here $\chi : \mathbf{R} \to \mathbf{R}$ is a C^∞ function $(0 \le \chi(t) \le 1)$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \le \delta_1/2, \\ 0 & \text{if } |t| \ge 2\delta_1/3, \end{cases}$$

and $\delta' > 0$ is a number chosen so that $\|\varphi(v) - v\| < \delta$ for $v \in \mathbf{R}^r$ and $\|D_v\varphi - id_{|\mathbf{R}^r}\| \le |e^{\delta'} - 1| < \delta$ for $v \in T_pM(\delta_1) \cap \mathbf{R}^r$. We shall denote by $\tilde{\varphi} : T_pM(\delta_1) \to T_pM(\delta_1)$ the extension of φ such that $\tilde{\varphi}(v) = \varphi(v)$ for $v \in E^c_p(\delta_1)$ and $\|\tilde{\varphi}(v) - v\| < \delta$ for $v \in T_pM(\delta_1)$. Put

$$\psi(x) = \begin{cases} \exp_p \circ \tilde{\varphi} \circ \exp_p^{-1}(x) & \text{if } x \in \exp_p(T_pM(\delta_1)), \\ x & \text{otherwise,} \end{cases}$$

and define $\tilde{g} = \psi \circ g$. Let $\hat{\delta} = \hat{\delta}(g, \hat{\varepsilon}) > 0$ be a number as in the definition of persistency of g. Then $d(\tilde{g}, g) < \hat{\delta}$ $(\tilde{g} \in \mathcal{H}(M))$ for a sufficiently small δ. Take and fix $v \in E^c_p(\delta_1)$ such that $d(\exp_p(v), p) = 2\hat{\varepsilon}$. Clearly $g^{mn}(\exp_p(v)) = \exp_p(v)$ for all $m \in \mathbf{Z}$. On the other hand, it is easy to see that for every $y \in B_{\hat{\varepsilon}}(\exp_p(v))$, there is $m(y) \in \mathbf{Z}$ such that

$$d(\tilde{g}^{m(y)n}(y), g^{m(y)n}(\exp_p(v))) = d(\tilde{g}^{m(y)n}(y), \exp_p(v)) > \hat{\varepsilon}.$$

This is a contradiction.

2. Proof of Proposition B

Before starting the proof of this propsition, we need some preparation. Throughout this section let $f \in \mathrm{Diff}(M)$ satisfy Axiom A with no-cycles. Take a basic set $\Lambda(f)$ of f and fix $\varepsilon_0 > 0$ sufficiently small. Since $\dim W^s_{\varepsilon_0}(x, f) = \dim W^s_{\varepsilon_0}(y, f)$ for $x, y \in \Lambda(f)$, we denote by $\mathrm{Ind}\,\Lambda(f)$ the dimension of $W^s_{\varepsilon_0}(x, f)$ for $x \in \Lambda(f)$. If $g \in \mathrm{Diff}(M)$ is C^1 close to f, then the number of basic sets $\{\Lambda_i(g)\}$ of g coincides with that of basic sets $\{\Lambda_i(f)\}$ because of Ω-stability of f.

The following lemma is induced by Franks's lemma (see [9, Lemma 3] for details).

Lemma 1. *Let $\Lambda_1(f)$ and $\Lambda_2(f)$ be basic sets for f. Suppose that there are $p = f^n(p) \in \Lambda_1(f)$ $(n > 0), q \in \Lambda_2(f)$ and $x \in M \setminus \Omega(f)$ satisfying $x \in W^s(p,f) \cap W^u(q,f)$. Then, for a neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}(M)$ of f, there are $0 < \varepsilon_1 < \varepsilon_0/2, g \in \mathcal{U}(f)$ and two distinct basic sets $\Lambda_1(g)$ and $\Lambda_2(g)$ such that*

$$\text{(i)} \qquad B_{4\varepsilon_1}(f^i(p)) \cap B_{4\varepsilon_1}(f^j(p)) = \phi \textit{ for } 0 \le i \ne j \le n-1,$$

(ii)

$$g(x) = \begin{cases} \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp^{-1}_{f^i(p)}(x) & \textit{if } x \in B_{\varepsilon_1}(f^i(p)) \textit{ for } 0 \le i \le n-1, \\ f(x) & \textit{if } x \notin \cup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$$

$$\text{(iii)} \qquad \begin{cases} g^n(p) = p \in \Lambda_1(g) \textit{ and } q \in \Lambda_2(g), \\ x \in W^s(p,g) \cap W^u(q,g), \\ T_xW^s(p,g) = T_xW^s(p,f) \textit{ and } T_xW^u(q,g) = T_xW^u(q,f). \end{cases}$$

Since f satisfies Axiom A, there exist a Df-invariant continuous splitting $T_{\Omega(f)}M = E^s \oplus E^u$ and a constant $0 < \lambda < 1$ such that

$$\|Df^m_{|E^s}\| < \lambda^m, \|Df^{-m}_{|E^u}\| < \lambda^m$$

for all $m \ge 0$. We denote by E^σ_x a fiber of E^σ at $x \in \Omega(f)$ and put $E^\sigma_x(\varepsilon) = \{v \in E^\sigma_x | \|v\| \le \varepsilon\}$ for $\varepsilon > 0$ $(\sigma = s,u)$. Let $g \in \mathrm{Diff}(M), p = g^n(p) \in \Lambda_1(g)$ and $\varepsilon_1 > 0$ be given as in Lemma 1. Then it is easily checked that for $0 < \varepsilon \le \varepsilon_1$, we have

$$\exp_p(E^\sigma_p(\varepsilon)) \subset W^\sigma_{\varepsilon_0}(p,g) \text{ and } \dim \exp_p(E^\sigma_p(\varepsilon)) = \dim W^\sigma_{\varepsilon_0}(p,g)$$

for $\sigma = s,u$.

Now we shall prove Proposition B. Fix $x \in M \setminus \Omega(f)$ and let $\Lambda_i(f)$ and $\Lambda_j(f)$ be basic sets of f such that $x \in W^s(\Lambda_i(f),f) \cap W^u(\Lambda_j(f),f)$. To simplify the proof we assume $i = 1$ and $j = 2$. If $\mathrm{Ind}\,\Lambda_1(f) = \dim M$ or $\mathrm{Ind}\,\Lambda_2(f) = 0$, then the conclusion of this proposition is clear. Thus we shall prove $T_xM = T_xW^s(x,f) + T_xW^u(x,f)$ when $\mathrm{Ind}\,\Lambda_1(f) \le \dim M - 1$ and $\mathrm{Ind}\,\Lambda_2(f) \ge 1$.

Since $f \in int\mathcal{P}(M)$ and $\Omega(f) = \overline{P(f)}$, there is $f' \in int\mathcal{P}(M)$ arbitrarily near to f in the C^1 topology such that

(a) $f(y) = f'(y)$ for all y outside of a small neighborhood of x,

(b) there are $p = f'^n(p) \in \Lambda_1(f)$ for some $n > 0$ and $q \in \Lambda_2(f)$ such that $x \in W^s(p,f') \cap W^u(q,f'), T_xW^s(p,f') = T_xW^s(x,f)$ and $W^u(q,f') = W^u(x,f)$.

By (a), there are basic sets $\Lambda_i(f')$ $(i = 1,2)$ for f' such that $\Lambda_i(f') = \Lambda_i(f)$ since f is Ω-stable. Let us prove $T_xM = T_xW^s(p,f') + T_xW^u(q,f')$. We identify f' with f for simplicity, and let $\mathcal{U}(f)$ be a small neighborhood of f such that $\mathcal{U}(f) \subset \mathcal{P}(M)$.

Then, by Lemma 1 there are $g \in \mathcal{U}(f)$ and basic sets $\Lambda_i(f)$ $(i = 1,2)$ satisfying Lemma 1 (i), (ii) and (iii). Thus $T_xW^s(p,g) = T_xW^s(x,f)$ and $W^u(q,g) = W^u(x,f)$. Pick $\ell > 0$ such that $g^\ell(x) \in W^s_{\varepsilon_1/2}(p,g)$ and $g^{-\ell}(x) \in W^u_{\varepsilon_0/2}(g^{-\ell}(q),g)$, and define

$$C^u(g^\ell(x)) = \text{ the connected component of } g^\ell(x) \text{ in } W^u(g^\ell(q),g) \cap B_{\varepsilon_1}(p).$$

Clearly, $\exp_p^{-1}(C^u(g^\ell(x))) \subset T_pM$.

For a linear subspace E of T_pM and $\nu > 0$, we write

$$E_\nu(g^\ell(x)) = \{v + \exp_p^{-1}(g^\ell(x)) | v \in E \text{ with } \|v\| \le \nu\}.$$

Then there are a linear subspace $E' \subset T_pM$ and a number $0 < \nu_0 \le \varepsilon_1$ such that

$$T_{g^\ell(x)} \exp_p(E'_\nu(g^\ell(x))) = T_{g^\ell(x)} C^u(g^\ell(x))$$

and $\exp_p(E'_\nu(g^\ell(x))) \subset B_{\varepsilon_1}(p)$ for $0 < \nu \le \nu_0$. Since $g^\ell(x) \notin \Omega(g)$, there is $0 < \nu_1 \le \nu_0$ such that $B_{\nu_1}(g^\ell(x)) \cap g^i(B_{\nu_1}(g^\ell(x))) = \phi$ for all $i \in \mathbf{Z} \setminus \{0\}$. If $\mathcal{U}(g) \subset \mathcal{U}(f)$ is a neighborhood of g, then there are $0 < \nu_2 < \nu_1/4$ and $\varphi \in \mathrm{Diff}(M)$ such that

$$\begin{cases} \varphi_{|(B_{4\nu_2}(g^\ell(x)))^c} = id, \\ \varphi(g^\ell(x)) = g^\ell(x), \\ \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) \subset C^u(g^\ell(x)), \\ \dim \varphi(\exp_p(E'_{\nu_2}(g^\ell(x)))) = \dim C^u(g^\ell(x)), \\ g' \in \mathcal{U}(g) \text{ where } g' = \varphi^{-1} \circ g. \end{cases}$$

We denote $\exp_p(E'_{\nu_2}(g^\ell(x)))$ by $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ because of $g'^\ell(x) = g^\ell(x)$. It is clear that there are two distinct basic sets $\Lambda_i(g')$ $(i = 1, 2)$ such that $\Lambda_i(g') = \Lambda_i(g)$ since g is Ω-stable, and such that

$$T_x W^\sigma(x, g') = T_x W^\sigma(x, g) \ (\sigma = s, u),$$
$$\exp_p(E'_{\nu_2}(g'^\ell(x))) \subset W^u(g'^\ell(q), g') \cap B_{\varepsilon_1}(p),$$
$$\dim \exp_p(E'_{\nu_2}(g'^\ell(x))) = \dim W^u(q, g') = \dim C^u(g^\ell(x)).$$

Lemma 2. *Under the above notation,* $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ *meets transversely* $W^s_{\varepsilon_1}(p, g')$ *at* $g'^\ell(x)$.

If this lemma is established, then we have $T_xM = T_xW^s(x, f) + T_xW^u(x, f)$ since $T_xW^\sigma(x, g') = T_xW^\sigma(x, g) = T_xW^\sigma(x, f)$ for $\sigma = s, u$.

Proof of Lemma 2. Put $C^u_\varepsilon(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W^u_{\varepsilon_0}(g'^{-\ell}(q), g'))$ for $\varepsilon > 0$. Take $0 < \tilde\varepsilon < \nu_2$ such that $C^u_\varepsilon(g'^\ell(x))$ is the connected component of $g'^\ell(x)$ in $B_{\tilde\varepsilon}(g'^\ell(x)) \cap g'^{2\ell}(W^u_{\varepsilon_0}(g'^{-\ell}(q), g'))$ for $0 < \varepsilon \le \tilde\varepsilon$, and such that $B_{\tilde\varepsilon}(g'^\ell(x)) \cap g'^{2\ell}(W^u_{\varepsilon_0}(g'^{-\ell}(q), g')) \subset \exp_p(E'_{\nu_2}(g'^\ell(x)))$.

Claim. For every $0 < \varepsilon \le \tilde\varepsilon$, *if* $d(g'^{-i}(g'^\ell(x)), g'^{-i}(w)) < \varepsilon$ *for all* $i \ge 0$, *then* $w \in C^u_\varepsilon(g'^\ell(x))$.

Indeed, it is clear that $d(g'^{-\ell-i}(x), g'^{-2\ell-i}(w)) < \varepsilon \le \varepsilon_0/2$ for all $i \ge 0$. On the other hand, since $d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0/2$ $(\forall i \ge 0)$,

$$d(g'^{-2\ell-i}(w), g'^{-\ell-i}(q)) \le d(g'^{-2\ell-i}(w), g'^{-\ell-i}(x)) + d(g'^{-\ell-i}(x), g'^{-\ell-i}(q)) < \varepsilon_0$$

for all $i \ge 0$ and so $g'^{-2\ell}(w) \in W^u_{\varepsilon_0}(g'^{-\ell}(q), g')$. Thus $w \in C^u_\varepsilon(g'^\ell(x)) = B_\varepsilon(g'^\ell(x)) \cap g'^{2\ell}(W^u_{\varepsilon_0}(g'^{-\ell}(q), g'))$ since $d(g'^\ell(x), w) < \varepsilon$. The claim is proved.

Suppose that $\exp_p(E'_{\nu_2}(g'^\ell(x)))$ does not meet transversely $W^s_{\varepsilon_1}(p, g')$ at $g'^\ell(x)$. Then there exist $0 < \nu_3 < \min\{\tilde\varepsilon, \nu_2/2\}$ such that for every $\delta > 0$ $(\delta \ll \nu_3)$ there is $\psi_\delta \in \mathrm{Diff}(M)$ satisfying

$$\begin{cases} \psi_\delta|_{(B_{\nu_3}(g'^\ell(x)))^c} = id, \\ d(\psi_\delta, id) < \delta, \\ \psi_\delta(\exp_p(E'_{\nu_3/2}(g'^\ell(x)))) \cap W^s_{\varepsilon_1}(p, g') = \phi. \end{cases}$$

Fix $0 < \varepsilon' < \nu_3/2$ and let $\delta' = \delta'(g', \varepsilon') > 0$ be a number as in the definition of persistency of g'. Take $\delta > 0$ such that $\tilde{g} = g' \circ \psi_\delta \in \mathcal{H}(M)$ and $d(\tilde{g}, g') < \delta'$. Then, for g'-orbit $\{g'^i(x)\}_{i\in\mathbf{Z}}$ of x, there is $y \in B_{\varepsilon'}(g'^\ell(x))$ such that $d(\tilde{g}^i(y), g'^i(g'^\ell(x))) < \varepsilon'$ for all $i \in \mathbf{Z}$. By the claim $y \in \exp_p(E'_{\nu_3/2}(g'^\ell(x)))$, from which $\tilde{g}(y) = g \circ \psi_\delta(y) \notin W^s_{\varepsilon_1}(p, g')$. Thus, by the hyperbolicity, $d(\tilde{g}^i(\tilde{g}(y)), g'^{i+1}(g'^\ell(x))) > \varepsilon'$ for some $i \geq 0$. This is a contradiction.

References

1. N. Aoki, *The set of Axiom A diffeomorphisms with no cycle*, Bol. Soc. Bras. Mat. 23-New Series (1992), 21-65. MR **94d:**58080
2. J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. A.M.S., 158 (1971), 301-308. MR **44:**1042
3. M. Gerber and A. Katok, *Smooth models of Thurston's pseudo-Anosov maps*, Ann. Scient. Éc. Norm. Sup., 4^e série, t. 15, 1982, 173-204. MR **84e:**58056
4. S. Hayashi, *Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A*, Ergod. Th. & Dyn. Sys. 12 (1992), 233-253. MR **94d:**58081
5. J. Lewowicz, *Persistence in expansive systems*, Ergod. Th. & Dyn. Sys. 3 (1983), 567-578. MR **85m:**58140
6. R. Mañé, *Expansive Diffeomorphisms*, Dynamical Systems-Warwick, 1974, (ed. by A. Manning), Lecture Notes in Math. 468, Springer, 1975, 162-174. MR **58:**31263
7. K. Moriyasu, *The topological stability of diffeomorphisms*, Nagoya Math. J. 123 (1991), 91-102. MR **92g:**58067
8. C. Robinson, *Stability theorems and hyperbolicity in dynamical systems*, Rocky Mountain J. Math., 7 (1977), 425-437. MR **58:**13200
9. K. Sakai, *Pseudo-orbit tracing property and strong transversality of diffeomorphisms on closed manifolds*, Osaka J. Math. 31(1994), 373-386. CMP 95:02

Department of Mathematics, Kanagawa University, Rokkakubashi, Kanagawa-Ku, Yokohama 221, Japan

E-mail address: kazsaka@kani.cc.kanagawa.ac.jp