PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 7, July 1996

A NEW PROOF OF KUNEN'S INCONSISTENCY

JINDŘICH ZAPLETAL

(Communicated by Andreas R. Blass)

ABSTRACT. Using a basic fact from Shelah's theory of possible cofinalities, we give a new proof of Kunen's inconsistency theorem: there is no nontrivial elementary embedding of the set-theoretical universe into itself.

Kunen's celebrated inconsistency theorem states that

Theorem [K]. *There is no nontrivial elementary embedding $j : V \to V$.*

Given the great importance of this result to the theory of large cardinals, several different proofs have emerged since 1970 [B,H]. We show that there is still another simple way to demonstrate the theorem, quite different from the previous ones.

We show that if $j : V \to M$ is an elementary embedding into a transitive model M and λ is the least ordinal above the critical point of j left fixed by j, then $j''\lambda \notin M$. It follows that $M \neq V$.

For contradiction fix $j : V \to M$ as above and assume that $j''\lambda \in M$. It is easily established [J1] that λ is a strong limit cardinal of cofinality ω and $j(\lambda^+) = \lambda^+$. We use the following fact due to Shelah.

Lemma. *If $\lambda > cof(\lambda) = \omega$ is a strong limit cardinal, there is a sequence of regular cardinals $\langle \lambda_i : i < \omega \rangle$ less than λ, with $sup(\langle \lambda_i : i < \omega \rangle) = \lambda$ and the true cofinality of $\prod_{i<\omega} \lambda_i$ modulo finite equal to λ^+, that is, there is a sequence $\langle g_\alpha : \alpha < \lambda^+ \rangle \subset \prod_{i<\omega} \lambda_i$ which is increasing and cofinal in the modulo finite ordering.*

For an elementary proof of this lemma see [J2]. Fix $\langle \lambda_i : i < \omega \rangle$ converging to λ as in the lemma. Without loss of generality $crit(j) < \lambda_i < \lambda$ for all $i < \omega$. Fix a sequence $G = \langle g_\alpha : \alpha < \lambda^+ \rangle \subset \prod_{i<\omega} \lambda_i$ increasing and cofinal in the modulo finite ordering of $\prod_{i<\omega} \lambda_i$, as in the lemma. By elementarity, $M \models$"$j(G)$ is modulo finite increasing and cofinal in $j(\prod_{i<\omega} \lambda_i)$". Define $g \in j(\prod_{i<\omega} \lambda_i) \cap M$ by $g(i) = sup(j''\lambda \cap j(\lambda_i)) = sup(j''\lambda_i) < j(\lambda_i)$. The last inequality follows from regularity of $j(\lambda_i)$ in M. If $f \in \prod_{i<\omega} \lambda_i$, then $g > j(f)$ pointwise as $j(f) = j \circ f$. Now $j''\lambda^+$ is cofinal in λ^+ since λ^+ is fixed by j. Therefore $j''G$ is cofinal in $j(G)$ and thus in $j(\prod_{i<\omega} \lambda_i)$ modulo finite. However, we have just seen that $g \in j(\prod_{i<\omega} \lambda_i)$ pointwise dominates every member of $j''G$, contradiction.

Received by the editors November 14, 1994 and, in revised form, January 20, 1995.
1991 *Mathematics Subject Classification*. Primary 03E55.

Our proof is similar to the previous ones in that it makes heavy use of the Axiom of Choice (in this case hidden in the proof of the lemma) and it provides for example that there is no nontrivial elementary embedding of $V_{\lambda+2}$ into itself, for an arbitrary λ.

References

[B] D. Burke, *Splitting stationary sets*, preprint.

[H] M. Harada, *Another proof for Kunen's theorem*, preprint.

[J1] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[J2] T. Jech, *On the cofinality of countable products of cardinal numbers*, A Tribute to P. Erdős (A. Baker, N. Bollobás and A. Hajnal, eds.), Cambridge University Press, Cambridge, 1990, pp. 289–306. MR **92m:**03083

[K] K. Kunen, *Elementary embeddings and infinitary combinatorics*, J. Symbolic Logic **36** (1971), 407–413. MR **47:**40

[S] S. Shelah, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, Clarendon Press, Oxford, 1994.

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Current address: M.S.R.I., 1000 Centennial Dr., Berkeley, California 94720

E-mail address: `jindra@msri.org`