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THE EULER CHARACTERISTIC IS STABLE UNDER COMPACT PERTURBATIONS

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ABSTRACT. We prove in the general case the stability under compact perturbations of the index (i.e. the Euler characteristic) of a Fredholm complex of Banach spaces. In particular, we obtain the corresponding stability property for Fredholm multioperators. These results are the consequence of a similar statement, concerning more general objects called Fredholm pairs.

1. Introduction

It is known that if the sequence

$$\alpha = (\alpha^p)_p : \quad 0 \to X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{n-1}} X^n \to 0$$

is a Fredholm complex of Banach spaces (Definition 1), consisting of bounded operators $\alpha^p : X^p \to X^{p+1}$, and if $\beta = (\beta^p)_p$ is another complex such that $\beta^p - \alpha^p : X^p \to X^{p+1}$ are compact operators, then β is Fredholm, too ([15], [2]; see also [9] for a generalization). It was conjectured that in this case α and β have the same index, but no proof was known in general. This problem was raised in [15] and received positive answers in several particular cases: for complexes of Hilbert spaces [16], for short complexes (with three spaces) [10], for complexes satisfying the hypothesis that $(\beta^{p+1} - \alpha^{p+1})(\beta^p - \alpha^p)$ are of finite rank [15], for complexes consisting of Banach spaces with the approximation property [4], and for complexes which split modulo compact operators [12] (this last assumption is equivalent to the existence of closed linear complements for all ranges and null-spaces of α^p, β^p [8]). See also [1], [6], [7], [13] for other related problems.

If $(T_1, \dots, T_n)$ and $(U_1, \dots, U_n)$ are two commuting multioperators on a Banach space X, such that one of them is Fredholm (Definition 3) and $U_i - T_i$ are compact, then the other is Fredholm, too [15]. A particular case of the previous problem is to prove that they have the same index. The above mentioned results provide solutions in the corresponding particular cases: if X is a Hilbert space [16], if $n = 2$ [10], if all the $2n$ operators $U_1, \dots, T_n$ commute (more generally, if the commutators $U_i T_j - T_j U_i$ are of finite rank) [15] etc.

By Theorems 5 and 8 we can now give an affirmative answer, in the general case, to these problems. This is possible via some results of stability under small or compact perturbations, concerning Fredholm pairs (S, T) (Definition 2). A particular

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case of this notion (namely for $ST = 0$, $TS = 0$) was introduced in [2]; see also [3], [4]. Most notation is similar to that in [2], [15].

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2. Main results

Let $\mathcal{G}(X)$ be the set of all closed linear subspaces of a Banach space X. If $Y \in \mathcal{G}(X)$ and $x \in X$, then $d(x, Y)$ is the distance from x to Y. The product of two Banach spaces X, Y is endowed with the norm $\|(x, y)\| := (\|x\|^2 + \|y\|^2)^{1/2}$, $x \in X$, $y \in Y$. If S is a closed operator from X to Y, then we denote by $D(S)$, $N(S)$, $R(S)$, $G(S)$ and $\gamma(S)$ its domain, null-space, range, graph and reduced minimum modulus, respectively. We denote respectively by $\mathcal{C}(X, Y)$, $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ the set of all closed, bounded and compact operators; the notations $\mathcal{B}(X)$, $\mathcal{K}(X)$ are used, too.

In Definition 1 we state a particular case of the corresponding notions defined in [2].

Definition 1 [2]**.** A family of operators $\alpha = (\alpha^p)_{p \in \mathbb{Z}}$ such that $\alpha^p \in \mathcal{C}(X^p, X^{p+1})$ and $R(\alpha^{p-1}) \subset N(\alpha^p)$ for all $p \in \mathbb{Z}$, where X^p are Banach spaces, will be called a *complex (of Banach spaces and closed operators)*. The complex α is called *of finite length* if $X^p \neq \{0\}$ only for a finite number of p. Set $H^p(\alpha) := N(\alpha^p)/R(\alpha^{p-1})$. The complex α is called *Fredholm* if $\inf_p \gamma(\alpha^p) > 0$ and the function

$$\mathbb{Z} \ni p \mapsto \dim H^p(\alpha) \in \mathbb{Z}_+ \cup \{\infty\}$$

is finite and has finite support. In this case, the number

$$\text{ind } \alpha := \sum_p (-1)^p \dim H^p(\alpha)$$

is called the *Euler characteristic* (or the *index*) of α.

Definition 2. Let X, Y be Banach spaces and let $S \in \mathcal{C}(X, Y)$, $T \in \mathcal{C}(Y, X)$. Then (S, T) is called a *Fredholm pair* if the following dimensions are finite:

$$a := \dim N(S)/N(S) \cap R(T),\ b := \dim R(T)/N(S) \cap R(T),$$

$$c := \dim N(T)/N(T) \cap R(S),\ d := \dim R(S)/N(T) \cap R(S).$$

In this case, the number

$$\text{ind } (S, T) := a - b - c + d$$

is called the *index* of (S, T). We denote by $\mathcal{F}(X, Y)$ the set of all such Fredholm pairs.

If $(S, T) \in \mathcal{F}(X, Y)$, then $R(T)$ (resp. $R(S)$) is contained in the sum of $N(S)$ (resp. $N(T)$) and a finite-dimensional space, and so it is closed. Namely, the space $R(T)$ for instance is closed since it is the range of a closed operator and has finite codimension a in the space $R(T) + N(S)$, which is closed since it is the sum of $N(S) \in \mathcal{G}(X)$ and a space of finite dimension b.

Any Fredholm complex can be associated with a Fredholm pair via Theorem 1, which is proved in [1] in a more general form.

Theorem 1 [1]. *If $\alpha = (\alpha^p)_p$ is a complex, then there are $S \in \mathcal{C}(X,Y)$, $T \in \mathcal{C}(Y,X)$ with the following properties:*

(a) $R(S) \subset N(T)$, $R(T) \subset N(S)$.

(b) $\gamma(\alpha) = \min\{\gamma(S), \gamma(T)\}$.

(c) *α is Fredholm if and only if $\gamma(S)$, $\gamma(T) > 0$ and $N(S)/R(T)$, $N(T)/R(S)$ are finite-dimensional. In this case,* $\operatorname{ind} \alpha = \dim N(S)/R(T) - \dim N(T)/R(S)$.

Proof. We give the definition of S, T for later use. We have $\alpha^p \in \mathcal{C}(X^p, X^{p+1})$ for some Banach spaces X^p, $p \in \mathbb{Z}$. Set

$$X := \{x = (x_{2k})_{k\in\mathbb{Z}} \in \textstyle\prod_k X^{2k};\ \|x\|^2 := \sum_k \|x_{2k}\|^2 < \infty\},$$
$$Y := \{y = (x_{2k+1})_{k\in\mathbb{Z}} \in \textstyle\prod_k X^{2k+1};\ \|y\|^2 := \sum_k \|x_{2k+1}\|^2 < \infty\};$$

and define the operators $S \in \mathcal{C}(X,Y)$, $T \in \mathcal{C}(Y,X)$ with the domains

$$D(S) := \{(x_{2k})_k \in X;\ x_{2k} \in D(\alpha^{2k}), (\alpha^{2k}x_{2k})_k \in Y\},$$
$$D(T) := \{(x_{2k+1})_k \in Y;\ x_{2k+1} \in D(\alpha^{2k+1}), (\alpha^{2k+1}x_{2k+1})_k \in X\}$$

by the equalities

$$S(x_{2k})_k := (\alpha^{2k}x_{2k})_k, \quad T(x_{2k+1})_k := (\alpha^{2k+1}x_{2k+1})_k. \tag{1}$$

Now (a), (b) and (c) can be easily verified. We omit the details. □

If the complex $\alpha = (\alpha^p)_p$ in Theorem 1 is of finite length and consists of (bounded) operators α^p with $D(\alpha^p) \in \mathcal{G}(X^p)$, then from (1) we obtain $S \in \mathcal{B}(D(S),Y)$, $T \in \mathcal{B}(D(T),X)$ with $D(S) \in \mathcal{G}(X)$, $D(T) \in \mathcal{G}(Y)$. For the sake of simplicity, from now on we consider only complexes and pairs of bounded operators, except for Theorems 6 and 7.

Theorem 2. *Let $S_j \in \mathcal{C}(E,F)$, $T_j \in \mathcal{C}(F,E)$, $j = 1,2$, be such that $D(S_2) = D(S_1)$, $D(T_2) = D(T_1)$ and* $\dim R(S_2 - S_1) < \infty$, $\dim R(T_2 - T_1) < \infty$. *If $(S_1,T_1) \in \mathcal{F}(E,F)$, then $(S_2,T_2) \in \mathcal{F}(E,F)$ and* $\operatorname{ind}\,(S_2,T_2) = \operatorname{ind}\,(S_1,T_1)$.

Proof. In the particular case $D(S_j) = E$, $D(T_j) = F$ this result is proved in [4]. Let $X := D(S_1) = D(S_2)$ and $Y := D(T_1) = D(T_2)$. Then X has finite codimension in $X + R(T_1) + R(T_2) \in \mathcal{G}(F)$. Indeed, since (S_1,T_1) is a Fredholm pair, then $R(T_1) \subset N(S_1) + B$ with $\dim B < \infty$. Since $N(S_1) \subset X$, it follows that $R(T_1)$ is contained in the sum of X and a finite-dimensional space. Since $T_2 - T_1$ has finite rank, the same holds for $R(T_2)$, as well as for $R(T_1) + R(T_2)$.

Similarly, we can see that Y has finite codimension in $Y + R(S_1) + R(S_2)$.

There exist some finite-dimensional spaces $C \subset E$, $D \subset F$ such that $C \cap X = \{0\}$, $D \cap Y = \{0\}$ and

$$X + R(T_1) + R(T_2) = X + C, \ \ Y + R(S_1) + R(S_2) = Y + D.$$

Then we define the spaces

$$L := (X + C) \times D, \ \ M := (Y + D) \times C$$

and the operators

$$u_j : L \to M, \ \ v_j : M \to L, \ \ j = 1,2,$$

by the equalities

$$u_j(x + c, d) := (S_j x, c), \ \ v_j(y + d, c) := (T_j y, d)$$

for all $x \in X$, $y \in Y$, $c \in C$, $d \in D$. Hence L, M and (u_j, v_j), $j = 1, 2$, satisfy the same hypothesis as E, F and (S_j, T_j), $j = 1, 2$, respectively. In addition, we have $D(u_j) = L$, $D(v_j) = M$. In this case, we may apply [4, Proposition 2.3], whence we obtain that $(u_2, v_2) \in \mathcal{F}(L, M)$ and

$$\text{ind }(u_1, v_1) = \text{ind }(u_2, v_2). \tag{2}$$

We also have the equalities

$$\text{ind }(u_j, v_j) = \text{ind }(S_j, T_j),\ j = 1, 2. \tag{3}$$

Indeed, since $N(u_j) = N(S_j) \times D$ and $R(v_j) = R(T_j) \times D$, then

$$N(u_j) \cap R(v_j) = (N(S_j) \cap R(T_j)) \times D$$

and

$$N(u_j)/N(u_j)\cap R(v_j) = (N(S_j)\times D)/((N(S_j)\cap R(T_j))\times D) \cong N(S_j)/N(S_j)\cap R(T_j),$$

and similarly one computes the other three terms which appear in the definition of the index. From (2) and (3), we obtain the conclusion of the theorem. □

Theorem 3. *For any $(S, T) \in \mathcal{F}(X, Y)$ there exists $\delta > 0$ such that if $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(X, Y)$ with $D(\widetilde{S}) = D(S)$, $D(\widetilde{T}) = D(T)$ and $\|\widetilde{S} - S\| < \delta$, $\|\widetilde{T} - T\| < \delta$, then* ind $(\widetilde{S}, \widetilde{T})$ = ind (S, T).

The proof of Theorem 3 follows the lines of the proof of [4, Theorem 3.1] (where the result is stated in the particular case $D(S) = X$, $D(T) = Y$) or of [5, Theorem 3.20] (a version in which the domains also are perturbed with respect to the gap topology).

Lemma 1. *If X, Y are Banach spaces and $K \in \mathcal{K}(X, Y)$, then the closure of $R(K)$ is a separable Banach space.*

Proof. Let X_1, Y_1 be the unit balls of X, Y respectively. Since the set $\overline{K(X_1)}$ is compact, then for any $n \geq 1$ it contains a finite set of elements y_{nj}, $j = 1, \ldots, N_n$, such that

$$\overline{K(X_1)} \subset \bigcup_{j=1}^{N_n} (y_{nj} + n^{-1} Y_1).$$

Hence the set $A := \{y_{nj}; n, j\}$ is dense in $\overline{K(X_1)}$. It follows that $\mathbb{Q}A := \{ra; r \in \mathbb{Q}, a \in A\}$ is dense in $\overline{R(K)}$ (where $\mathbb{Q}$ is the set of the rational numbers). Indeed, for any $y \in \overline{R(K)}$ with $y \neq 0$ there exists a sequence $x_n \in X$, $n \geq 1$, such that $Kx_n \to y$. We may assume $x_n \neq 0$. For each n we choose $r_n \in \mathbb{Q}$ such that $|\|x_n\| - r_n| < n^{-1}$ (we may also assume $r_n > 0$). Take a sequence $a_n \in A$, $n \geq 1$, such that

$$\|K(\|x_n\|^{-1} x_n) - a_n\| < (n r_n)^{-1}.$$

Then $r_n a_n \to y$, since

$$r_n a_n = K x_n - (\|x_n\| - r_n) K(\|x_n\|^{-1} x_n) - (r_n K(\|x_n\|^{-1} x_n) - r_n a_n)$$

and we have $Kx_n \to y$, as well as $(\|x_n\| - r_n)K(\|x_n\|^{-1}x_n) \to 0$ and

$$\|r_n K(\|x_n\|^{-1}x_n) - r_n a_n\| = r_n \|K(\|x_n\|^{-1}x_n) - a_n\| \leq r_n(nr_n)^{-1} = n^{-1} \to 0.$$

So, the countable set $\mathbb{Q}A$ is dense in $\overline{R(K)}$ and the lemma is proved. $\square$

The main result of stability under compact perturbations (Theorem 4) is proved via Theorems 2 and 3. To this aim, we consider a compact perturbation $(S+K, T+L)$ of $(S,T) \in \mathcal{F}(X,Y)$ and try to approximate K, L by finite-rank operators, which is not always possible in general Banach spaces. This holds, for instance, when we deal with Banach spaces with the approximation property. Since $\overline{R(K)}$ is separable, then it can be embedded by $i : \overline{R(K)} \to C[0,1]$ into the space $C[0,1]$ of all continuous functions on the interval $[0,1]$, endowed with the sup-norm. Note that $C[0,1]$ has the approximation property. However, $D(T) \subset Y$ and the inclusion $\overline{R(K)} \hookrightarrow Y$ also holds. Then we shall embed both $D(T)$ and $\overline{R(K)}$ in a single Banach space F containing Y and $C[0,1]$ and in which we identify any $y \in \overline{R(K)}$ with $iy \in C[0,1]$. Strictly speaking, K will be approximated by finite-rank operators which take values in the space $Y \times C[0,1]$ factorized through the equivalence relation $(y,0) \sim (0,iy)$, $y \in \overline{R(K)}$, namely in $F := (Y \times C[0,1])/G(-i)$.

Theorem 4. *Let X, Y be Banach spaces and let $(S_j, T_j) \in \mathcal{F}(X,Y)$ with $D(S_j) = X$, $D(T_j) = Y$, $j = 1,2$. If $S_2 - S_1 \in \mathcal{K}(X,Y)$ and $T_2 - T_1 \in \mathcal{K}(Y,X)$, then* $\operatorname{ind}(S_1,T_1) = \operatorname{ind}(S_2,T_2)$.

Proof. Set $K := S_2 - S_1$, $L := T_2 - T_1$ and $(S,T) := (S_1,T_1)$. By Lemma 1, $\overline{R(K)}$ and $\overline{R(L)}$ are separable Banach spaces. According to a classical result (see for instance [11]), there exist some isometrical embeddings $i : \overline{R(K)} \to C$, $j : \overline{R(L)} \to C$, where $C := C[0,1]$. Then

$$G(-i) \in \mathcal{G}(\overline{R(K)} \times C) \subset \mathcal{G}(Y \times C),\ G(-j) \in \mathcal{G}(\overline{R(L)} \times C) \subset \mathcal{G}(X \times C).$$

We define the Banach spaces

$$E := (X \times C)/G(-j)\,,\ F := (Y \times C)/G(-i)$$

and let $\pi : X \times C \to E$, $\lambda : Y \times C \to F$ denote the canonical mappings. Let $\alpha \in \mathcal{B}(X,E)$ and $\beta \in \mathcal{B}(Y,F)$ be defined by $\alpha x := \pi(x,0)$, $x \in X$, and $\beta y := \lambda(y,0)$, $y \in Y$, respectively. Let $X' := R(\alpha)$ and $Y' := R(\beta)$. There exists a finite constant $c > 0$ such that

$$\|\alpha x\| \geq c\,\|x\|,\ x \in X. \tag{4}$$

Otherwise, we can find a sequence $(x_n)_n$ such that $x_n \in X$, $\|x_n\| = 1$ and $\alpha x_n \to 0$. Hence $d((x_n,0), G(-j)) \to 0$ and so there are $z_n \in \overline{R(L)}$, such that $\|(x_n,0) - (z_n, -jz_n))\| \to 0$, whence we obtain $\|x_n - z_n\| \to 0$ and $\|jz_n\| \to 0$. Since j is an isometry, it follows that $x_n \to 0$, which is false. Hence (4) holds.

From (4) it follows that $\gamma(\alpha) > 0$, and so $X' \in \mathcal{G}(E)$. Similarly, we have $Y' \in \mathcal{G}(F)$.

We introduce the following notation: for any $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}(Y,X)$, let $A' \in \mathcal{B}(X',Y')$ and $B' \in \mathcal{B}(Y',X')$ be defined by $A' := \beta A \alpha^{-1}$ and $B' := \alpha B \beta^{-1}$

respectively. Note that A', B' are similar to A, B respectively, via α and β. By an elementary algebraic computation, this implies that

$$(S', T') \in \mathcal{F}(E, F)\,,\ \text{ind}\ (S', T') = \text{ind}\ (S, T)$$

and

$$((S+K)', (T+L)') \in \mathcal{F}(E, F)\,,\ \text{ind}\ ((S+K)', (T+L)') = \text{ind}\ (S+K, T+L).$$

We obviously have $(S+K)' = S' + K'$ and $(T+L)' = T' + L'$. Therefore, in order to prove the conclusion of the theorem (i.e. the equality ind $(S+K, T+L) =$ ind (S, T)) it suffices to prove that

$$\text{ind}\ (S' + K', T' + L') = \text{ind}\ (S', T'). \tag{5}$$

Note that C has the approximation property since it has a Schauder basis (see for instance [11]). Hence for $iK \in \mathcal{K}(X, C)$ there exists a sequence $U_n \in \mathcal{B}(X, C)$ such that $\dim R(U_n) < \infty$ and

$$\|iK - U_n\| \to 0,\ n \to \infty. \tag{6}$$

Similarly, we can find $V_n \in \mathcal{B}(Y, C)$ such that $\dim R(V_n) < \infty$ and

$$\|jL - V_n\| \to 0,\ n \to \infty. \tag{7}$$

For any $f \in C$, set $\gamma \in \mathcal{B}(C, F)$, $\gamma f := \lambda(0, f)$ and $\delta \in \mathcal{B}(C, E)$, $\delta f := \pi(0, f)$. We define the finite-rank operators

$$U_n' := \gamma U_n \alpha^{-1} \in \mathcal{B}(X', F)\,,\ V_n' := \delta V_n \beta^{-1} \in \mathcal{B}(Y', E).$$

By Theorem 2, it follows that $(S' + U_n', T' + V_n') \in \mathcal{F}(E, F)$ and

$$\text{ind}\ (S' + U_n', T' + V_n') = \text{ind}\ (S', T'),\ n \geq 1. \tag{8}$$

Let us prove the inequalities

$$\|K' - U_n'\| \leq 2\|iK - U_n\|,\ \ \|L' - V_n'\| \leq 2\|jL - V_n\|. \tag{9}$$

To verify for instance the first estimate from (9), let $x' \in X'$ be arbitrary with $\|x'\| < 1$. Take $x \in X$ such that $x' = \alpha x = \pi(x, 0)$. Since $\|x'\| < 1$, then $d((x, 0), G(-j)) < 1$. Hence there exists $x_0 \in \overline{R(L)}$ such that $\|(x, 0) - (x_0, -jx_0)\| < 1$, i.e. $\|x - x_0\|^2 + \|jx_0\|^2 < 1$. Since j is an isometry, it follows that $\|x_0\| < 1$, $\|x - x_0\| < 1$, and so $\|x\| < 2$. Therefore, by the equalities

$$(K' - U_n')x' = \beta K \alpha^{-1} \alpha x - \gamma U_n \alpha^{-1} \alpha x$$
$$= \beta K x - \gamma U_n x = \lambda(Kx, 0) - \lambda(0, U_n x) = \lambda(Kx, -U_n x)$$

we obtain the estimates

$$\|(K' - U_n')x'\| = d((Kx, -U_n x), G(-i)) \leq \|(Kx, -U_n x) - (Kx, -iKx)\|$$
$$= \|(0, iKx - U_n x)\| = \|(iK - U_n)x\| \leq \|iK - U_n\|\,\|x\| \leq 2\|iK - U_n\|.$$

If we now take the supremum over x', then we obtain the desired estimate. The inequality $\|L' - V_n'\| \leq 2\|jL - V_n\|$ holds similarly, and so we have (9).

If $n \geq 1$ is sufficiently large, then we have

$$\text{ind}\ (S' + U_n', T' + V_n') = \text{ind}\ (S' + K', T' + L'), \tag{10}$$

via (6), (7), (9) and Theorem 3.

By (8) and (10), we obtain (5) and the theorem is proved. $\square$

Theorem 5. *Let X^p, $p \in \mathbb{Z}$, be Banach spaces and let $\alpha = (\alpha^p)_p$, $\beta = (\beta^p)_p$ be complexes of bounded operators and of finite length such that $\alpha^p, \beta^p \in \mathcal{B}(X^p, X^{p+1})$. If α is Fredholm and $\beta^p - \alpha^p \in \mathcal{K}(X^p, X^{p+1})$, $p \in \mathbb{Z}$, then β is Fredholm and* $\operatorname{ind} \beta = \operatorname{ind} \alpha$.

Proof. If a complex β is a compact perturbation of a Fredholm complex α as above, then β is Fredholm, too [15]. By the construction in the proof of Theorem 1, the Fredholm complexes α, β can be respectively associated with some Fredholm pairs (S, T), $(S', T') \in \mathcal{F}(X, Y)$ such that $D(S) = D(S') = X$, $D(T) = D(T') = Y$, where $X = \prod_p X^{2p}$ and $Y = \prod_p X^{2p+1}$. Moreover, by (c) we have

$$\operatorname{ind}(S, T) = \operatorname{ind} \alpha, \ \operatorname{ind}(S', T') = \operatorname{ind} \beta. \tag{11}$$

Since $\beta^p - \alpha^p \in \mathcal{K}(X^p, X^{p+1})$, $p \in \mathbb{Z}$, then $S' - S \in \mathcal{K}(X, Y)$ and $T' - T \in \mathcal{K}(Y, X)$. According to Theorem 4, it follows that

$$\operatorname{ind}(S', T') = \operatorname{ind}(S, T). \tag{12}$$

From (11) and (12), we obtain the desired conclusion. □

Lemma 2 [9]. *Let $S \in \mathcal{B}(X, Y)$ and $T \in \mathcal{B}(Y, Z)$ be such that $TS = 0$. We have $\dim N(T)/R(S) < \infty$ and $R(T)$ closed if and only if for every bounded sequence $(y_n)_n \subset Y$ such that $(Ty_n)_n$ is totally bounded in Z, we can find a bounded sequence $(x_n)_n \subset X$ such that $(y_n - Sx_n)_n$ is totally bounded.*

The following version of Theorem 4 is concerned with a particular case of Fredholm pairs of closed (not necessarily bounded) operators.

Theorem 6. *Let $S_j \in \mathcal{C}(E, F)$, $T_j \in \mathcal{C}(F, E)$ be such that $R(S_j) \subset N(T_j)$, $R(T_j) \subset N(S_j)$, $j = 1, 2$, and $D(S_1) = D(S_2)$, $D(T_1) = D(T_2)$. If $(S_1, T_1) \in \mathcal{F}(E, F)$ and the operators*

$$K := S_2 - S_1 : D(S_1) \to D(T_1), \ L := T_2 - T_1 : D(T_1) \to D(S_1)$$

extend to some compact operators $\overline{K} \in \mathcal{K}(\overline{D(S_1)}, F)$, $\overline{L} \in \mathcal{K}(\overline{D(T_1)}, E)$ respectively, then $(S_2, T_2) \in \mathcal{F}(E, F)$ and $\operatorname{ind}(S_2, T_2) = \operatorname{ind}(S_1, T_1)$.

Proof. Set $S := S_1$ and $T := T_1$. Let X be the space $D(S)$ endowed with the graph norm $| x |^2 := \|x\|^2 + \|Sx\|^2$, $x \in X$. Since S is closed, then X is complete. Let Y be the (Banach) space $D(T)$ endowed with the norm $| y |^2 := \|y\|^2 + \|Ty\|^2$, $y \in Y$. Then $SX \subset Y$ and $S \in \mathcal{B}(X, Y)$, since

$$| Sx |^2 = \|Sx\|^2 + \|TSx\|^2 = \|Sx\|^2 \leq \|x\|^2 + \|Sx\|^2 = | x |^2, \ x \in X.$$

Similarly, we obtain $T \in \mathcal{B}(Y, X)$.

We prove now that $K \in \mathcal{K}(X, Y)$, $L \in \mathcal{K}(Y, X)$. Let $x_n \in X$, $n \geq 1$, with $(| x_n |)_n$ bounded. Then $\|x_n\|$, $\|Sx_n\|$ are bounded. Since K is compact with respect to the norms on E, F, then the sequence Kx_n contains a subsequence which is convergent in F. We may denote it also by Kx_n. Hence there is $y \in F$ with $Kx_n \to y$ in F, and so $y \in \overline{R(K)} \subset \overline{D(T)}$. Since $\overline{L}$ is bounded, then we obtain $LKx_n \to \overline{L}y$. Afterwards, since $\|Sx_n\|$ is bounded, then by considering again a subsequence we may assume that LSx_n is convergent in E. Then the equality

$$TKx_n = -LSx_n - LKx_n$$

obtained via $T_2S_2 = 0$ implies that TKx_n is convergent in E. Together with $Kx_n \to y$, this implies $y \in D(T)$ and $TKx_n \to Ty$, since T is closed. Hence

$$| Kx_n - y |^2 = \|Kx_n - y\|^2 + \|T(Kx_n - y)\|^2 \to 0,$$

and so we proved that the sequence Kx_n contains a subsequence which is convergent in the norm $|\cdot|$, for any sequence x_n which is bounded in the norm $|\cdot|$. Therefore, $K \in \mathcal{K}(X, Y)$ (and similarly one obtains $L \in \mathcal{K}(Y, X)$).

Since $(S, T) \in \mathcal{F}(X, Y)$, then by Lemma 2 it follows that $(S + K, T + L) \in \mathcal{F}(X, Y)$, and so $(S + K, T + L) \in \mathcal{F}(E, F)$. Indeed, the characterization of the Fredholmness of a pair in the terms of Lemma 2 is invariant under compact perturbations. To see this, note that compact operators take bounded sequences into totally bounded sequences, the sum of two totally bounded sequences is totally bounded, etc.

By Theorem 4, applied to the pairs (S, T) and $(S + K, T + L)$ from $\mathcal{F}(X, Y)$, it follows that ind $(S + K, T + L) =$ ind (S, T), and so the theorem is proved. $\square$

Prof. F.-H. Vasilescu pointed out to me that versions of Theorem 5 could be stated in the more general context of Definition 1 if it is assumed that $\beta^p - \alpha^p \to 0$ as $| p | \to \infty$, the convergence and the compactness of $\beta^p - \alpha^p$ being considered with respect to either the norm topology, or the graph norm induced by α^p on $D(\alpha^p)$. Via Theorem 6, one obtains Theorem 7 which is an example in this sense.

Theorem 7. *Let $\alpha = (\alpha^p)_p$, $\beta = (\beta^p)_p$ be complexes of Banach spaces and closed operators (Definition 1) such that $D(\alpha^p) = D(\beta^p) \subset X^p$, $p \in \mathbb{Z}$. If α is a Fredholm complex, all $\beta^p - \alpha^p$ are compact with respect to the norms on X^p, $p \in \mathbb{Z}$, and $\|\beta^p - \alpha^p\| \to 0$ as $| p | \to \infty$, then β is Fredholm, too and* ind $\beta =$ ind α.

Proof. We can apply Theorem 6 to the pairs obtained from α and β by the construction in the proof of Theorem 1. Let $\theta^p \in \mathcal{K}(\overline{D(\alpha^p)}, X^{p+1})$ be the extension of $\beta^p - \alpha^p$, $p \in \mathbb{Z}$. The compact extensions $\overline{K}$ and $\overline{L}$ are direct sums of the operators θ^{2p}, $p \in \mathbb{Z}$, and θ^{2p+1}, $p \in \mathbb{Z}$, respectively. We omit the details. $\square$

To state the classical definition of the Koszul complex (see for instance [14]), fix $n \geq 1$ and denote by Λ the *exterior algebra* with n generators, namely the algebra generated by n elements $\sigma_1, \ldots, \sigma_n$ with $\sigma_i \wedge \sigma_j = -\sigma_j \wedge \sigma_i$, $i, j = 1, \ldots, n$. Let $\Lambda^p \subset \Lambda$ be the linear space of the homogeneous p-forms. We define $S_j : \Lambda \to \Lambda$, $j = 1, \ldots, n$, by $S_j\xi := \sigma_j \wedge \xi$.

A *(commuting) multioperator* on the Banach space X is a family

$$T = (T_1, \ldots, T_n) \in \mathcal{B}(X)^n$$

such that $T_iT_j = T_jT_i$, $i, j = 1, \ldots, n$.

Definition 3. The *Koszul complex* of a multioperator $T = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ is the complex $(\alpha_T^p)_p$ defined by

$$\alpha_T^p := \alpha_T | X \otimes \Lambda^p \to X \otimes \Lambda^{p+1},$$

where

$$\alpha_T := \sum_{j=1}^n T_j \otimes S_j : X \otimes \Lambda \to X \otimes \Lambda.$$

The multioperator T is called *Fredholm* if its Koszul complex is Fredholm. In this case, the number $\operatorname{ind} T := \operatorname{ind} (\alpha_T^p)_p$ is called the *index* of T.

Theorem 8. *Let $T = (T_1, \dots, T_n)$ and $U = (U_1, \dots, U_n)$ be two commuting multioperators on a Banach space X. If T is Fredholm and $U_j - T_j \in \mathcal{K}(X)$, $j = 1, \dots, n$, then U is Fredholm, too and* $\operatorname{ind} S = \operatorname{ind} T$.

Proof. Let $(\alpha_T^p)_p$ and $(\alpha_U^p)_p$ be the Koszul complexes associated with T and U, respectively. Since $U_j - T_j \in \mathcal{K}(X)$, $j = 1, \dots, n$, then

$$\alpha_U^p - \alpha_T^p = \sum_{j=1}^n (U_j - T_j) \otimes S_j \in \mathcal{K}(X \otimes \Lambda^p, X \otimes \Lambda^{p+1}),\ p = 0, \dots, n-1,$$

and so we obtain the desired conclusion via Theorem 5. □

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