

$\delta_{\sim 2}^1$ WITHOUT SHARPS

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ABSTRACT. We show that the supremum of the lengths of $\Delta_{\sim 2}^1$ prewellorderings of the reals can be ω_2 , with ω_1 inaccessible to reals, assuming only the consistency of an inaccessible.

$\delta_{\sim 2}^1$ denotes the supremum of the lengths of $\Delta_{\sim 2}^1$ prewellorderings of the reals. A result of Kunen and Martin (see Martin [77]) states that $\delta_{\sim 2}^1$ is at most ω_2 , and it is known that in the presence of sharps the assumption $\delta_{\sim 2}^1 = \omega_2$ is strong: it implies the consistency of a strong cardinal (see Steel-Welch [?]).

In this paper we show how to obtain the consistency of $\delta_{\sim 2}^1 = \omega_2$ in the absence of sharps, without strong assumptions.

Theorem. *Assume the consistency of an inaccessible. Then it is consistent that $\delta_{\sim 2}^1 = \omega_2$ and ω_1 is inaccessible to reals (i.e., $\omega_1^{L[x]}$ is countable for each real x).*

The proof is obtained by combining the Δ_1 -coding technique of Friedman-Velickovic [96] with the use of a product of Jensen codings of Friedman [94].

We begin with a description of the Δ_1 -coding technique.

Definitions. Suppose x is a set, $\langle x, \epsilon \rangle$ satisfies the axiom of extensionality and $A \subseteq \text{ORD}$. x preserves A if $\langle x, A \cap x \rangle \cong \langle \bar{x}, A \cap \bar{x} \rangle$ where \bar{x} = transitive collapse of x . For any ordinal δ , $x[\delta] = \{f(\gamma) \mid \gamma < \delta, f \in x, f \text{ a function}, \gamma \in \text{Dom}(f)\}$. x strongly preserves A if $x[\delta]$ preserves A for every cardinal δ . A sequence $x_0 \subseteq x_1 \subseteq \dots$ is tight if it is continuous and for each i , $\langle \bar{x}_j \mid j < i \rangle$ belongs to the least ZF^- -model which contains \bar{x}_i as an element and correctly computes $\text{card}(\bar{x}_i)$.

Condensation Condition for A. Suppose t is transitive, δ is regular, $\delta \in t$ and $x \in t$. Then:

- There exists a continuous, tight δ -sequence $x_0 \prec x_1 \prec \dots \prec t$ such that $\text{card}(x_i) = \delta$, $x \in x_0$ and x_i strongly preserves A , for each i .
- If δ is inaccessible, then there exist x_i 's as above but where $\text{card}(x_i) = \aleph_i$.

The following is proved in Friedman-Velickovic [96].

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Δ_1 -Coding. Suppose $V = L$ and the Condensation Condition holds for A . Then A is Δ_1 in a class-generic real R , preserving cardinals.

Now we are ready to begin the proof of the Theorem. Suppose κ is the least inaccessible and $V = L$. Let $\langle \alpha_i \mid i < \kappa^+ \rangle$ be the increasing list of all $\alpha \in (\kappa, \kappa^+)$ such that $L_\alpha = \text{Skolem hull}(\kappa)$ in L_α . For each $i < \kappa^+$ define $f_i : \kappa \rightarrow \kappa$ by $f_i(\gamma) = \text{ordertype}(ORD \cap \text{Skolem hull}(\gamma) \text{ in } L_{\alpha_i})$. By identifying f_i with its graph and using a pairing function we can think of f_i as a subset of κ . The following is straightforward.

Lemma 1. *Each f_i obeys the Condensation Condition. Indeed $\langle f_i \mid i < \kappa^+ \rangle$ jointly obeys the Condensation Condition in the following sense: Suppose t is transitive, δ is regular, $\delta \in t$, $x \in t$. Then there exists a tight δ -sequence $x_0 \prec x_1 \prec \dots \prec t$ such that $\text{card}(x_i) = \delta$, $x \in x_0$ and each x_i strongly preserves all f_j for $j \in x_i$ (and if $\delta = \kappa$, then we can alternatively require $\text{card}(x_i) = \aleph_i$).*

Now, following Friedman [94] we use a “diagonally-supported” product of Jensen-style codings. For each $i < \kappa^+$ let $\mathcal{P}(i)$ be the forcing from Friedman-Velickovic [96] to make f_i Δ_1 -definable in a class-generic real. Then \mathcal{P} consists of all $p \in \prod_{i < \kappa^+} \mathcal{P}(i)$ such that for infinite cardinals γ , $\{i \mid p(i)(\gamma) \neq (\phi, \phi)\}$ has cardinality at most γ and in addition $\{i \mid p(i)(0) \neq (\phi, \phi)\}$ is finite.

Now note that for successor cardinals $\gamma < \kappa$ the forcing \mathcal{P} factors as $\mathcal{P}_\gamma * \mathcal{P}^{G_\gamma}$ where \mathcal{P}_γ forces that \mathcal{P}^{G_γ} has the γ^+ -CC. Also the joint Condensation Condition of Lemma 1 implies that the argument of Theorem 3 of Friedman-Velickovic [96] can be applied here to show that \mathcal{P}_γ is $\leq \gamma$ -distributive, and also that \mathcal{P} is Δ -distributive (if $\langle D_i \mid i < \kappa \rangle$ is a sequence of predense sets, then it is dense to reduce each D_i below \aleph_{i+1}). So \mathcal{P} preserves cofinalities.

Thus in a cardinal-preserving forcing extension of L we have produced κ^+ reals $\langle R_i \mid i < \kappa^+ \rangle$ where R_i Δ_1 -codes f_i and hence there are wellorderings of κ of any length $< \kappa^+$ which are Δ_1 in a real. Finally Lévy collapse to make $\kappa = \omega_1$ and we have $\delta^1_{\sim_2} = \omega_2$, ω_1 inaccessible to reals. □

The above proof also shows the following, which may be of independent interest.

Theorem 2. *Let $\delta_{\sim_1}(\kappa)$ be the sup of the lengths of wellorderings of κ which are Δ_{\sim_1} over $L_\kappa[x]$ for some x , a bounded subset of κ . Then (relative to the consistency of an inaccessible) it is consistent that κ be weakly inaccessible and $\delta_{\sim_1}(\kappa) = \kappa^+$.*

Remark. The conclusion of Theorem 2 cannot hold in the context of sharps: if κ is weakly inaccessible and every bounded subset of κ has a sharp, then $\delta_{\sim_1}(\kappa) < \kappa^+$. This is because $\delta_{\sim_1}(\kappa)$ is then the second uniform indiscernible for bounded subsets of κ , which can be written as the direct limit of the second uniform indiscernible for subsets of δ , as δ ranges over cardinals less than κ ; so $\delta_{\sim_1}(\kappa)$ has cardinality κ .

Using the least inner model closed under sharp, we can also obtain the following.

Theorem 3. *Assuming it is consistent for every set to have a sharp, then this is also consistent with $\delta^1_{\sim_3} = \omega_2$.*

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