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PRODUCTS OF QUASI-MEASURES

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Abstract. A quasi-state is a positive functional on $C(X)$ that is only assumed to be linear on singly-generated subalgebras. We consider the "iterated integral" of two quasi-states and determine when this gives a quasi-state on the product space. We also provide explicit formulas for the corresponding quasi-measures in case it does. Finally, we show the general failure of Fubini's Theorem for quasi-states.

If X is a compact, Hausdorff space, we let $C(X)$ denote the collection of real-valued continuous functions on X. We let sp f denote the range of f. A quasi-state is a function $\rho : C(X) \to \mathbb{R}$ such that:

(i) If $f \geq 0$, then $\rho(f) \geq 0$.
(ii) $\rho(1) = 1$.
(iii) If $r \in \mathbb{R}$, then $\rho(rf) = r\rho(f)$.
(iv) If $\varphi, \psi \in C(\text{sp } f)$, then $\rho(\varphi \circ f + \psi \circ f) = \rho(\varphi \circ f) + \rho(\psi \circ f)$.

In [1], Aarnes answered the question of whether every quasi-state must be linear in the negative. He did this by establishing a correspondence between quasi-states and certain set functions. In particular, a function μ defined for the subsets of X that are either open or closed is a quasi-measure if:

a) $\mu(A) \geq 0$ for all A.
b) $A \subseteq B$ implies that $\mu(A) \leq \mu(B)$.
c) $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$.
d) If U is open, $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ closed }\}$.
e) $\mu(X) = 1$.

The primary difference between a quasi-measure and a finitely additive measure is that quasi-measures do not have to be subadditive. The quasi-state ρ corresponds to the quasi-measure μ if

$$\mu(U) = \sup\{\rho(f) : 0 \leq f \prec U\} \qquad \text{for } U \text{ open}$$

and

$$\mu(K) = \inf\{\rho(f) : K \prec f\} \qquad \text{for } K \text{ closed.}$$

Here, $f \prec U$ means that $0 \leq f \leq 1$ and the support of f is contained in U. Also, $K \prec f$ means that $f \geq 0$ and $f \geq 1$ on K. This construction is detailed in [1], where a particular example of a quasi-measure that is not a measure is given. Other examples may be found in [2, 3] and [5].

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There are some basic properties of quasi-states that we will need for this paper. In particular, we will require the fact that if f and g are such that $fg = 0$, then $\rho(f+g) = \rho(f) + \rho(g)$. We also use the fact that ρ is continuous on $C(X)$. These facts may be found in [1]. For notational convenience, we will write $\langle f, \rho\rangle = \rho(f)$.

We will call a quasi-state simple if $\langle \varphi \circ f, \rho\rangle = \varphi(\langle f, \rho\rangle)$ when $\varphi \in C(\text{sp } f)$. It is shown in [2] that the quasi-state ρ is simple if and only if the corresponding quasi-measure μ satisfies $\mu(A) \in \{0, 1\}$ for all A. We call such quasi-measures $\{0, 1\}$-quasi-measures.

From now on, we will let X and Y be compact, Hausdorff spaces, ρ a quasi-state on $C(X)$ with corresponding quasi-measure μ and η a quasi-state on $C(Y)$ with corresponding quasi-measure ν. We are interested in considering the functions on $C(X \times Y)$ obtained from "repeated integration". To do so, we need the following definitions.

Definition 1. For $f \in C(X \times Y)$, define $f^y(x) = f(x, y) = f_x(y)$. Then $f^y \in C(X)$,$f_x \in C(Y)$ and the functions $y \mapsto f^y$ and $x \mapsto f_x$ are continuous when $C(X)$ and $C(Y)$ are given the uniform topologies.

Define $T_\rho(f)(y) = \langle f^y, \rho\rangle$ and $S_\eta(f)(x) = \langle f_x, \eta\rangle$. By continuity of ρ and η, we see that $T_\rho(f) \in C(Y)$ and $S_\eta(f) \in C(X)$.

Finally, define $\rho\times_l\eta$ and $\rho\times_r\eta$ on $C(X \times Y)$ by $\langle f, \rho\times_l\eta\rangle = \langle T_\rho(f), \eta\rangle$ and $\langle f, \rho\times_r\eta\rangle = \langle S_\eta(f), \rho\rangle$.

Thus, $\rho\times_l\eta$ is obtained by first "integrating" with the use of ρ and then with η, while $\rho\times_r\eta$ integrates with η first and then ρ. There is a real difference between $\rho\times_l\eta$ and $\rho\times_r\eta$ as will be seen later. It is easily seen that $\rho\times_r\eta$ and $\rho\times_l\eta$ satisfy (i)–(iii) in the definition of quasi-state.

Proposition 1. *If $g \in C(X)$ and $h \in C(Y)$, let $g \otimes h(x, y) = g(x)h(y)$. Then*

$$\langle g \otimes h, \rho\times_l\eta\rangle = \langle g \otimes h, \rho\times_r\eta\rangle = \langle g, \rho\rangle\langle h, \eta\rangle.$$

Proof. In fact, $(g \otimes h)^y = h(y) \cdot g$, so $T_\rho(g \otimes h)(y) = h(y)\langle g, \rho\rangle$. Thus, $\langle g \otimes h, \rho\times_l\eta\rangle = \langle T_\rho(g \otimes h), \eta\rangle = \langle h\langle g, \rho\rangle\ , \eta\rangle = \langle g, \rho\rangle\langle h, \eta\rangle$. Similarly for the other equality. □

One fact that is perhaps surprising is that the functions $\rho\times_l\eta$ and $\rho\times_r\eta$ do not always give quasi-states. The exact situations when they do are given in the following theorem.

Theorem 1. *The function $\rho\times_l\eta$ is a quasi-state if and only if η is linear or ρ is simple.*

Proof. The only consideration is whether property (iv) in the definition of a quasi-state is satisfied. Let $f \in C(X \times Y)$ and $\varphi, \psi \in C(\text{sp } f)$. Then

$$\begin{aligned} T_\rho(\varphi \circ f + \psi \circ f)(y) &= \langle (\varphi \circ f + \psi \circ f)^y, \rho\rangle \\ &= \langle \varphi \circ f^y + \psi \circ f^y, \rho\rangle \\ &= \langle \varphi \circ f^y, \rho\rangle + \langle \psi \circ f^y, \rho\rangle \\ &= T_\rho(\varphi \circ f)(y) + T_\rho(\psi \circ f)(y). \end{aligned}$$

So $T_\rho(\varphi \circ f + \psi \circ f) = T_\rho(\varphi \circ f) + T_\rho(\psi \circ f)$.

If η is linear, we have

$$\begin{aligned}\langle \varphi\circ f+\psi\circ f,\rho\times_l\eta\rangle &= \langle T_\rho(\varphi\circ f+\psi\circ f),\eta\rangle\\ &= \langle T_\rho(\varphi\circ f)+T_\rho(\psi\circ f),\eta\rangle\\ &= \langle T_\rho(\varphi\circ f),\eta\rangle+\langle T_\rho(\psi\circ f),\eta\rangle\\ &= \langle \varphi\circ f,\rho\times_l\eta\rangle+\langle \psi\circ f,\rho\times_l\eta\rangle,\end{aligned}$$

so $\rho\times_l\eta$ is a quasi-state in this case.

If, instead, ρ is simple, we have that $T_\rho(\varphi\circ f)(y)=\langle\varphi\circ f^y,\rho\rangle=\varphi(\langle f^y,\rho\rangle)=\varphi\circ T_\rho(f)(y)$. So we have for this case, $T_\rho(\varphi\circ f+\psi\circ f)=\varphi\circ T_\rho(f)+\psi\circ T_\rho(f)$, which gives

$$\begin{aligned}\langle \varphi\circ f+\psi\circ f,\rho\times_l\eta\rangle &= \langle T_\rho(\varphi\circ f+\psi\circ f),\eta\rangle\\ &= \langle \varphi\circ T_\rho(f)+\psi\circ T_\rho(f),\eta\rangle\\ &= \langle \varphi\circ T_\rho(f),\eta\rangle+\langle \psi\circ T_\rho(f),\eta\rangle\\ &= \langle T_\rho(\varphi\circ f),\eta\rangle+\langle T_\rho(\psi\circ f),\eta\rangle\\ &= \langle \varphi\circ f,\rho\times_l\eta\rangle+\langle \psi\circ f,\rho\times_l\eta\rangle,\end{aligned}$$

so again, $\rho\times_l\eta$ is a quasi-state.

Conversely, assume that ρ is not simple, but that $\rho\times_l\eta$ is a quasi-state. We will show that η is linear.

Since ρ is not simple, the corresponding quasi-measure, μ, is not a $\{0,1\}$-quasi-measure. Let $A\subseteq X$ be closed with $0<\mu(A)<1$. Use inner regularity of μ on $X\setminus A$ to find a closed set B with $0<\mu(B)<1$ and B disjoint from A. Now pick two positive functions k_1 and k_2 such that $k_1k_2=0$, $A\prec k_1$ and $B\prec k_2$. Then $a=\langle k_1,\rho\rangle\neq 0$ and $b=\langle k_2,\rho\rangle\neq 0$. Define $f_1=k_1/a$ and $f_2=k_2/b$. Notice that $\langle f_1,\rho\rangle=\langle f_2,\rho\rangle=1$.

Now, for $g,h\in C(Y)$, we obtain

$$\begin{aligned}\langle g,\eta\rangle+\langle h,\eta\rangle &= \langle f_1,\rho\rangle\langle g,\eta\rangle+\langle f_2,\rho\rangle\langle h,\eta\rangle\\ &= \langle f_1\otimes g,\rho\times_l\eta\rangle+\langle f_2\otimes h,\rho\times_l\eta\rangle\\ &= \langle f_1\otimes g+f_2\otimes h,\rho\times_l\eta\rangle\\ &= \langle T_\rho(f_1\otimes g+f_2\otimes h),\eta\rangle,\end{aligned}$$

where we have used $(f_1\otimes g)(f_2\otimes h)=0$ and the fact that $\rho\times_l\eta$ is a quasi-state. But we check that $(f_1\otimes g+f_2\otimes h)^y=g(y)f_1+h(y)f_2$. Since $f_1f_2=0$, we have

$$\begin{aligned}T_\rho(f_1\otimes g+f_2\otimes h)(y) &= \langle (f_1\otimes g+f_2\otimes h)^y,\rho\rangle\\ &= \langle g(y)f_1+h(y)f_2,\rho\rangle\\ &= g(y)\langle f_1,\rho\rangle+h(y)\langle f_2,\rho\rangle\\ &= g(y)+h(y),\end{aligned}$$

so the previous calculation yields

$$\langle g+h,\eta\rangle=\langle g,\eta\rangle+\langle h,\eta\rangle.$$

This states the linearity of η. ☐

Corollary 1. *If ρ and η are both simple quasi-states, then $\rho\times_l\eta$ is also a simple quasi-state.*

Proof. If $f\in C(X\times Y)$ and $\varphi\in C(\operatorname{sp} f)$, we have that $T_\rho(\varphi\circ f)=\varphi\circ T_\rho(f)$, so that $\langle\varphi\circ f,\rho\times_l\eta\rangle=\langle\varphi\circ T_\rho(f),\eta\rangle=\varphi(\langle T_\rho(f),\eta\rangle)=\varphi(\langle f,\rho\times_l\eta\rangle)$. ☐

Thus, if μ is a quasi-measure on X and ν is a quasi-measure on Y, we may define a product quasi-measure $\mu\times_l\nu$ on $X \times Y$ if either ν is a measure or μ is a $\{0,1\}$-quasi-measure. If both μ and ν are $\{0,1\}$-quasi-measures, so is $\mu\times_l\nu$. It should be pointed out that there are analogous results to those above for the function $\rho\times_r\eta$, which will give a quasi-measure $\mu\times_r\nu$ if ν is a $\{0,1\}$-quasi-measure or if μ is a measure.

It is of interest to see how the quasi-measure $\mu\times_l\nu$ acts on sets when it is defined. This is the content of the next two results. The first is a generalization of a construction first considered in [4]. There, the relevant quasi-measure was obtained from a weak-* limit procedure. However, no description of how this quasi-measure acts on sets was given.

Theorem 2. *Let μ be a quasi-measure on X and ν a measure on Y. Then for A either open or closed in $X \times Y$, we have*

$$\mu\times_l\nu(A) = \int_Y \mu(A^y)\ d\nu(y)$$

where $A^y = \{x : (x,y) \in A\}$.

Proof. Claim 1. If $U \subseteq X \times Y$ is open, then the function $y \to \mu(U^y)$ is lower semi-continuous and so is ν-measurable.

Suppose that $\mu(U^{y_0}) > \alpha$. Pick $K \subseteq U^{y_0}$ compact such that $\mu(K) > \alpha$. Then $K \times \{y_0\} \subseteq U$, so there is a neighborhood V of y_0 such that $K \times V \subseteq U$. Then for $y \in V$, we have $K \subseteq U^y$, so $\alpha < \mu(U^y)$.

Claim 2. If $U \subseteq X \times Y$ is open, then $\mu\times_l\nu(U) \leq \int_Y \mu(U^y)\ d\nu(y)$.

If $f \prec U$, then $f^y \prec U^y$ for all $y \in Y$. Thus $\rho(f^y) \leq \mu(U^y)$, which gives $\rho\times_l\eta(f) = \int_Y \rho(f^y)\ d\nu(y) \leq \int_Y \mu(U^y)\ d\nu(y)$. Now use the fact that $\mu\times_l\nu(U)$ is the supremum of such $\rho\times_l\eta(f)$.

Claim 3. If $U \subseteq X \times Y$ is open, then $\mu\times_l\nu(U) = \int_Y \mu(U^y)\ d\nu(y)$.

Since the function $y \to \mu(U^y)$ is lower semi-continuous, we have that

$$\int_Y \mu(U^y)\ d\nu(y) = \sup\{\int_Y g\ d\nu : 0 \leq g(y) \leq \mu(U^y), g \in C(Y)\}.$$

Let $g \in C(Y)$ such that $0 \leq g(y) \leq \mu(U^y)$ for $y \in Y$. Let $\varepsilon > 0$. For each $y \in Y$, let $K_y \subseteq U^y$ be compact with $\mu(K_y) > g(y) - \varepsilon/2$. Then $K_y \times \{y\} \subseteq U$, so there is a neighborhood V_y of y such that $K \times \overline{V}_y \subseteq U$ and for $z \in V_y$ we have $|g(z) - g(y)| < \varepsilon/2$. Choose $V_{y_1}, \ldots, V_{y_n}$ that cover Y. Define

$$E = \bigcup_{i=1}^n K_{y_i} \times \overline{V}_{y_i} \subseteq U.$$

Then E is compact. Choose f so that $E \prec f \prec U$. Then, if $y \in Y$, say $y \in V_{y_i}$, we have $K_{y_i} \subseteq E^y \prec f^y$, so

$$g(y) - \varepsilon < g(y_i) - \varepsilon/2 \leq \mu(K_{y_i}) \leq \rho(f^y).$$

Thus

$$\int_Y g\ d\nu - \varepsilon \leq \int_Y \rho(f^y)\ d\nu(y) = \rho\times_l\eta(f) \leq \mu\times_l\nu(U).$$

Now let $\varepsilon \to 0$ to get $\int_Y g\ d\nu \leq \mu\times_l\nu(U)$. Since this happens with every g as above, we are done.

The case where A is closed in $X \times Y$ follows by taking complements. $\square$

The proof of the next result is very similar in conception to that of the previous theorem. The differences arise from the lack, as yet, of a suitable integration theory for lower semi-continuous functions with respect to a quasi-measure.

Theorem 3. *Let μ be a $\{0,1\}$-quasi-measure on X and ν a quasi-measure on Y. Construct $\mu\times_l\nu$ as above. Then for A either open or closed in $X\times Y$, we have*

$$\mu\times_l\nu(A)=\nu(\{y:\mu(A^y)=1\}).$$

Proof. For $A\subseteq X\times Y$ either open or closed, we define $B(A)=\{y:\mu(A^y)=1\}$. We use the notation of A^c for the complement of A. Notice that $B(A)^c=\{y:\mu(A^y)=0\}=\{y:\mu((A^y)^c)=1\}=\{y:\mu((A^c)^y=1\}=B(A^c)$.

Claim 1. If A is open, then $B(A)$ is open. If A is closed, $B(A)$ is closed.

It is enough to show this for A open. Suppose this is so and assume that $y_0\in B(A)$, i.e. $\mu(A^{y_0})=1$. Since A^{y_0} is open, and μ is a $\{0,1\}$-quasi-measure, there is a compact set $K\subseteq A^{y_0}$ such that $\mu(K)=1$. Then $K\times\{y_0\}\subseteq A$, so there are open sets $K\subseteq U\subseteq X$ and $y_0\in V\subseteq Y$ such that $U\times V\subseteq A$. Now, if y is in the neighborhood V of y_0, we have $K\subseteq U\subseteq A^y$, so $\mu(A^y)=1$, i.e. $V\subseteq B(A)$.

Claim 2. If A is closed, then $\nu(B(A))\le\mu\times_l\nu(A)$.

We have that $\mu\times_l\nu(A)=\inf\{\langle f,\rho\times_l\eta\rangle : A\prec f\}$. Suppose that $A\prec f$. Then $A^y\prec f^y$ for all $y\in Y$, so $\mu(A^y)\le\langle f^y,\rho\rangle=T_\rho(f)(y)$. Thus
$B(A)\prec T_\rho(f)$. This shows that $\nu(B(A))\le\langle T_\rho(f),\eta\rangle=\langle f,\rho\times_l\eta\rangle$. This gives the claim.

Claim 3. If A is open, then $\nu(B(A))\ge\mu\times_l\nu(A)$.

In fact, $\nu(B(A))=1-\nu(B(A)^c)=1-\nu(B(A^c))\ge1-\mu\times_l\nu(A^c)=\mu\times_l\nu(A)$.

Claim 4. If A is open, $\nu(B(A))=\mu\times_l\nu(A)$.

We use the fact that $\nu(B(A))=\sup\{\nu(K):K\subseteq B(A),\mathrm{K\ closed}\}$. Suppose that $K\subseteq B(A)$ is closed. For each $y\in K$, $\mu(A^y)=1$, so there is a compact set $C_y\subseteq A^y$ such that $\mu(C_y)=1$. Then $C_y\times\{y\}\subseteq A$, so there are open sets $C_y\subseteq U_y\subseteq X$ and $y\in W_y\subseteq Y$ such that $U_y\times\overline{W_y}\subseteq A$. Finitely many $W_{y_1},W_{y_2},\dots,W_{y_n}$ cover K.

Let $D=\bigcup_{i=1}^n C_{y_i}\times\overline{W_{y_i}}$. Then D is compact and $D\subseteq A$. Choose f such that $D\prec f\prec A$. Then for $y\in K$, say $y\in W_{y_i}$, we have $C_{y_i}\subseteq D^y\prec f^y$, so $1=\mu(C_{y_i})\le\langle f^y,\rho\rangle=T_\rho(f)(y)$. Thus $K\prec T_\rho(f)$. Finally, this shows that

$$\nu(K)\le\langle T_\rho(f),\eta\rangle=\langle f,\rho\times_l\eta\rangle\le\mu\times_l\nu(A).$$

If we take the supremum over $K\subseteq B(A)$ and use the previous claim, we are finished.

Claim 5. For A closed, $\nu(B(A))=\mu\times_l\nu(A)$.

This is now easy. □

Corollary 2. *Let μ be a quasi-measure on X and ν a quasi-measure on Y such that $\mu\times_l\nu$ is a quasi-measure. If $A\subseteq X$ and $B\subseteq Y$ are either both open or both closed, then $\mu\times_l\nu(A\times B)=\mu(A)\nu(B)$.*

We now turn to the question of when a version of Fubini's Theorem holds for quasi-measures and quasi-states. In other words, in what circumstances does $\rho\times_l\eta=\rho\times_r\eta$? If $\rho\times_l\eta$ is not a quasi-state, in other words, if ρ is not simple and η is not linear, we choose $f_1,f_2\in C(X)$ such that $f_1f_2=0$ and $1=\langle f_1,\rho\rangle=\langle f_2,\rho\rangle$ as in the proof of Theorem 1. Also pick $g, h \in C(Y)$ such that $\langle g+h, \eta\rangle \neq \langle g, \eta\rangle + \langle h, \eta\rangle$. If we let $k = f_1 \otimes g + f_2 \otimes h$, we see that $\langle k, \rho\times_l\eta\rangle = \langle g+h, \eta\rangle$, while $\langle k, \rho\times_r\eta\rangle = \langle g, \eta\rangle + \langle h, \eta\rangle$. Thus $\rho\times_l\eta \neq \rho\times_r\eta$ in this case.

If, however, $\rho\times_l\eta$ is a quasi-state, we would need for $\rho\times_r\eta$ to be one also. An enumeration of cases shows that this situation occurs only if one of μ or ν is a point-mass measure, when both of ρ and η are simple, or when both μ and ν are measures. In the last case, $\rho\times_l\eta = \rho\times_r\eta$ by Fubini's Theorem. If either μ or ν is a point-mass, an easy calculation shows that $\rho\times_l\eta = \rho\times_r\eta$. In the final case when ρ and η are both simple, but non-linear, we have the following.

Corollary 3. *If μ and ν are both $\{0,1\}$-quasi-measures that are not measures, then $\mu\times_l\nu \neq \mu\times_r\nu$.*

Proof. Since neither μ nor ν are measures, they must violate subadditivity. Thus we may find $A, C \subseteq X$ and $B, D \subseteq Y$ open such that $\mu(A) = \mu(C) = \nu(B\cap D) = 0$ and $\mu(A\cup C) = \nu(B) = \nu(D) = 1$. If we set $E = (A\times B)\cup(C\times D)$, we see that $\mu\times_l\nu(E) = 0$, but $\mu\times_r\nu(E) = 1$. $\square$

An interesting consequence of this is that both $\mu\times_l\nu$ and $\mu\times_r\nu$ are $\{0,1\}$-quasi-measures on $X\times Y$ that agree on rectangles, but are distinct. This occurs even if $X = Y$ and $\mu = \nu$. If we consider quasi-measures of the form $\alpha\cdot\mu\times_l\nu + (1-\alpha)\cdot\mu\times_r\nu$, we get an uncountable family of quasi-measures that agree on rectangles, but are distinct. In contrast, a measure on the product space is determined by its action on rectangles.

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