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A SIMPLE PROOF OF THE ELLIPTICAL RANGE THEOREM

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Abstract. A short proof is given of the elliptical range theorem concerning the numerical range of a 2×2 complex matrix.

Let A be a bounded linear operator acting on a Hilbert space. The *numerical range* (also known as the *field of values*) of A is the collection of complex numbers of the form (Ax, x) with x ranging through the unit vectors in the Hilbert space. As pointed out by many authors, the numerical range is very useful in studying operators and has applications to many subjects (e.g., see [AL], [H], [HJ], [I]). One of the basic properties, which is important in developing the theory, of numerical range is that the numerical range of an operator is always convex. There are many proofs of this interesting result, and many of them are done by reducing the problem to considering the numerical range of a 2×2 complex matrix, viewed as an operator acting on $\mathbb{C}^2$. In this particular case, we have the following result, known as the Elliptical Range Theorem.

Theorem. *Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2. The numerical range of A is an elliptical disk with λ_1 and λ_2 as foci, and $\{\mathrm{tr}\ (A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}$ as minor axis.*

Most proofs of this theorem are computational and are quite involved (e.g., see [D], [M], and [HJ, §1.3]). The purpose of this note is to present a simple proof of it.

Proof of Theorem. Let $W(A)$ be the numerical range of A. It is easy to verify that $W(U^*AU) = W(A)$ for any unitary matrix U, and $W(\mu A - \eta I) = \mu W(A) - \eta$ for any $\mu, \eta \in \mathbb{C}$. These types of transformations will not change the conclusion of the theorem, and will be used in our proof.

If A is normal, then A is unitarily similar to a diagonal matrix $D = \mathrm{diag}\ (\lambda_1, \lambda_2)$. One readily checks that

$$W(A) = W(D) = \{\lambda_1|x_1|^2 + \lambda_2|x_2|^2 : x_1, x_2 \in \mathbb{C},\ |x_1|^2 + |x_2|^2 = 1\}$$

is a line segment, which can be viewed as a degenerate ellipse with λ_1 and λ_2 as foci, and 0 as the length of the minor axis.

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Suppose A is not normal. Replace A by $A-(\text{tr } A)I/2$ so that we have tr $A=0$. If both eigenvalues of A equal zero, then A is unitarily similar to $B=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, and hence

$$W(A)=W(B)=\{bx_1x_2 : x_1,x_2\in\mathbb{C},\ |x_1|^2+|x_2|^2=1\}$$

is a circular disk, which can be viewed as an elliptical disk with both foci equal to 0, and $|b|$ as the length of the minor axis.

Suppose A has nonzero eigenvalues a and $-a$. Replace A by A/a so that we may assume $a=1$. Since A is not normal, by a suitable unitary similarity transform, we may assume that $A=\begin{pmatrix} 1 & 2c \\ 0 & -1 \end{pmatrix}$ with $c>0$. Let $C=\{(A+A^*)+\gamma(A-A^*)\}/2$ with $\gamma=\sqrt{1+c^2}/c$. Then both eigenvalues of C equal zero, and C is unitarily similar to $\begin{pmatrix} 0 & 2\sqrt{1+c^2} \\ 0 & 0 \end{pmatrix}$. By the previous argument, $W(C)$ is a circular disk with radius $\sqrt{1+c^2}$ centered at the origin. Note that $x+iy\in W(A)$ if and only if $x+i\gamma y\in W(C)$. Since $W(C)$ has boundary $\{\sqrt{1+c^2}e^{it} : t\in\mathbb{R}\}$, $W(A)$ has boundary $\{\sqrt{1+c^2}\cos t+ic\sin t : t\in\mathbb{R}\}$. Hence $W(A)$ is an elliptical disk with $2\sqrt{1+c^2}$ and $2c$ as the lengths of the major and minor axes, respectively, and 1 and -1 as foci.

References

[AL] T. Ando and C.K. Li (Editors), *Special Issue: The Numerical Range and Numerical Radius*, Linear and Multilinear Algebra 37, nos. 1 – 3, 1994. CMP 95:07

[D] W.F. Donoghue, *On the numerical range of a bounded operator*, Michigan J. Math. **4** (1957), 261-263. MR **20:**2622

[H] P.R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967. MR **34:**8178

[HJ] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991. MR **95c:**15001

[I] V. Istratescu, *Introduction to Linear Operator Theory*, Marcel Dekker, New York, 1981. MR **83d:**47002

[M] F.D. Murnaghan, *On the field of values of a square matrix*, Proc. Natl. Acad. Sci. **18** (1932), 246-248.

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