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SCHOTTKY'S FORM AND THE HYPERELLIPTIC LOCUS

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ABSTRACT. We show that Schottky's modular form, J_g, has in every genus an irreducible divisor which contains the hyperelliptic locus. We also improve a corollary of Igusa concerning Siegel modular forms that must necessarily vanish on the hyperelliptic locus.

§1. INTRODUCTION

In 1888 Schottky gave the famous modular cusp form J_4 which vanishes on the Jacobian locus in $\mathcal{H}_4$, the Siegel upper half space of degree four. In 1981 Igusa represented J_4 as a rational multiple of $\vartheta^2_{D_8^+} - \vartheta_{D_{16}^+}$ where $\vartheta_{D_8^+}$ and $\vartheta_{D_{16}^+}$ are the theta series associated to the lattices D_8^+ and D_{16}^+, respectively. This representation could be "accidental" in that the dimension of cusp forms of weight 8 on $\mathcal{H}_4$ is small enough to make the the proportionality of forms arising from different sources likely. On the other hand this representation could point to a deeper relationship between differences of theta series and geometrically interesting loci in $\mathcal{H}_g$.

This paper provides a piece of data which supports the hypothesis of a deeper relationship. We show that $\vartheta^2_{D_8^+} - \vartheta_{D_{16}^+}$ vanishes on the hyperelliptic locus in $\mathcal{H}_g$ for every degree g. If we view $\vartheta_{D_8^+}$ and $\vartheta_{D_{16}^+}$ as *complete invariants* of their associated lattices, then we see that Jacobians of hyperelliptic curves of any genus cannot distinguish the $D_8^+ \oplus D_8^+$ lattice from the D_{16}^+ lattice. The proof we give is a simple modification of an argument due to Igusa in [3, page 845] that uses his homomorphism $\rho_g : A(\Gamma_g) \to S(2, 2g+2)$. Theorem 3.8 shows that if $f \in A(\Gamma_g)$ vanishes at the cusps of the hyperelliptic locus, then $\rho_g(f)$ is divisible by the discriminant in $S(2, 2g+2)$. Theorem 3.8 provides a brief proof of the more interesting Corollary 3.10 that the modular form $\vartheta^2_{D_8^+} - \vartheta_{D_{16}^+}$ always vanishes on the hyperelliptic locus. The author is presently investigating whether or not this form vanishes on the Jacobian locus for $g \geq 5$. I thank William Duke for the interesting talk at Columbia University on Siegel modular forms and codes which led me to this investigation. I also thank my colleague Armand Brumer for his explanations on these topics.

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§2. Notation

We first review the notation concerning modular forms and subvarieties of the moduli space of principally polarized abelian varieties. We let $\mathcal{H}_g$ denote the Siegel upper half space of degree $g \geq 1$, and let $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$ denote the Siegel modular group which acts on $\mathcal{H}_g$. Let $A_k(\Gamma_g)$ denote the Siegel modular forms of weight k for Γ_g, and let $A(\Gamma_g) = \bigoplus_{k\geq 0} A_k(\Gamma_g)$ be the graded ring of Siegel modular forms. For $g \geq 2$ a homomorphism of graded rings $\Phi_g : A(\Gamma_g) \to A(\Gamma_{g-1})$ is defined for $f \in A(\Gamma_g)$ and $\Omega \in \mathcal{H}_{g-1}$ by

$$(\Phi_g f)(\Omega) = \lim_{\lambda\to+\infty} f\begin{pmatrix}\Omega & 0\\ 0 & i\lambda\end{pmatrix} . \tag{2.1}$$

Elements in the kernel of Φ_g are called *cusp forms.* We view $\mathcal{A}_g = \mathcal{H}_g/\Gamma_g$ as the moduli space of principally polarized abelian varieties. The Torelli map sends a compact Riemann surface of genus g to its Jacobian's class in $\mathcal{A}_g$. We let $\mathcal{J}_g$ denote the closure in $\mathcal{A}_g$ of the image of the Torelli map and refer to $\mathcal{J}_g$ as the *Jacobian locus.* In the same way we let h_g denote the closure of the image of the restriction of the Torelli map to hyperelliptic Riemann surfaces, and call h_g the *hyperelliptic locus.* We say that a Siegel modular form $f \in A(\Gamma_g)$ vanishes on h_g if for all $\Omega \in \mathcal{H}_g$ such that $[\Omega] \in h_g$ we have $f(\Omega) = 0$.

We now discuss lattices in $\mathbb{R}^n$ and their associated theta series. A lattice $\Lambda \subseteq \mathbb{R}^n$ is called *integral* if for any $\ell_1, \ell_2 \in \Lambda$ the value of the inner product $\langle \ell_1, \ell_2\rangle$ is an integer. The following sequence of analytic functions ϑ_Λ are invariant under isometries of the lattice Λ.

2.2 **Definition.** Let Λ be a lattice in $\mathbb{R}^n$. For each $g \geq 1$ the *theta series* of Λ, $\vartheta_\Lambda : \mathcal{H}_g \to \mathbb{C}$, is defined for $\Omega \in \mathcal{H}_g$ by

$$\vartheta_\Lambda(\Omega) = \sum_{\ell_1,\dots,\ell_g\in\Lambda} \exp\left(i\pi \sum\nolimits_{j,k=1}^{g} \Omega_{jk}\langle \ell_j, \ell_k\rangle\right).$$

An integral lattice Λ is called *even* if for all $\ell \in \Lambda$ we have $\langle \ell, \ell\rangle \in 2\mathbb{Z}$. If Λ is an even self–dual lattice of dimension n we have $\vartheta_\Lambda \in A_{n/2}(\Gamma_g)$ for each g. For such Λ we necessarily have that 8 divides n, and the examples relevant here are the lattices D_n^+ for $n \in 8\mathbb{Z}^+$ in the notation of Conway and Sloane [1, page 119]. For $n = 8$ there is one isometry class of even self–dual lattices, given by D_8^+; for $n = 16$ there are two isometry classes, given by $D_8^+ \oplus D_8^+$ and D_{16}^+. Theta series satisfy $\Phi_g(\vartheta_\Lambda \text{ on } \mathcal{H}_g) = \vartheta_\Lambda$ on $\mathcal{H}_{g-1}$.

2.3 **Definition.** For $g \geq 1$, define $J_g \in A_8(\Gamma_g)$ by $J_g = \vartheta_{(D_8^+\oplus D_8^+)} - \vartheta_{D_{16}^+}$.

From the work of Igusa in [4] we know that the vanishing of J_4 defines $\mathcal{J}_4$ in $\mathcal{A}_4$.

Finally, we shall use the standard notation $\mathbb{C}[a_1, \dots, a_r]$, $\mathbb{C}(a_1, \dots, a_r)$ for polynomial domains and their quotient fields in r variables. Let $\mathbb{C}[a_1, \dots, a_r]^{\mathrm{sym}}$ and $\mathbb{C}(a_1, \dots, a_r)^{\mathrm{sym}}$ denote, respectively, the polynomial domain and rational function field fixed under the action of the symmetric group S_r on $\mathbb{C}(a.)$ induced by permutations of $a_1, \dots, a_r$. For $s \geq 0$, we denote by $\mathbb{C}[a.]_s^{\mathrm{sym}}$ the polynomials in $\mathbb{C}[a.]^{\mathrm{sym}}$ of degree s in any one, hence in any, of the a_i. Elements of $\mathbb{C}[a.]_s^{\mathrm{sym}}$ are said to have *weight* s. We note that $\mathbb{C}[a.]^{\mathrm{sym}} = \bigoplus_{s\geq 0} \mathbb{C}[a.]_s^{\mathrm{sym}}$ but that this is not the usual grading on $\mathbb{C}[a.]^{\mathrm{sym}}$ given by the homogeneous degree in the a_i; rather it is the grading given by the homogeneous degree in the elementary symmetric functions of the a_i. If we let $\Delta_r = \prod_{i<j}(a_i - a_j) \in \mathbb{C}[a_1, \ldots, a_r]$ as usual, then the element $\Delta_r^2 \in \mathbb{C}[a_1, \ldots, a_r]_s^{\mathrm{sym}}$ has weight $s = 2(r-1)$.

§3. $\mathbf{J}_g$ vanishes on $\mathbf{h}_g$

In this section we prove that a hyperelliptic cusp form of weight less than $8+4/g$ must vanish on the hyperelliptic locus, h_g. The main tools are Igusa's homomorphism $\rho_g : A(\Gamma_g) \dashrightarrow S(2, 2g+2)$ from a subring of Siegel modular forms to a graded ring of "binary invariants", and Tsuyumine's map $T_g : S(2, 2g+2) \dashrightarrow \mathbb{C}(a_1, \ldots, a_{2g})$ that gives the ρ-induced action of Φ_g on $S(2, 2g+2)$.

3.1 **Definition.** For $r \geq 1$, $s \geq 0$, let $S(2, r)_s$ be the set of $f \in \mathbb{C}[a_1, \ldots, a_r]$ such that both 1. and 2. hold.

1. For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ we have

$$f\left(\frac{aa_1+b}{ca_1+d}, \ldots, \frac{aa_i+b}{ca_i+d}, \ldots, \frac{aa_r+b}{ca_r+d}\right) = \left(\prod_{i=1}^{r}(ca_i+d)\right)^{-s} f(a_1, \ldots, a_i, \ldots, a_r).$$

2. $f \in \mathbb{C}[a_1, \ldots, a_r]_s^{\mathrm{sym}}$.

3.2 **Definition.** For $r \geq 1$, let $S(2, r) = \bigoplus_{s \geq 0} S(2, r)_s$.

3.3 *Remarks.* If we apply condition 1. of Definition 3.1 to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we see that $S(2, r) \subseteq \mathbb{C}[a_i - a_j \,|\, i, j \in \{1, \ldots, r\}] \cap \mathbb{C}[a_1, \ldots, a_r]^{\mathrm{sym}}$. Also notice that Δ_r satisfies condition 1. with $s = r-1$.

3.4 **Theorem** (Igusa, [3, page 844]). *Let $g \geq 1$. There exists a homomorphism of graded rings $\rho_g : \mathrm{Dom}(\rho_g) \subseteq A(\Gamma_g) \to S(2, 2g+2)$ such that conditions* 1.–3. *hold.*

1. $\mathrm{Dom}(\rho_g)$ *contains all modular forms of even weight.*
2. $\mathrm{Ker}(\rho_g)$ *is the ideal of* $\mathrm{Dom}(\rho_g)$ *vanishing on h_g.*
3. ρ_g *multiplies weights by $\frac{1}{2}g$.*

3.5 **Definition** [7, page 762]. Let $g, m \geq 1$. Define $T_g : \bigoplus_{m \geq 1} S(2, 2g+2)_{gm} \to \mathbb{C}(a_1, \ldots, a_{2g})$ by, for $I \in S(2, 2g+2)_{gm}$,

$$(T_g I)(a_1, \ldots, a_{2g}) = \left(\prod_{i=1}^{2g} a_i\right)^{-m} I(a_1, \ldots, a_{2g}, 0, 0).$$

In [7] the domain of T_g (designated by Φ in [7]) is given as $S(2, 2g+2)$ and the range space of T_g is given as $\mathbb{C}(a_1, \ldots, a_{2g})$, but the formula defining T_g gives $T_g I$ in the algebraic extension $\mathbb{C}(a_1, \ldots, a_{2g})(\sqrt[g]{a_1 \ldots a_{2g}})$. One easy way to mend this discrepancy is to define $\mathrm{Dom}(T_g) = \bigoplus_{m \geq 1} S(2, 2g+2)_{gm}$ so that the range space of T_g is $\mathbb{C}(a_1, \ldots, a_{2g})$. This is what has been done in Definition 3.5. The $\mathrm{Dom}(T_g)$ in Definition 3.5 includes the ρ_g-images of even weight modular forms; this inclusion is all we use here and all used in [7] to prove the following proposition.

3.6 **Proposition** (Tsuyumine [7, page 786]). *Let $g \geq 1$. There is a choice of ρ_g in Theorem 3.4 such that for even k and for all $f \in A_k(\Gamma_g)$ we have $(T_g \circ \rho_g)(f) = (\rho_{g-1} \circ \Phi_g)(f)$ in $S(2,2g)$.*

3.7 **Definition.** Let $g \geq 2$, and let $f \in A(\Gamma_g)$. We say that f is a hyperelliptic cusp form when $\Phi_g(f) \equiv 0$ on h_{g-1}.

3.8 **Theorem.** *Let $k \in 2\mathbb{Z}^+$. Let $f \in A_k(\Gamma_g)$ be a hyperelliptic cusp form. Then we have Δ_{2g+2}^2 divides $\rho_g(f)$ in $S(2, 2g+2)$.*

Proof. Let f be a hyperelliptic cusp form so that we have $\Phi_g(f) \equiv 0$ on h_{g-1} by Definition 3.7. From 2. of Theorem 3.4 we have $\Phi_g(f) \in \mathrm{Ker}(\rho_{g-1})$. From Tsuyumine's Proposition 3.6 we have $T_g(\rho_g(f)) = \rho_{g-1}(\Phi_g(f)) = 0$ in $\mathbb{C}(a_1, \ldots, a_{2g})$. From the definition of T_g we see that $\rho_g(f)$ is in the ideal (a_{2g+1}, a_{2g+2}) of the ring $\mathbb{C}[a_1, \ldots, a_{2g+2}]$. Since $\rho_g(f) \in S(2, 2g+2) \subseteq \mathbb{C}[a_i - a_j \mid i, j \in \{1, \ldots, 2g+2\}]$, we have $(a_{2g+1} - a_{2g+2})$ divides $\rho_g(f)$ in $\mathbb{C}[a.]$. Since $\mathbb{C}[a.]$ is a unique factorization domain, the facts that $\rho_g(f)$ is divisible by $(a_{2g+1} - a_{2g+2})$ and that $\rho_g(f) \in \mathbb{C}[a.]^{\mathrm{sym}}$ imply that Δ_{2g+2} divides $\rho_g(f)$ in $\mathbb{C}[a.]$. Since $\frac{\rho_g(f)}{\Delta_{2g+2}}$ is an alternating polynomial in the $a.$, we have that Δ_{2g+2} divides $\frac{\rho_g(f)}{\Delta_{2g+2}}$ in $\mathbb{C}[a.]$. Since Δ_{2g+2}^2 and $\rho_g(f)$ are both in $S(2, 2g+2)$, their quotient is as well, and we have that Δ_{2g+2}^2 divides $\rho_g(f)$ in $S(2, 2g+2)$. $\square$

3.9 **Corollary.** *Let $k \in 2\mathbb{Z}^+$. Let $f \in A_k(\Gamma_g)$ be a hyperelliptic cusp form. If $k < 8 + \frac{4}{g}$, then f vanishes identically on h_g.*

Proof. We have that Δ_{2g+2}^2 divides $\rho_g(f)$ in $S(2, 2g+2)$ by Theorem 3.8. The weight of Δ_{2g+2}^2 is $2(2g+1)$ and the weight of $\rho_g(f)$ is $\frac{1}{2}gk$ when $\rho_g(f)$ is nontrivial. However, we have $\frac{1}{2}gk < 2(2g+1)$ by hypothesis so that $\rho_g(f)$ being divisible by Δ_{2g+2}^2 implies that $\rho_g(f) = 0$. Therefore we have $f \equiv 0$ on h_g by Igusa's Theorem 3.4. $\square$

3.10 **Corollary.** *For all $g \geq 1$, the Siegel modular form J_g vanishes on the hyperelliptic locus h_g.*

Proof. For g such that $1 \leq g \leq 4$ this is known from the work of Witt [8], Kneser and Igusa [4], [5]. Corollary 3.10 follows by induction on g. Suppose that we have $J_g \equiv 0$ on h_g; then we have $\Phi_{g+1}(J_{g+1}) = J_g \equiv 0$ on h_g so that J_{g+1} is a hyperelliptic cusp form. J_{g+1} is of even weight 8 so that we may apply Corollary 3.9 to conclude that $J_{g+1} \equiv 0$ on h_{g+1}. $\square$

3.11 *Remark.* Corollary 3.10 may also be proven using Thomæ's formula [3, pg. 838] and the theta identities in Lemma 1 of [5, pg. 354]. It then reduces to the following interesting polynomial identity which can be proven inductively by letting $a_{2g+1} = a_{2g+2}$. For $g \geq 1$ we have the polynomial identity in $\mathbb{Z}[a_1, \ldots, a_{2g+2}]$:

$$\left(\sum_{\{T \amalg T^{\mathbf{c}}\}} \prod_{i<j;\ i,j \in T} (a_i - a_j)^2 \prod_{i<j;\ i,j \in T^{\mathbf{c}}} (a_i - a_j)^2 \right)^2 = 2^g \sum_{\{T \amalg T^{\mathbf{c}}\}} \prod_{i<j;\ i,j \in T} (a_i - a_j)^4 \prod_{i<j;\ i,j \in T^{\mathbf{c}}} (a_i - a_j)^4.$$

The above sum is over the $\frac{1}{2}\binom{2g+2}{g+1}$ partitions $T \amalg T^{\mathbf{c}}$ of $\{1, 2, \ldots, 2g+2\}$ for which both T and $T^{\mathbf{c}}$ have $g+1$ elements. This formula was our original method of proof and was also noted by the referee.

Finally, we mention that J_g is irreducible in $A(\Gamma_g)$.

3.12 **Proposition.** *For all $g \geq 4$, the divisor of J_g in $\mathcal{A}_g$ is irreducible.*

Proof. We will show by induction on g that the divisor of J_g is proper and irreducible in $A(\Gamma_g)$ for $g \geq 4$. The case $g = 4$ is due to Igusa [4]. We use a principle of Freitag which he deduces from his "Satz 2" in [2, page 256]. "For $g \geq 3$ an $f \in A(\Gamma_g)$ has an irreducible divisor, $\operatorname{div}(f)$, if f may not be written as the product of modular forms of lower weight." If we had $J_g = \psi_1\psi_2$ in $A(\Gamma_g)$ where $0 < \deg\psi_1, \deg\psi_2 < 8$, then applying the map Φ_g to J_g and using the induction hypothesis show that J_g is nontrivial on $\mathcal{H}_g$ and that $J_{g-1} = \Phi_g(\psi_1)\Phi_g(\psi_2)$ in $A(\Gamma_{g-1})$ where $0 < \deg\Phi(\psi_1), \deg\Phi(\psi_2) < 8$. This is impossible because $\operatorname{div}(J_{g-1})$ is irreducible by the induction hypothesis. This shows that $\operatorname{div}(J_g)$ is both proper and irreducible in $\mathcal{A}_g$. $\square$

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