PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 7, July 1996

p-SEQUENTIALITY AND p-FRÉCHET-URYSOHN PROPERTY OF FRANKLIN COMPACT SPACES

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(Communicated by Franklin D. Tall)

ABSTRACT. Franklin compact spaces defined by maximal almost disjoint families of subsets of ω are considered from the view of its p-sequentiality and p-Fréchet-Urysohn-property for ultrafilters $p \in \omega^*$. Our principal results are the following: CH implies that for every P-point $p \in \omega^*$ there are a Franklin compact p-Fréchet-Urysohn space and a Franklin compact space which is not p-Fréchet-Urysohn; and, assuming CH, for every Franklin compact space there is a P-point $q \in \omega^*$ such that it is not q-Fréchet-Urysohn. Some new problems are raised.

INTRODUCTION

Let us recall some definitions. All spaces are assumed to be completely regular and Hausdorff. The Greek letter ω will denote the first infinite ordinal. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. The Stone-Čech compactification of ω with the discrete topology is denoted by $\beta(\omega)$, which can be viewed as the set of all ultrafilters on ω, where $\widehat{A} = \mathrm{cl}_{\beta(\omega)}(A) = \{q \in \beta(\omega) : A \in q\}$, for $A \subseteq \omega$, is a basic clopen subset of $\beta(\omega)$. The remainder $\omega^* = \beta(\omega)\backslash\omega$ is the set of all free ultrafilters on ω. For $A \in [\omega]^\omega$, let A^* stand for $\widehat{A}\backslash A$. If $f\colon\ X \to Y$ is a continuous function, then $\overline{f}\colon\ \beta(X) \to \beta(Y)$ stands for the Stone-Čech extension of f. The *Rudin-Keisler* (pre-)order on ω^* is defined by $q \leq_{\mathrm{RK}} p$ if there is a function $f\colon\ \omega \to \omega$ such that $\overline{f}(p) = q$, for $p, q \in \omega^*$. The set of RK-predecessors of $p \in \omega^*$ is denoted by $P_{\mathrm{RK}}(p) = \{q \in \omega^*\colon q \leq_{\mathrm{RK}} p\}$. We say that p is RK-equivalent to q and write $p \approx_{\mathrm{RK}} q$, if $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$. The *Type* of $p \in \omega^*$ is the set $T(p) = \{q \in \omega^* : p \approx_{\mathrm{RK}} q\}$ and it is not hard to see that $p \approx_{\mathrm{RK}} q$ iff there is a bijection $f\colon \omega \to \omega$ such that $\overline{f}(p) = q$. If $p \in \omega^*$, then $T(p)$ is a dense subset of ω^*. A subset $N \subseteq \omega^*$ is said to be *RK-dense* in ω^* if for every $p \in \omega^*$ there is $q \in N$ such that $q \leq_{\mathrm{RK}} p$. If $\Sigma \subseteq [\omega]^\omega$, then $\Sigma^* = \{A^* : A \in \Sigma\}$. An *almost disjoint* (AD) family of subsets of ω is a family Σ of $[\omega]^\omega$ such that if $A, B \in \Sigma$ and $A \neq B$, then $|A \cap B| < \omega$ or $A^* \cap B^* = \varnothing$. If Σ is not a proper subset of any AD family, then Σ is called a *maximal almost disjoint* (*MAD*) family.

Our fundamental concept is the following.

Received by the editors July 5, 1993 and, in revised form, January 27, 1995.
1991 *Mathematics Subject Classification.* Primary 54A20, 54A35.
Key words and phrases. Ultrafilter, MAD family, Franklin compact space, Rudin-Keisler order, p-sequential, p-Fréchet Urysohn, ultra-sequential, ultra-Fréchet-Urysohn.

1.1. Definition (Katětov [Ka])**.** Let $\mathcal{F}$ be a filter on ω. A point x of a topological space X is a $\mathcal{F}$*-limit point* of the sequence $(x_n)_{n<\omega}$ in X, we write $x = \mathcal{F}\text{-}\lim x_n$ if $\{n < \omega : x_n \in V\} \in \mathcal{F}$ for every neighborhood V of x.

Let us note that if M is a nonempty closed subset of ω^* and $\mathcal{F}_M = \{A \in [\omega]^\omega : M \subseteq A^*\}$, then $M = \{p \in \omega^* : \mathcal{F}_M \subseteq p\} = \bigcap\{A^* : A \in \mathcal{F}_M\}$ and $\mathcal{F}_M$ is a free filter on ω. Conversely, if $\mathcal{F}$ is a free filter on ω and $M_\mathcal{F} = \bigcap\{A^* : A \in \mathcal{F}\}$, then $\mathcal{F} = \{A \in [\omega]^\omega : M_\mathcal{F} \subseteq A^*\}, M_\mathcal{F} = \{p \in \omega^* : \mathcal{F} \subseteq p\}$ and $M_\mathcal{F}$ is a nonempty closed subset of ω^*. Hence, we have that $M = M_{\mathcal{F}_M}$ and $\mathcal{F} = \mathcal{F}_{M_\mathcal{F}}$ for every nonempty closed subset M of ω^* and for every free filter $\mathcal{F}$ on ω. Thus, throughout this paper, we will not distinguish between nonempty closed subsets of ω^* and the corresponding free filters on ω. This correspondence allows us to see that $\mathcal{F}$-limits, for an arbitrary free filter $\mathcal{F}$ on ω, may be defined in terms of p-limits for $p \in \omega^*$ as is stated in the next lemma.

1.2. Lemma. *Let $\mathcal{F}$ be a free filter on ω, and let $(x_n)_{n<\omega}$ be a sequence in a space X. Then, $x = \mathcal{F}\text{-}\lim x_n$ iff $x = p\text{-}\lim x_n$ for all $p \in M_\mathcal{F}$.*

Proof. Only the sufficiency requires proof. Assume that there is a neighborhood V of x such that $A = \{n < \omega : x_n \in V\} \notin \mathcal{F}$. Then, $M_\mathcal{F}$ is not contained in A^*. Hence, we pick $p \in M_\mathcal{F} \backslash A^*$. But, we have that $\{n < \omega : x_n \in V\} \in p$, since $x = p\text{-}\lim x_n$, which is a contradiction.

By using the $\mathcal{F}$-limits we may generalize sequentiality and Fréchet-Urysohn-property as follows: ☐

1.3. Definition. Let $\mathcal{F}$ be a free filter on ω and X a space. Then

(1) (Kombarov [Ko]) X is $\mathcal{F}$*-sequential* if for every nonclosed subset A of X there are a sequence $(x_n)_{n<\omega}$ in A and $x \in X\backslash A$ such that $x = \mathcal{F}\text{-}\lim x_n$;

(2) (Comfort-Ponomarov-Savchenko) X is a $\mathrm{FU}(\mathcal{F})$*-space* if for each $A \subseteq X$ and each $x \in \mathrm{cl}(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = \mathcal{F}\text{-}\lim x_n$;

(3) (Malykhin) X is *ultra-sequential* (resp., *ultra*-FU) if X is p-sequential (resp., $\mathrm{FU}(p)$-space) for every $p \in \omega^*$.

The next lemma is a direct consequence of Lemma 1.2.

1.4. Lemma. *Let $\mathcal{F}$ be a free filter on ω and X a space. Then*

(1) *X is $\mathcal{F}$-sequential iff for every nonclosed subset A of X there are a sequence $(x_n)_{n<\omega}$ and $x \in X\backslash A$ such that $x = p\text{-}\lim x_n$ for all $p \in M_\mathcal{F}$;*

(2) *X is a $\mathrm{FU}(\mathcal{F})$-space iff for each $A \subseteq X$ and each $x \in \mathrm{cl}(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p\text{-}\lim x_n$ for all $p \in M_\mathcal{F}$.*

Notice from 1.4 that if X is $\mathcal{F}$-sequential (resp., a $\mathrm{FU}(\mathcal{F})$-space) for a free filter $\mathcal{F}$ on ω, then X is p-sequential (resp., a $\mathrm{FU}(p)$-space) for all $p \in M_\mathcal{F}$. The converse does not hold in general; for instance, if $q <_{\mathrm{RK}} p$, then the subspace $\xi(q) = \omega \cup \{q\}$ of $\beta(\omega)$ is both a $\mathrm{FU}(q)$-space and a $\mathrm{FU}(p)$-space, but it is not a $\mathrm{FU}(\mathcal{F}_M)$-space, where $M = \{p, q\}$. If $\mathcal{F}$ is a free filter on ω, then $\mathcal{F}$-sequentiality and $\mathrm{FU}(\mathcal{F})$-property coincide with the concepts of strong $M_\mathcal{F}$-sequentiality (see [Ko]) and $\mathrm{SFU}(M_\mathcal{F})$-property (see [GT], [Koč]), respectively. If $\mathcal{F}_r$ is the Fréchet filter on ω (i.e., the filter generated by the cofinite subsets of ω), then sequentiality and Fréchet-Urysohn-property agree with $\mathcal{F}_r$-sequentiality and $\mathrm{FU}(\mathcal{F}_r)$-property, respectively. More general, if $\mathrm{int}_{\omega^*}(M_\mathcal{F}) \neq \varnothing$ for a free filter $\mathcal{F}$ on ω, then sequentiality = $\mathcal{F}$-sequentiality and Fréchet-Urysohn-property = $\mathrm{FU}(\mathcal{F})$-property. For a nonempty closed subset M of ω^*, $\mathrm{FU}(M)$-space will mean $\mathrm{FU}(\mathcal{F}_M)$-space.

The basic relationship between Rudin-Keisler order and the concepts given in Definition 1.3 is established in the following lemma.

1.5. Lemma (Ferreira [G]). *For $p, q \in \omega^*$, the following are equivalent.*

(1) $q \leq_{\mathrm{RK}} p$;
(2) *every q-sequential space is p-sequential*;
(3) *every* $\mathrm{FU}(q)$*-space is a* $\mathrm{FU}(p)$*-space.*

For $p \in \omega^*$ and a subset Y of a space X, we define $Y^p = \{x \in X : x = p\text{-}\lim x_n$ for some sequence $(x_n)_{n<\omega}$ in $Y\}$. By induction, we define $Y_0^p = Y$ and, for every ordinal number $\nu, Y_\nu^p = (Y_\mu^p)^p$ if $\nu = \mu + 1$ and $Y_\nu^p = \bigcup_{\lambda<\nu} Y_\lambda^p$ if ν is a limit ordinal. Notice that if X is a p-sequential space, then $\mathrm{cl}_X(Y) = Y_{\omega_1}^p = \bigcup_{\nu<\omega_1} Y_\nu^p$ for each $Y \subseteq X$. Hence, we have that a space X is p-sequential iff the ordinal number

$$\sigma_p(X) = \min\{\nu \leq \omega_1 : \mathrm{cl}_X(Y) = Y_\nu^p \text{ for all } Y \subseteq X\}$$

exists. If X is p-sequential, then $\sigma_p(X)$ is called the *degree of p-sequentiality* of X. Observe that a space X is a $\mathrm{FU}(p)$-space iff $\sigma_p(X) = 1$.

2. Franklin compact spaces

We start with the definition of Franklin compact space: For a MAD family Σ, the Franklin compact space $\mathcal{F}(\Sigma)$ of Σ is defined as follows: $\mathcal{F}(\Sigma)$ is a factor-space of $\beta(\omega)$ in which every A^*, for $A \in \Sigma$, is identified with a single point and $\omega^* \setminus \bigcup \Sigma^*$ is also identified with a single point ∞; hence, the point set of $\mathcal{F}(\Sigma)$ is $\omega \cup \{\{A^*\} : A \in \Sigma\} \cup \{\infty\}$. Franklin compact spaces were introduced in [F], and they have been studied in many papers, for instance, [B$_1$], [B$_2$], [Ma$_1$] and [Ma$_2$]. It is known that Franklin compact spaces are sequential spaces with degree of sequentiality equal to 2 and are not Fréchet-Urysohn. In this section, we will consider these spaces from the view of p-sequentiality and $\mathrm{FU}(p)$-property for $p \in \omega^*$. So $\mathcal{F}(\Sigma)$ is ultrasequential and its degree of p-sequentiality is not greater than 2, for every $p \in \omega^*$. The next two lemmas are fundamental in the study of the p-sequentiality and the $\mathrm{FU}(p)$-property of Franklin compact spaces.

2.1. Lemma. *Let $p \in \omega^*$ and Σ a MAD family. Then, $\mathcal{F}(\Sigma)$ is not a* $\mathrm{FU}(p)$*-space if and only if there is $C \in [\omega]^\omega$ such that*

$$C^* \setminus \bigcup \Sigma^* \neq \varnothing \quad \text{and} \quad P_{\mathrm{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*.$$

Proof. Necessity: If $\sigma_p(\mathcal{F}(\Sigma)) = 2$, then there must be $C \in [\omega]^\omega$ such that $\infty \in \mathrm{cl}(C)$ and no sequence of C p-converges to ∞. Let $q \in P_{\mathrm{RK}}(p) \cap C^*$. Then, we may find $f : \omega \to C$ such that $\overline{f}(p) = q$. Since $(f(n))_{n<\omega}$ is a sequence in C and $q = p\text{-}\lim f(n)$, we have that $q \in A^*$ for some $A \in \Sigma$. This shows that $P_{\mathrm{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$.

Sufficiency: Let $C \in [\omega]^\omega$ satisfy that $C^* \setminus \bigcup \Sigma^* \neq \varnothing$ and $P_{\mathrm{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$. It is enough to prove that no sequence of C p-converges to ∞ in $\mathcal{F}(\Sigma)$. Indeed, let $f : \omega \to C$ be such that $\overline{f}(p) = q \in \omega^*$. Then $q \in P_{\mathrm{RK}}(p) \cap C^* \subseteq \bigcup \Sigma^*$ and so there is $A \in \Sigma$ for which $q \in A^*$; hence, $A^* = p\text{-}\lim f(n)$. □

2.2. Lemma (Boldjiev and Malykhin [BM]). *Let Σ be an AD family. If $p \in \omega^*$ is not a P-point, then $T(p)$ is not contained in $\bigcup \Sigma^*$.*

From 2.2 it follows that if $T(p) \subseteq \bigcup \Sigma^*$ for some AD family Σ, then p must be a P-point. The following theorem is proved in [BM]: now it can be considered as a consequence of Lemmas 2.1 and 2.2.

2.3. Theorem (Boldjiev and Malykhin)**.** *Every Franklin compact space is a* $\mathrm{FU}(p)$*-space for each non-*P*-point* $p \in \omega^*$.

2.4. Corollary (Boldjiev and Malykhin)**.** *If there are not* P*-points in* ω^**, then every Franklin compact space is ultra-*FU.

The existence of P-points in ω^* is independent of the axioms of ZFC. In fact, W. Rudin [R] showed that CH implies the existence of P-points in ω^* and Shelah [W], [Mi] constructed a model of ZFC in which there are not P-points in ω^*. Hence, in Shelah's model the Franklin compact spaces are ultra-FU. So, it is natural to ask whether every Franklin compact space is not a $\mathrm{FU}(p)$-space whenever $p \in \omega^*$ is a P-point. We shall show that CH implies that for every P-point $p \in \omega^*$ there is a MAD family Σ such that $\mathcal{F}(\Sigma)$ is not a $\mathrm{FU}(p)$-space.

Let ρ be the smallest cardinal, such that if $\mathcal{A}$ is a centered family of clopen subsets of ω^* and $|\mathcal{A}| < \mathfrak{c}$, then $\mathrm{int}(\bigcap \mathcal{A}) \neq \varnothing$. It is well known that $\rho = \mathfrak{c}$ under MA.

2.5. Theorem ($\rho = \mathfrak{c}$)**.** *If* N *is a nowhere dense closed subset of* ω^**, then there is a MAD family* Σ *such that* $N \subseteq \omega^* \setminus \bigcup \Sigma^*$ *and* $\mathcal{F}(\Sigma)$ *is a* $\mathrm{FU}(N)$*-space; hence,* $\mathcal{F}(\Sigma)$ *is a* $\mathrm{FU}(p)$*-space for all* $p \in N$.

Proof. The desired family will be constructed by transfinite induction. Enumerate the set $\{C \in [\omega]^\omega : C^* \cap N = \varnothing\}$ as $\{C_\alpha : \alpha < \mathfrak{c}\}$. We define $A_0 = C_0$ and $N_0 = N$. Suppose that $A_\beta \in [\omega]^\omega$ and a nowhere dense closed subset N_β have been defined for each $\beta < \alpha < \mathfrak{c}$, where α is a fixed ordinal smaller than $\mathfrak{c}$, so that

(1) $\Sigma_\beta = \{A_\gamma : \gamma < \beta\}$ is AD for each $\beta < \alpha$;
(2) $(\bigcup \Sigma_\beta^*) \cap (\bigcup_{\gamma<\beta} N_\gamma) = \varnothing$ for each $\beta < \alpha$;
(3) $N_\beta \subseteq \bigcup\{P_{\mathrm{RK}}(p) : p \in N\}$ and $N_\beta \cap P_{\mathrm{RK}}(p) \neq \varnothing$ for each $\beta < \alpha$; and
(4) if C_β^* is not contained in $\bigcup \Sigma_\beta^*$, then $A_\beta^* \subseteq C_\beta^* \setminus [(\bigcup \Sigma_\beta^*) \cup (\bigcup_{\gamma \le \beta} N_\gamma)]$.

Consider C_α^* and $\Sigma_\alpha = \{A_\beta : \beta < \alpha\}$. If $C_\alpha^* \subseteq \bigcup \Sigma_\alpha^*$, then we let $N_\alpha = \varnothing$. If this is not the case, then put $W = \mathrm{int}(C_\alpha^* \setminus [(\bigcup \Sigma_\alpha^*) \cup (\bigcup_{\beta<\alpha} N_\beta)])$. Since $\rho = \mathfrak{c}$, we have that $W \neq \varnothing$. Hence, we choose $B \in [\omega]^\omega$ so that $B^* \subseteq W$. Now, we fix a one-to-one function $f_\alpha \colon\ \omega \to B$ and set $N_\alpha = \overline{f}_\alpha(N)$. Then we choose $A_\alpha \in [\omega]^\omega$ such that $A_\alpha^* \subseteq B^* \setminus N_\alpha$. Thus, we have defined A_α and N_α. It is not hard to verify that $\Sigma = \{A_\alpha \colon\ \alpha < \mathfrak{c}\}$ is the required MAD family.

It follows from 2.5, in particular, that if $\rho = \mathfrak{c}$ and $K \subseteq \omega^*$ satisfies $|K| < \mathfrak{c}$, then there is a MAD family Σ such that $\mathcal{F}(\Sigma)$ is a $\mathrm{FU}(p)$-space for every $p \in \mathrm{cl}(K)$. $\square$

The next result, for only one P-point of ω^*, is contained in [BM, Example 1], but there is a small defect in its proof there. Here, we give a more general statement and the correct proof.

2.6. Theorem (CH)**.** *If* $p_\alpha \in \omega^*$ *is a* P*-point for every* $\alpha < \mathfrak{c}$*, then there exists a MAD family* Σ *such that* $\bigcup_{\alpha<\mathfrak{c}} P_{\mathrm{RK}}(p_\alpha) \subseteq \bigcup \Sigma^*$ *and hence* $\sigma_{p_\alpha}(\mathcal{F}(\Sigma)) = 2$ *for each* $\alpha < \mathfrak{c}$.

Proof. Let $p_\alpha \in \omega^*$ be a P-point for every $\alpha < \mathfrak{c}$. Notice that each element of $\bigcup_{\alpha<\mathfrak{c}} P_{\mathrm{RK}}(p_\alpha)$ is also a P-point of ω^*. We may enumerate $\bigcup_{\alpha<\mathfrak{c}} P_{\mathrm{RK}}(p_\alpha)$ as $\{q_\mu : \mu < \omega_1\}$; this is possible because $|\bigcup_{\alpha<\mathfrak{c}} P_{\mathrm{RK}}(p_\alpha)| = 2^\omega = \omega_1$. To define the MAD family we proceed by transfinite induction. Let $\{A_n : n < \omega\}$ be a partition of ω in infinite subsets. Suppose that A_μ has been defined for $\mu < \nu < \omega_1$ so that

(1) $\{A_\mu : \mu < \nu\}$ is an AD family; and

(2) $\{q_\mu : \mu < \nu\} \subseteq \bigcup\{A^*_\mu : \mu < \nu\}$.

If $q_\nu \in \bigcup\{A^*_\mu : \mu < \nu\}$, then we choose any $A_\nu \in [\omega]^\omega$ such that $|A_\nu \cap A_\mu| < \omega$ for all $\mu < \nu$. Otherwise, we take $A_\nu \in q_\nu$ so that $A^*_\nu \cap A^*_\mu = \varnothing$ for every $\mu < \nu$. Set $\Sigma = \{A_\mu : \mu < \nu\}$. By construction, we have that $\bigcup_{\alpha<\mathfrak{c}} P_{\mathrm{RK}}(p_\alpha) \subseteq \bigcup\{A^*_\mu : \mu < \omega_1\}$. Now, if $A \in [\omega]^\omega$, then there are $\mu, \nu < \omega_1$ such that $q_\mu \in A^* \cap A^*_\nu$ and so $A \cap A_\nu$ is infinite. This shows that Σ is a MAD family. From 2.1 it follows that $\sigma_{p_\alpha}(\mathcal{F}(\Sigma)) = 2$ for each $\alpha < \mathfrak{c}$.

A solution to the following problem could provide some information about the FU(p)-property of Franklin compact spaces for the case when $p \in \omega^*$ is a P-point. □

2.7. Problem. Is it possible that an ultra-FU Franklin compact space exists if P-points of ω^* exist?

The proof of the next theorem is analogous to the previous one.

2.8. Theorem (MA)**.** *For every $P_{\mathfrak{c}}$-point $p \in \omega^*$ there exists a MAD family Σ such that $P_{\mathrm{RK}}(p) \subseteq \bigcup \Sigma^*$ and hence $\sigma_p(\mathcal{F}(\Sigma)) = 2$.*

2.9. Problem. Assume $[MA + \neg CH]$ and let $p \in \omega^*$ be a P_κ-point for some $\omega < \kappa < \mathfrak{c}$ (such P_κ-point exists under $[MA + \neg CH]$, see [S]).

Is there a MAD family Σ family that:

(a) $T(p) \subseteq \bigcup \Sigma^*$ or

(b) $P_{\mathrm{RK}}(p) \subseteq \bigcup \Sigma^*$?

We turn now to prove that CH implies that for every MAD family Σ there is a P-point $q \in \omega^*$ such that $\mathcal{F}(\Sigma)$ is not a FU(q)-space. We need the next lemma.

2.10. Lemma. *Let Σ be a MAD family. If $f\colon\ \omega \to \omega$ is finite-to-one and onto, then $f^{-1}[\Sigma] = \{f^{-1}(A) : A \in \Sigma\}$ is also a MAD family.*

Proof. Let $T = f^{-1}[\Sigma]$. Since Σ is AD and f is finite-to-one, we have that $|f^{-1}(A) \cap f^{-1}(B)| < \omega$ whenever $A, B \in \Sigma$ and $A \neq B$. Now, let $B \in [\omega]^\omega$. Since $f[B] \in [\omega]^\omega$, there is $A \in \Sigma$ such that $|A \cap f[B]| = \omega$ and so $|f^{-1}(A) \cap B| = \omega$. Thus, T is a MAD family. □

2.11. Theorem (CH)**.** *For every MAD family Σ there is a P-point $p \in \omega^*$ such that $P_{\mathrm{RK}}(p) \subseteq \bigcup \Sigma^*$ and hence $\sigma_p(\mathcal{F}(\Sigma)) = 2$.*

Proof. Let $\mathcal{F} = \{f \mid f : \omega \to \omega$ is finite-to-one and onto$\}$ and $\Sigma = \{A_\alpha : \alpha < \omega_1\}$. Enumerate $\mathcal{F}$ as $\{f_\alpha : \alpha < \omega_1\}$. We proceed by transfinite induction. Set $B_0 = A_0$, and assume that B_β and $A_{\gamma_\beta} \in \Sigma$ have been defined for each $\beta < \alpha$ so that

(1) $B^*_\beta \subseteq [B_0 \cap f^{-1}_\beta(A_{\gamma_\beta})]^*$ for $\beta < \alpha$; and

(2) $\{B_\beta : \beta < \alpha\}$ has the finite intersection property. Since $\{B_\beta : \beta < \alpha\}$ has the finite intersection property, there is $D \in [\omega]^\omega$ for which $D^* \subseteq \bigcap_{\beta<\alpha} B^*_\beta$. By Lemma 2.10, we may find $A_{\gamma_\alpha} \in \Sigma$ such that $|D \cap f^{-1}_\alpha(A_{\gamma_\alpha})| = \omega$. Then, we put $B_\alpha = D \cap f^{-1}_\alpha(A_{\gamma_\alpha})$. Thus, $\{B_\alpha : \alpha < \omega_1\}$ has the finite intersection property and hence there is $p \in \omega^*$ such that $\{B_\alpha : \alpha < \omega_1\} \subseteq p$. So $p \in A^*_0$. If $f : \omega \to \omega$ is a bijection, then $f = f_\alpha$ for some $\alpha < \omega_1$; hence, $\overline{f}(p) \in A^*_{\gamma_\alpha}$. This shows that $T(p) \subseteq \bigcup \Sigma^*$. According to 2.2, we have that p must be a P-point of ω^*. Notice, in general, that if $q \leq_{\mathrm{RK}} p$ and $p \in \omega^*$ is a P-point, then there is a finite-to-one surjection $f : \omega \to \omega$ such that $\overline{f}(p) = q$. We claim that $P_{\mathrm{RK}}(p) \subseteq \bigcup \Sigma^*$. Indeed, fix $q \in P_{\mathrm{RK}}(p)$. Then, there is $\alpha < \omega_1$ such that $\overline{f}_\alpha(p) = q$. Since $p \in [f^{-1}_\alpha(A_{\gamma_\alpha})]^*, q = \overline{f}_\alpha(p) \in A^*_{\gamma_\alpha}$. This proves the claim. □

In a similar way we may prove the following.

2.12. Theorem (MA). *For every MAD family Σ there is a $P_{\mathfrak{c}}$-point $p \in \omega^*$ such that $P_{\mathrm{RK}}(p) \subseteq \bigcup \Sigma^*$ and hence $\sigma_p(\mathcal{F}(\Sigma)) = 2$.*

We could predict something like Theorems 2.14 and 2.15 after knowing Malykhin's result [Ma$_3$, Theorem 1], about the coincidence of ultra-FU and ultra-sequentiality with Fréchet-Urysohn property and sequentiality, respectively, under the assumption $n(\omega^*) > \mathfrak{c}$, where $n(X)$ is the Novak number of X (i.e., the smallest power of a family of nowhere dense subsets of X covering X). Indeed, under CH, MA or $n(\omega^*) > \mathfrak{c}$ a Franklin compact space $\mathcal{F}(\Sigma)$ cannot be ultra-FU (i.e., there is $p \in \omega^*$ such that $\sigma_p(\mathcal{F}(\Sigma)) > 1$) since it is not Fréchet-Urysohn. As a consequence of this result and Lemma 1.5 we have the following two theorems.

2.13. Problem. Under CH, is it true that for every MAD family Σ there is a P-point $p \in \omega^*$ such that $\mathcal{F}(\Sigma)$ is a FU(p)-space?

2.14. Theorem. *Let Σ be a MAD family. Then, $\mathcal{F}(\Sigma)$ is ultra-*FU *if and only if $\{p \in \omega^* : \mathcal{F}(\Sigma)$ is a* FU(p)*-space$\}$ is* RK*-dense in ω^*.*

The next result is a direct application of Malykhin's Theorem quoted above and 2.14.

2.15. Theorem $(n(\omega^*) > \mathfrak{c})$**.** *Let Σ be a MAD family. Then, the set $\{p \in \omega^* : \mathcal{F}(\Sigma)$ is a* FU(p)*-space$\}$ is not* RK*-dense in ω^*.*

We formulate and prove the following proposition that is stronger than Malykhin's theorem.

2.16. Proposition $(n(\omega^*) > \mathfrak{c})$**.** *Let $\mathfrak{C}$ be a family of nonempty closed subsets of ω^* such that $|\mathfrak{C}| \leq \mathfrak{c}$ and $\bigcup \mathfrak{C}$ is* RK*-dense in ω^*. Then,* $\mathrm{int}(F) \neq \varnothing$ *for some $F \in \mathfrak{C}$.*

Proof. We have that for every $p \in \omega^*$ there are $F_p \in \mathfrak{C}$ and a function $f_p\colon\ \omega \to \omega$ such that $p \in \overline{f}_p^{-1}(F_p)$. Hence, $\{\overline{f}_p^{-1}(F_p) : p \in \omega^*\}$ is a family of nonempty closed subsets of ω^* and covers ω^*. But this family has power less than or equal to $\mathfrak{c}$. So there must be $F \in \mathfrak{C}$ such that $\mathrm{int}(F) \neq \varnothing$.

We proceed to prove, under the assumption of $n(\omega^*) > \mathfrak{c}$, a result more general than the one given in Theorem 2.11: it is an immediate consequence of 2.1 and 2.16. ☐

2.17. Theorem $(n(\omega^*) > \mathfrak{c})$**.** *Let Σ_α be a MAD family for every $\alpha < \mathfrak{c}$. Then, there is (a P-point) $p \in \omega^*$ such that $\mathcal{F}(\Sigma_\alpha)$ is not a* FU(p)*-space for all $\alpha < \mathfrak{c}$.*

Proof. For each $\alpha < \mathfrak{c}$, put $F_\alpha = \omega^* \setminus \bigcup \Sigma_\alpha^*$. Consider the set $\mathfrak{C} = \{F_\alpha : \alpha < \mathfrak{c}\}$. Since $\mathrm{int}(F_\alpha) = \varnothing_\alpha$ for each $\alpha < \mathfrak{c}$, by 2.16, there is $p \in \omega^*$ such that $P_{\mathrm{RK}}(p) \cap \bigcup \mathfrak{C} = \varnothing$ and so $P_{\mathrm{RK}}(p) \subseteq \omega^* \setminus \bigcup \mathfrak{C} = \bigcap_{\alpha<\mathfrak{c}} [\bigcup \Sigma_\alpha^*]$. Lemma 2.1 implies that $\mathcal{F}(\Sigma_\alpha)$ is not a FU(p)-space for all $\alpha < \mathfrak{c}$. ☐

The next problem now appears to be natural.

2.18. Problem. Is it possible that some $p \in \omega^*$ exists such that every Franklin compact space is not a FU(p)-space?

Perhaps it will be so in models obtained from GCH and by adding greater than ω_1 of new Cohen reals.

We recall that, for $p\in\omega^*$, a space X is a *strictly* FU(p)*-space* if $x\in\bigcap_{n<\omega}\mathrm{cl}_X(Y_n)$, where $Y_n\subseteq X$ for each $n<\omega$; then for each $n<\omega$ there is $x_n\in Y_n$ such that $x=p\text{-}\lim x_n$.

2.19. Problem. Let $p\in\omega^*$, and let Σ be a MAD family. Can $\mathcal{F}(\Sigma)$ be a strictly FU(p)-space?

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