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REAL INSTANTONS, DIRAC OPERATORS AND QUATERNIONIC CLASSIFYING SPACES

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Abstract. Let $M(k, SO(n))$ be the moduli space of based gauge equivalence classes of $SO(n)$ instantons on principal $SO(n)$ bundles over S^4 with first Pontryagin class $p_1 = 2k$. In this paper, we use a monad description (Y. Tian, *The Atiyah-Jones conjecture for classical groups*, preprint, S. K. Donaldson, Comm. Math. Phys. **93** (1984), 453–460) of these moduli spaces to show that in the limit over n, the moduli space is homotopy equivalent to the classifying space $BSp(k)$. Finally, we use Dirac operators coupled to such connections to exhibit a particular and quite natural homotopy equivalence.

1. Introduction

Let $M(k, SO(n))$ be the moduli space of based gauge equivalence classes of $SO(n)$ instantons on principal $SO(n)$ bundles over S^4 with first Pontryagin class $p_1 = 2k$. By adding a trivial connection on a trivial line bundle, there are natural maps $M(k, SO(n)) \hookrightarrow M(k, SO(n+1))$, and one can define the direct limit space $M(k, SO)$. In this paper we show that there is a homotopy equivalence $M(k, SO) \simeq BSp(k)$, where $Sp(k)$ denotes the symplectic group of norm preserving automorphisms of the quaternionic vector space H^k. We also show that this equivalence can be realized by a "Dirac-type" map, constructed by coupling a Dirac operator to an $SO(n)$ connection. More precisely, the coupling of a Dirac operator to a connection associates to each element of $M(k, SO(n))$ an operator acting on the space of sections of a certain vector bundle. Associated to each selfdual connection is the vector space of sections in the kernel of its associated operator. This procedure defines a complex vector bundle, which for $SO(n)$ connections has a symplectic structure, and this bundle is classified by a map which we shall refer to as the Dirac map, $\partial_{SO(n)} : M(k, SO(n)) \to BSp(k)$. The topological properties of these Dirac maps for $SU(n)$ connections were first studied by Atiyah and Jones [AJ], and more recently it was shown in [S] that the limit map $\partial_{SU} : M(k, SU) \to BU(k)$ realizes Kirwan's [K] homology isomorphism $H_*(M(k, SU)) \cong H_*(BU(k))$, and is, therefore, a homotopy equivalence. It also makes sense to define such Dirac maps on the limit spaces $M(k, G)$, where G is either SO or Sp, and in [S] it was shown that the limit map $\partial_{Sp} : M(k, Sp) \to BO(k)$ is a homotopy equivalence. In this

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paper we complete the picture for the classical groups by showing that the limit map $\partial_{SO} : M(k, SO) \to BSp(k)$ is also a homotopy equivalence.

Our proof will be fairly direct. In Section 1 we review Tian's [Ti] version of Donaldson's [D] monad description of $M(k, SO(n))$. Tian exhibits this moduli space as the quotient of a set of triples of certain complex matrices by an action of $Sp(k; C)$, the complex symplectic group. We shall show that this action is free, that there are natural maps $M(k, SO(n)) \hookrightarrow M(k, SO(n+1))$, and that in the limit over n the space of triples is contractible. Hence, $M(k, SO)$ will be shown to be the quotient of a contractible space by a free $Sp(k; C)$ action. In Section 2, we use a comparison between $SO(n)$ and $SU(n)$ connections to show that the Dirac map $\partial_{SO(n)} : M(k, SO(n)) \to BSp(k)$ induces a surjection in integral homology through a range of dimension increasing with n. Since $H_*(M(k, SO); Z) \cong H_*(BSp(k); Z)$ by results of Section 1, the limit map ∂_{SO} must be a homology isomorphism and therefore a homotopy equivalence.

Notice that the Sp and SO duality in these moduli spaces is foreshadowed in Bott Periodicity. Since the entire space of based gauge equivalence classes of $SO(n)$ connections is homotopy equivalent to $\Omega^3 SO(n)$, the limit over n is homotopy equivalent to $Z \times BSp$. Similarly, the space of $Sp(n)$ connections is homotopy equivalent to $\Omega^3 Sp(n)$ which, after passing to the limit, is homotopy equivalent to $Z \times BO$. Alternatively, as we will see in Section 2, this duality comes from the fact that the bundle of real spinors over S^4 is naturally a symplectic vector bundle. Recently, in fact, Tian [Ti] has shown that by comparing the two possible limit processes which one can apply to these moduli spaces, viz., fixing k and taking the limit over n or fixing n and taking the limit over k, one actually can prove Bott Periodicity. This consequence alone demonstrates the beauty and complexity of these moduli spaces.

2. $M(k, SO)$ AND $BSp(k)$

The ADHM construction [ADHM] identifies the space of instantons with certain holomorphic bundles over complex projective space, and Donaldson [D] used a monad construction to characterize such bundles in terms of a quotient of a set of sequences of complex matrices by a natural group action. For $SO(n)$ instantons, Tian [Ti] carried out this procedure explicitly.

Let σ denote the standard skew form on $\mathbf{C}^{2k}$,

$$\sigma = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix. The complexified symplectic group, $Sp(k, \mathbf{C}) \subset Gl(2k, \mathbf{C})$, consists of those matrices g such that $g^{-1} = -\sigma g^T \sigma$. The maximal compact subgroup of $Sp(k, \mathbf{C})$ is the compact symplectic group $Sp(k)$.

Proposition 1 (Donaldson [D] and Tian [Ti]). *Let $A(k, SO(n))$ be the space of triples of complex matices (γ_1, γ_2, c), where γ_i is $2k \times 2k$ and c is $n \times 2k$, satisfying:*

a) $\gamma_1^T = -\sigma \gamma_1 \sigma$,
b) $\gamma_2^T = -\gamma_2$,
c) $2(\gamma_1^T \gamma_2 + \gamma_2^T \gamma_1) + c^T c = 0$,
d) $\begin{pmatrix} \gamma_1 + xI_{2k} \\ \gamma_2 + y\sigma \\ c \end{pmatrix}$ *has rank $2k$ for all $x, y \in \mathbf{C}$.*

Then there is a natural action of $Sp(k,\mathbf{C})$ on $A(k,SO(n))$ given by

$$g\cdot(\gamma_1,\gamma_2,c)=(g\gamma_1 g^{-1},(g^{-1})^T\gamma_2 g^{-1},cg^{-1}),$$

and $M(k,SO(n))$ is homeomorphic to the quotient $A(k,SO(n))/Sp(k,\mathbf{C})$.

It has been shown [S] that in the analogous description of $SU(n)$ and $Sp(n)$ instantons, this group action is free. This is, in some sense, already implicit in the monad construction. Not surprisingly, then, it is also true in the $SO(n)$ case. For the sake of completeness, however, we now give the proof.

Lemma 2. *The natural action of $Sp(k,\mathbf{C})$ on $A(k,SO(n))$ is free.*

Proof. Assume the converse, so we have

$$(g\gamma_1 g^{-1},(g^{-1})^T\gamma_2 g^{-1},cg^{-1})=(\gamma_1,\gamma_2,c)$$

for a particular $g\neq I$ and triple (γ_1,γ_2,c), and note that elements of $Sp(k,\mathbf{C})$ satisfy $g^{-1}=-\sigma g^T\sigma$. Consider the subspace $im(g-I)=V\subset\mathbf{C}^{2k}$. By assumption it is non-empty. Thus, from the definition of the action we have

$$\begin{aligned}c(g-I)&=0\ ,\\ \gamma_1(g-I)&=(g-I)\gamma_1\ ,\\ \sigma\gamma_2(g-I)&=(g-I)\sigma\gamma_2\ .\end{aligned}$$

This last fact is proved as follows:

$$\begin{aligned}\gamma_2 &= (g^{-1})^T\gamma_2 g^{-1}\\ \Rightarrow\gamma_2 g &= (g^{-1})^T\gamma_2\\ \Rightarrow\sigma\gamma_2 g &= -\sigma(g^{-1})^T\sigma\sigma\gamma_2\\ \Rightarrow\sigma\gamma_2 g &= g\sigma\gamma_2\ .\end{aligned}$$

Equivalently c annihilates V and γ_1 and $\sigma\gamma_2$ preserve V. Using conditions a), b), and c), we see that on V

$$\begin{aligned}&\gamma_1^T\gamma_2+\gamma_2^T\gamma_1=0\\ &\Rightarrow -\sigma\gamma_1\sigma\gamma_2-\gamma_2\gamma_1=0\\ &\Rightarrow \gamma_1\sigma\gamma_2-\sigma\gamma_2\gamma_1=0\ .\end{aligned}$$

Hence γ_1 and $\sigma\gamma_2$ have a common eigenvector in V.

Choose $v\in V$ satisfying $\gamma_1 v=\lambda v$ and $\sigma\gamma_2 v=\mu v$. Then

$$\begin{pmatrix}\gamma_1-\lambda I_{2k}\\ \gamma_2+\mu\sigma\\ c\end{pmatrix}v=0,$$

contradicting condition d). Thus, the image of $g-I$ must be empty, so $g=I$ and the action is free.

We now construct the limit space $M(k,SO)$ and show that it is homotopy equivalent to $BSp(k)$. First notice that there is an $Sp(k,\mathbf{C})$ equivariant map from $A(k,SO(n))\hookrightarrow A(k,SO(n+1))$ which sends each γ_i to itself and sends c to the $(n+1)\times k$ matrix made up of c with an extra row of zeros on top. On the level of monads, this adds to the bundle over CP^2 the trivial holomorphic line bundle (see [Ti]). Thus this map induces the natural inclusion $M(k,SO(n))\hookrightarrow M(k,SO(n+1))$ sending the connection ω to the connection $\omega\oplus d$, where d is ordinary exterior differentiation. We now prove the main theorem of this section.

Theorem 3. *$A(k, SO)$ is a contractible space with a free $Sp(k, \mathbf{C})$ action. Thus, $M(k, SO) \cong A(k, SO)/Sp(k, \mathbf{C}) \simeq BSp(k)$.*

Proof. To show that $A(k, SO)$ is contractible it suffices to show that all of its homotopy groups are zero. To this end we show that for any k and n there is an $r > n$ such that inclusion $A(k, SO(n)) \hookrightarrow A(k, SO(r))$ is homotopically trivial (cf. [S], sections 2 and 3).

For $0 \leq t \leq 1$ define $\tilde{I}_k(t)$ to be the $4k \times 2k$ matrix whose j^{th} column is the vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ it \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where there are $2j - 2$ zeroes before the t. Note that $(\tilde{I}_k(t))^T \cdot \tilde{I}_k(t)$ is the zero matrix. Now consider the homotopy $H_t : A(k, SO(n)) \to A(k, SO(4k + n))$ defined as follows:

$$H_t(\gamma_1, \gamma_2, c) = ((1 - t)\gamma_1, (1 - t)\gamma_2, c_t)$$

where

$$c_t = \begin{pmatrix} \tilde{I}_k(t) \\ \\ (1 - t)c \end{pmatrix}.$$

It is easy to check that for any $x \in A(k, SO(n))$ we have $H_t(x) \in A(k, SO(4k + n))$ because c_t clearly has rank $2k$ and $c_t^T \cdot c_t = c^T \cdot c(1 - t)^2$. Finally, notice that H_0 is just the natural inclusion $A(k, SO(n)) \hookrightarrow A(k, SO(4k + n))$, and H_1 is a constant map. This finishes the proof of the theorem.

3. The Dirac map $\partial_{SO(n)} : M(k, SO(n)) \to BSp(k)$

In this section we review the construction of the Dirac map, and show that after passing to the limit over n it is a homotopy equivalence. To define this map it is instructive to first consider $SU(n)$ instantons. Let ω be a connection on the $SU(n)$ vector bundle E_k, where the second Chern class $c_2(E_k) = k$, and let S denote the canonical bundle of complex spinors over S^4 with its canonical connection ∇_s. The tensor product bundle $S \otimes E_k$ inherits a Clifford module structure from the one on S, and we can view $\nabla_s \otimes \omega$ as a connection on this bundle. This connection gives rise to a Dirac operator

$$D_\omega : \Gamma(S \otimes E_k) \longrightarrow \Gamma(S \otimes E_k),$$

where $\Gamma(S \otimes E_k)$ is the space of smooth sections of $S \otimes E_k$. There is a splitting $S \cong S^+ \oplus S^-$ and the Dirac operator interchanges the two summands. The operator

$$D_\omega^+ : \Gamma(S^+ \otimes E_k) \longrightarrow \Gamma(S^- \otimes E_k)$$

is Fredholm, in an appropriate Sobolev completion, and of index k [AJ]. Furthermore, if ω is selfdual, then $Coker(D_\omega^+) = 0$ [AHS]. Therefore, the space of sections in the kernel of D_ω^+ gives a well-defined vector space associated to the connection

ω. There is an equivariance of the kernel under gauge transformation in the sense that $\sigma \in Ker(D^+_\omega)$ implies $g\sigma \in Ker(D^+_{gw})$, for any g in the based gauge group of bundle automorphisms of E_k. Passing to gauge equivalence classes gives a k-dimensional complex vector bundle over $M(k, SU(n))$. This bundle is classified by a map, $\partial_{SU(n)} : M(k, SU(n)) \to BU(k)$, which we shall refer to as the Dirac map.

A similar construction can be used to define the Dirac map for $SO(n)$ connections. Given an $SO(n)$ bundle E with $p_1(E) = 2k$ and an $SO(n)$ instanton ω on E, we can complexify the bundle and connection, denoted ω_C and E_C, and then use the unitary Dirac map to obtain

$$M(k, SO(n)) \longrightarrow M(2k, SU(n)) \stackrel{\partial_{SU(n)}}{\longrightarrow} BU(2k).$$

(Note that $c_2(E_C) = 2k$.) However, because E_C has by definition an underlying real structure, given by some bundle involution J_E, and the complex spinor bundle S has a quaternionic structure, given by some complex anti-linear bundle automorphism J_s, where $J_s \circ J_s = -1$, the tensor product bundle $S \otimes E_C$ will also have a quaternionic structure. Moreover, the Dirac operator will respect this extra structure because the tensor product connection $\nabla_s \otimes \omega_C$ will commute with $J_s \otimes J_E$. Thus, the kernel bundle, defined by coupling a Dirac operator to a real $SO(n)$ instanton, will be a k-dimensional quaternionic bundle over $M(k, SO(n))$. In other words, the composition

$$M(k, SO(n)) \longrightarrow M(2k, SU(n)) \stackrel{\partial_{SU(n)}}{\longrightarrow} BU(2k)$$

factors through $BSp(k)$. We denote this lifting by $\partial_{SO(n)}$. In short, we have the homotopy commutative diagram

$$\begin{array}{ccc} M(k, SO(n)) & \stackrel{\partial_{SO(n)}}{\longrightarrow} & BSp(k) \\ \downarrow & & \downarrow \\ M(2k, SU(n)) & \stackrel{\partial_{SU(n)}}{\longrightarrow} & BU(2k) \end{array}$$

We now show that we can define the limit map $\partial_{SO} : M(k, SO) \to BSp(k)$. From the matrix description of $M(k, SO(n))$, we see that the natural inclusion $M(k, SO(n)) \hookrightarrow M(k, SO(n+1))$, mapping (ω, E) to $(\omega \oplus d, E \oplus R)$, embeds $M(k, SO(n))$ as a closed submanifold of $M(k, SO(n+1))$. It follows that the direct limit $M(k, SO)$ is homotopy equivalent to the homotopy direct limit $M(k, SO)_h$. Thus, it suffices to define ∂_{SO} on $M(k, SO)_h$. To this end, let $\mathcal{A}(k, SO(n))$ denote the space of instantons before passing to gauge equivalence classes, and let $G_{k,SO(n)}$ denote the based gauge group of bundle automorphisms of the $SO(n)$ bundle E, where $p_1(E) = 2k$. Let $\eta(k, SO(n))$ denote the bundle classified by the map $\partial_{SO(n)}$: $M(k, SO(n)) \to BSp(k)$. By definition,

$$\eta(k, SO(n)) = \left\{ [(\omega, \tau)] : \tau \in ker(D^+_\omega) \right\} \ \subset \ \mathcal{A}(k, SO(n)) \times_{G_{k,SO(n)}} \Gamma(S^+ \otimes E_C).$$

Since the untwisted Dirac operator on S^4 has no kernel (S^4 has no harmonic spinors), the natural inclusion of bundles

$$\begin{array}{ccc} \eta(k,SO(n)) & \hookrightarrow & \eta(k,SO(n+1)) \\ \downarrow & & \downarrow \\ M(k,SO(n)) & \hookrightarrow & M(k,SO(n+1)) \end{array}$$

defined by $(\omega,\tau) \to (\omega \oplus d, \tau \oplus 0)$ is an isomorphism on fibers. Thus the pullback of $\eta(k,SO(n+1))$ via the inclusion $M(k,SO(n)) \hookrightarrow M(k,SO(n+1))$ is isomorphic to $\eta(k,SO(n))$. Hence, the diagram

$$\begin{array}{ccc} M(k,SO(n)) & \hookrightarrow & M(k,SO(n+1)) \\ \partial_{SO(n)} \downarrow & & \downarrow \partial_{SO(n+1)} \\ BSp(k) & = & BSp(k) \end{array}$$

commutes up to homotopy. So there exists a map $\partial_{SO} : M(k,SO)_h \to BSp(k)$. Precomposing with the equivalence $M(k,SO) \simeq M(k,SO)_h$ gives a map

$$\partial_{SO} : M(k,SO) \to BSp(k).$$

This map is not necessarily uniquely determined. Nevertheless, any two choices, when restricted to $M(k,SO(n))$, will classify the bundle $\eta(k,SO(n))$, and this is the only property of the limit map which we will use. In particular, any such choice will give a homotopy commutative diagram

$$\begin{array}{ccc} M(k,SO(n)) & \longrightarrow & M(k,SO) \\ & \partial_{SO(n)} \searrow & \downarrow \partial_{SO} \\ & & BSp(k). \end{array}$$

Now, since $H_*(M(k,SO)) \cong H_*(BSp(k))$ by Theorem 3, ∂_{SO} will induce a homology isomorphism, and therefore be a homotopy equivalence, if and only if it induces a surjection in homology. By the homotopy commutativity of the previous diagram, it suffices to show that $\partial_{SO(n)} : M(k,SO(n)) \to BSp(k)$ induces a surjection in homology through a range increasing with n.

Theorem 4. *The Dirac map $\partial_{SO(n)}$ induces a surjection in homology through dimension $2n-4$. Thus, the limit map $\partial_{SO} : M(k,SO) \to BSp(k)$ is a homotopy equivalence.*

We begin by proving the following lemma:

Lemma 5. *There is a commutative diagram*

$$\begin{array}{ccc} H_*(M(k,SU(n))) & \xrightarrow{(\partial_{SU(n)})_*} & H_*(BU(k)) \\ \downarrow & & \downarrow \\ H_*(M(k,SO(2n))) & \xrightarrow{(\partial_{SO(2n)})_*} & H_*(BSp(k)), \end{array}$$

where $Sp(k) \subset U(2k)$ consists of all matrices of the form

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

for any $A, B \in End(C^k)$, and the map $BU(k) \to BSp(k)$ is induced from the inclusion $U(k) \hookrightarrow Sp(k)$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

Proof. First notice that the natural map of Lie algebras $su(n) \hookrightarrow so(2n)$ induces a map $M(k, SU(n)) \to M(k, SO(2n))$. The self-duality condition is preserved because the Hodge star operator is complex linear. Locally, the connection matrix $\gamma = \gamma_1 + i\gamma_2$, where γ_j is a real matrix-valued one form, will map to the matrix

$$\begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix}.$$

Also notice that, as mentioned previously, the complexification of a real connection on an $SO(r)$ bundle induces a natural map $M(k, SO(2n)) \to M(2k, SU(2n))$. Locally, the composition of these two maps

$$M(k, SU(n)) \to M(k, SO(2n)) \to M(2k, SU(2n))$$

is given by

$$\gamma = \gamma_1 + i\gamma_2 \quad \mapsto \quad \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix},$$

where the last matrix is viewed as taking values in the Lie algebra $su(2n)$. Since there is a $g \in SU(2n)$ such that

$$g^{-1} \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix} g = \begin{pmatrix} \gamma_1 + i\gamma_2 & 0 \\ 0 & \gamma_1 - i\gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{pmatrix},$$

the connections represented by these matrix-valued one forms are gauge equivalent. Thus, the composition $M(k, SU(n)) \to M(k, SO(2n)) \to M(2k, SU(2n))$ sends the equivalence class of the selfdual connection ω on the bundle E to the equivalence class of the selfdual connection $\omega \oplus \bar{\omega}$ on the bundle $E \oplus \bar{E}$.

Now consider the diagram

$$\begin{array}{ccc} M(k, SU(n)) & \xrightarrow{\partial_{SU(n)}} & BU(k) \\ \downarrow & & \downarrow \\ M(k, SO(2n)) & \xrightarrow{\partial_{SO(2n)}} & BSp(k) \\ \downarrow & & \downarrow \\ M(2k, SU(2n)) & \xrightarrow{\partial_{SU(2n)}} & BU(2k). \end{array}$$

By the definition of $\partial_{SO(2n)}$, the bottom square homotopy commutes. Since the map $BSp(k) \to BU(2k)$ induces an injection in homology, the top square will induce a commutative diagram in homology if the large outer "square"

$$\begin{array}{ccc} M(k, SU(n)) & \stackrel{\partial_{SU(n)}}{\longrightarrow} & BU(k) \\ \downarrow & & \downarrow \\ M(2k, SU(2n)) & \stackrel{\partial_{SU(2n)}}{\longrightarrow} & BU(2k) \end{array}$$

commutes in homology. Note that on the level of bundles the right vertical map sends a complex vector bundle F to the complex bundle $F \oplus \bar{F}$. Let $\eta(r, SU(l))$ denote the Dirac bundle classified by the map $\partial_{SU(l)} : M(r, SU(l)) \to BU(r)$. The proof of the lemma will be complete if the composition

$$M(k, SU(n)) \longrightarrow M(2k, SU(2n)) \stackrel{\partial_{SU(2n)}}{\longrightarrow} BU(2k)$$

classifies the bundle $\eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n))$. There is a natural bundle map

$$\begin{array}{ccc} \eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n)) & \longrightarrow & \eta(2k, SU(2n)) \\ \downarrow & & \downarrow \\ M(k, SU(n)) & \longrightarrow & M(2k, SU(2n)) \end{array}$$

defined by

$$[(\omega, \psi_1 \oplus \psi_2)] \mapsto [(\omega \oplus \bar{\omega}, \psi_1 \oplus \bar{\psi}_2)]$$

where $\bar{\psi}_2$ is the section ψ_2 viewed as a section of the conjugate bundle. Since ψ is in the kernel of D^+_ω if and only if $\bar{\psi}$ is in the kernel of $D^+_{\bar{\omega}}$, this bundle map is a surjection on fibers. Since the fibers have the same dimension, this map is an isomorphism. Thus $\eta(k, SU(n)) \oplus \bar{\eta}(k, SU(n))$ is isomorphic to the pullback of $\eta(2k, SU(2n))$, and the lemma is proved.

The proof of Theorem 4 is now easy. In [S], section 5, it was shown that the map

$$(\partial_{SU(n)})_* : H_*(M(k, SU(n))) \longrightarrow H_*(BU(k))$$

is a surjection through dimension $2n - 4$. Furthermore, we know that the map $BU(k) \to BSp(k)$ induces a surjection in homology. Thus, by the commutativity of the diagram

$$\begin{array}{ccc} H_*(M(k, SU(n))) & \stackrel{(\partial_{SU(n)})_*}{\longrightarrow} & H_*(BU(k)) \\ \downarrow & & \downarrow \\ H_*(M(k, SO(2n))) & \stackrel{(\partial_{SO(2n)})_*}{\longrightarrow} & H_*(BSp(k)), \end{array}$$

$(\partial_{SO(2n)})_* : H_*(M(k, SO(2n))) \to H_*(BSp(k))$ must also be a surjection through this range. In particular, then, the limit map $\partial_{SO} : M(k, SO) \to BSp(k)$ is a homotopy equivalence.

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