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K-THEORY AND THE ANTI-AUTOMORPHISM OF THE STEENROD ALGEBRA

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Abstract. We give simple proofs of some relations in the Steenrod algebra involving the powers $\mathcal{P}^i$ and their duals $\chi\mathcal{P}^i$ and show how these relations arise from K-theory.

1. Introduction

Barratt and Miller in [3] derived some relations in the Steenrod algebra $\mathcal{A}(p)$ using expressions for the Adem relations due to Bullett and Macdonald [4]. In Section 2 we reformulate and give a simple proof of the former in the spirit of the latter. These relations arise naturally from properties of the Adams operations in complex K-theory, as was explained in [7, 6] for the prime 2; alternative and shorter K-theoretic derivations valid for all primes are given in Section 3.

2. Results of Barratt and Miller

We use standard notation for the mod p Steenrod algebra $\mathcal{A}(p)$ with the convention that $\mathcal{P}^i$ denotes Sq^i when $p = 2$. For each non-negative integer n, let

$$\phi(n) = \min\{k \geq 0 \mid k + \nu_p(k!) \geq n\}.$$

Another, more illuminating description of $\phi(n)$ is given in Proposition 2.6.

Theorem 2.1. *The polynomial*

$$F_n(T) = \sum_{i=0}^{n} \mathcal{P}^{n-i}\chi(\mathcal{P}^i)T^i \in \mathcal{A}(p)[T]$$

is divisible by $(T-1)^{\phi(n)}$.

This result can be restated in many ways. Let $T^m F_n(T)$, for $m \in \mathbb{Z}$, be expressed as a formal power series in $S = T - 1$. By considering the coefficient of S^j with $0 \leq j < \phi(n)$, it follows that

$$\sum_{i=0}^{n} \binom{m+i}{j} \mathcal{P}^{n-i}\chi(\mathcal{P}^i) = 0 \quad \text{for } 0 \leq j < \phi(n). \tag{2.2}$$

Looking at $T^n F(T^{-1})$ instead of $F(T)$ (or applying the anti-automorphism χ) we obtain

$$\sum_{i=0}^{n} \binom{m+i}{j} \mathcal{P}^{i}\chi(\mathcal{P}^{n-i}) = 0 \quad \text{for } 0 \leq j < \phi(n). \tag{2.3}$$

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This is Theorem 1 of [3]. (The condition $0 \le j < \phi(n)$ is equivalent to $(p-1)n > jp - \alpha(j)$ in the notation of [3] and setting $(N, K, L) = (n, j-m-1, j)$ gives $\binom{m+i}{j} = (-1)^L \binom{K-i}{L}$.)

It will be convenient to write $\gamma(r) = (p^r - 1)/(p-1)$.

Corollary 2.4. *Let $n \ge \gamma(r+1) - r$ and $0 \le k < p^r$. Then*

$$\sum_{j \ge 0} \mathcal{P}^{k+p^r j} \chi(\mathcal{P}^{n-k-p^r j}) = 0.$$

This is Theorem 3.2 of [3]. The condition $n \ge \gamma(r+1) - r$ is equivalent to $p^r \le \phi(n)$; see Proposition 2.6. From Theorem 2.1, $F_n(T)$ is divisible by $T^{p^r} - 1 = (T-1)^{p^r}$ and the result follows as a polynomial $\sum a_i T^i$ is divisible by $T^N - 1$ if and only if $\sum a_{k+Nj} = 0$ for each k, $0 \le k < N$.

From Corollary 2.4, one obtains Straffin's formula [10]:

$$\textstyle\sum_{j=0}^{p} \mathcal{P}^{p^r j} \chi(\mathcal{P}^{p^r(p-j)}) = 0. \tag{2.5}$$

In the next proposition we collect some properties of $\phi(n)$. (We are grateful to the referee for drawing our attention to the characterization (d) of ϕ.)

Proposition 2.6. *The function ϕ has the following properties.*
(a) *$\phi(\gamma(r)) = p^{r-1}$, for $r \ge 1$.*
(b) *$\phi(i+j) \le \phi(i) + \phi(j)$ for i, $j \ge 0$.*
(c) *Let $\gamma(r) < n < \gamma(r+1)$. Then $\phi(n) = \phi(n - \gamma(r)) + \phi(\gamma(r))$.*
(d) *Let $\psi(n)$, defined for non-negative integers n, satisfy: $\psi(0) = 1$; $\psi(i+j) \le \psi(i) + \psi(j)$ for i, $j \ge 0$; $\psi(\gamma(r)) \le p^{r-1}$ for $r \ge 1$. Then $\psi(n) \le \phi(n)$ for all $n \ge 0$.*

Proof. Recall that $\nu_p(k!) = (k - \alpha_p(k))/(p-1)$, where $\alpha_p(k)$ is the sum of the coefficients in the p-adic expansion of k. One finds that $p^{r-1} + \nu_p(p^{r-1}!) = \gamma(r)$, and this establishes (a).

Part (b) follows from the inequality $k + \nu_p(k!) + l + \nu_p(l!) \le (k+l) + \nu_p((k+l)!)$.

In (c) we have $p^{r-1} < \phi(n) \le p^r$. Consider $k = \phi(n) - p^{r-1}$. Now

$$\alpha_p(k) = \alpha_p(\phi(n)) + \begin{cases} -1 & \text{if } \phi(n) < p^r, \\ p-2 & \text{if } \phi(n) = p^r \end{cases}$$

and

$$k + \nu_p(k!) = \phi(n) + \nu_p(\phi(n)!) - \gamma(r) - \begin{cases} 0 & \text{if } \phi(n) < p^r, \\ 1 & \text{if } \phi(n) = p^r. \end{cases}$$

In both cases we have $k \ge \phi(n - \gamma(r))$, since $n < \gamma(r+1)$ when $\phi(n) = p^r$. The reverse inequality is supplied by (b).

Part (d) is then an immediate consequence of (a)-(c). □

The properties of ψ postulated in (d) above are precisely those required of ϕ in the proof of Theorem 2.1 which follows.

Proof of Theorem 2.1. Both the hypothesis and the conclusion of the theorem are multiplicative: ϕ, as we have noted above, is subadditive and the $F_n(T)$ satisfy a Cartan formula. For let $P = \sum_{i \ge 0} \mathcal{P}^i w^i \in \mathcal{A}(p)[[w]]$ be the total Steenrod power and $P_T^{-1} = \sum_{i \ge 0} \chi(\mathcal{P}^i)(wT)^i$ in $\mathcal{A}(p)[T][[w]]$. So $P \cdot P_T^{-1} = \sum_{n \ge 0} F_n(T) w^n$. As both P and its inverse are multiplicative on cohomology classes, so is $P \cdot P_T^{-1}$.

It is, therefore, sufficient to verify the assertion in the theorem on a generator $e \in H^2(\mathbb{C}P^\infty; \mathbb{F}_p)$ for p odd or $e \in H^1(\mathbb{R}P^\infty; \mathbb{F}_2)$ for $p = 2$. (See, for example, Chapters I and VI of [9].)

A short calculation shows that

$$(P \cdot P_T^{-1})e = \sum_{r \geq 0} (-1)^r e^{p^r} (1 + we^{p-1})^{p^r} (wT)^{\gamma(r)}.$$

So

$$F_n(T)e = \begin{cases} e, & n = 0, \\ (-1)^r e^{p^r} (T-1)^{p^{r-1}} T^{\gamma(r-1)}, & n = \gamma(r),\ r \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\phi(\gamma(r)) = p^{r-1}$, this completes the proof. $\square$

The proof of Theorem 1 in [3] uses an auxiliary result which can be established by similar reasoning.

Proposition 2.7. *The polynomial*

$$G_n(T) = \sum_{i=0}^{n} \chi(\mathcal{P}^{n-i}) \mathcal{P}^i T^i \in \mathcal{A}(p)[T]$$

is of the form $(T-1)^r g(T^p)$*, where* $r \equiv n \,(\mathrm{mod}\, p)$.

Again statements are multiplicative and it suffices to check the result on the class e. One verifies that

$$G_n(T)e = \begin{cases} e, & n = 0, \\ (-1)^r e^{p^r} (1-T), & n = \gamma(r),\ r \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

So, setting $S = T - 1$ as above and expanding $T^{pm} G_n(T)$, it follows that

(2.8) $$\textstyle\sum_{i=0}^{n} \binom{i+pm}{j} \chi(\mathcal{P}^{n-i}) \mathcal{P}^i = 0, \quad \text{when } j \not\equiv n \ (\mathrm{mod}\, p).$$

The basic technique in these proofs can be found already in work of Atiyah and Hirzebruch [2]. Working in the category of finite CW-complexes we set

(2.9) $$\mathcal{V}^*(-) = H^*(-; \mathbb{F}_p) \otimes \mathbb{F}_p[w],$$

where w has degree -1 if $p = 2$ and $-2(p-1)$ if p is odd, and equip the cohomology theory $\mathcal{V}^*$ with the obvious product.

Any automorphism A of this multiplicative theory which fixes w is of the form

$$A = \sum_{i \geq 0} \alpha_i w^i, \quad \text{with } \alpha_i \in \mathcal{A}(p), \text{ and } \alpha_0 = 1.$$

Now $A(e) = \sum_{r \geq 0} a_r e^{p^r} w^{\gamma(r)}$, $a_r \in \mathbb{F}_p$, with $a_0 = 1$. (As we have restricted attention to finite complexes, we should work with the skeleta of the infinite-dimensional projective space.) We associate to A the formal power series

$$\xi_A(X) = \sum_{r \geq 0} a_r w^{\gamma(r)} X^{p^r} \in \mathbb{F}_p[w][[X]], \quad a_0 = 1,$$

and notice that $\xi_{AB} = \xi_B \circ \xi_A$.

Theorem 2.10 (Atiyah-Hirzebruch)**.** *The map $A \mapsto \xi_A(X)$ gives an anti-isomorphism between the group of automorphisms of the multiplicative cohomology theory $\mathcal{V}^*(-)$ which fix w and the group of formal power series of the form*

$$\sum_{r\geq 0} a_r w^{\gamma(r)} X^{p^r} \in \mathbb{F}_p[w][[X]], \quad a_0 = 1.$$

The basic example of such an automorphism is the total Steenrod power P. More generally, given a commutative $\mathbb{F}_p$-algebra R, one can consider R-automorphisms of the cohomology theory $\mathcal{V}^*(-) \otimes R$. Theorem 2.10 generalizes immediately. We have been working with the polynomial ring $R = \mathbb{F}_p[T]$.

For reference we write

$$\pi(X) = \xi_P(X), \qquad \pi_T(X) = \xi_{P_T}(X). \tag{2.11}$$

3. Connections with Adams operations in K-theory

The arithmetic result used to translate statements about complex K-theory with p-local coefficients into statements about cohomology operations with $\mathbb{F}_p$-coefficients is the following.

Lemma 3.1. *Let $f(T) \in \mathbb{Z}_{(p)}[T]$ be a polynomial such that p^n divides $f(k^{p-1})$ for every integer k prime to p. Then the* mod p *reduction of $f(T)$ in $\mathbb{F}_p[T]$ is divisible by $(T-1)^{\phi(n)}$.*

Proof. We recall that when $g(U) \in \mathbb{Q}[U]$ is a rational polynomial of degree d which has the property that $g(l) \in \mathbb{Z}_{(p)}$ for every $l \in \mathbb{Z}$, then for some $b_i \in \mathbb{Z}_{(p)}$

$$g(U) = \sum_{i=0}^{d} b_i \binom{U}{i}.$$

Given (non-zero) $f(T)$ of degree d, we define $g(U) = p^{-n} f(1+pU)$. If k is prime to p, then $k^{p-1} = 1 + pl$ for some $l \in \mathbb{Z}$. On the other hand, for any integer l there is some k prime to p with $k^{p-1} \equiv 1 + pl \pmod{p^n}$. So $g(l) \in \mathbb{Z}_{(p)}$ for all $l \in \mathbb{Z}$. Hence

$$f(T) = \sum_{i=0}^{d} p^n b_i \binom{(T-1)/p}{i}$$

for some $b_i \in \mathbb{Z}_{(p)}$.

By comparing coefficients of T^d, we see that $p^{n-d-\nu_p(d!)} b_d$ is equal to the leading coefficient $a_d \in \mathbb{Z}_{(p)}$ of $f(T)$. So

$$p^n b_d \binom{(T-1)/p}{d} \in \mathbb{Z}_{(p)}[T]$$

and has mod p reduction $a_d(T-1)^d$. If $d < \phi(n)$, then a_d is divisible by p. Otherwise, there is no restriction on a_d mod p. The proof is completed by induction on d. □

We note that for each integer $n \geq 0$ there is a polynomial $f(T)$ satisfying the hypotheses of Lemma 3.1 with mod p reduction equal to $(T-1)^{\phi(n)}$, namely

$$f(T) = p^k k! \binom{(T-1)/p}{k},$$

where $k = \phi(n)$. So the exponent $\phi(n)$ in Lemma 3.1 cannot be improved.

Until the final paragraph of this section we will now assume that p is an odd prime. We give two closely related ways of seeing that the relations of Theorem 2.1 arise from standard properties of Adams operations.

The first follows the approach of [7, 6]. For a finite CW-complex Z without homology p-torsion, it was shown in [8], by pulling the unstable Adams operations apart, that there exist homomorphisms $S^i : H^*(Z;\mathbb{Z}_{(p)}) \to H^{*+2(p-1)i}(Z;\mathbb{Z}_{(p)})$, with S^0 the identity, satisfying certain properties. (These operations are not natural: their construction depends upon a choice of isomorphism between K-theory and cohomology.) When reduced mod p, S^i coincides with the Steenrod power $\mathcal{P}^i$. Let us introduce

$$\Phi_n(T) = \sum_{i=0}^{n} S^{n-i}\chi(S^i)T^i,$$

where $\chi(S^i)$ is defined inductively by $\sum_{i=0}^{n} S^{n-i}\chi(S^i) = 0$ if $n > 0$ and is the identity homomorphism if $n = 0$. Corollary 2.12 of [8] asserts that the homomorphism $\Phi_n(k^{p-1})$ is divisible by p^n, as a linear operator, for each integer k prime to p.

Now Lemma 3.1 extends immediately to the polynomial $f(T) = \Phi_n(T)$ with values in the endomorphism ring of $H^*(Z;\mathbb{Z}_{(p)})$. Hence $F_n(T)$ is divisible by $(T-1)^{\phi(n)}$ as an endomorphism of $H^*(Z;\mathbb{F}_p)$. Since relations among the Steenrod powers are detected by torsion-free spaces, Theorem 2.1 follows.

The second approach is closer to that in Section 2. Let ℓ^* be the Adams summand of p-local connective complex K-theory. Again restricting attention to finite complexes, we define a multiplicative cohomology theory $\mathcal{W}^*$ by

$$\mathcal{W}^*(-) = H^*(-;\mathbb{Z}_{(p)}) \otimes \mathbb{Z}_{(p)}[w] = \bigoplus_{j\geq 0} H^{*+2(p-1)j}(-;\mathbb{Z}_{(p)})w^j,$$

where w has degree $-2(p-1)$. Both the Adams integral Chern character

$$\mathrm{ch} : \ell^*(-) \to \mathcal{W}^*(-)$$

and the Thom homomorphism $\dim : \ell^*(-) \to H^*(-;\mathbb{Z}_{(p)})$ are multiplicative transformations. We have $\mathrm{ch}(v) = pw$, where v is a Bott generator of $\ell^{-2(p-1)}$.

According to Adams there is the commutative diagram:

$$\begin{tikzcd}
\ell^*(-) \arrow[r, "\mathrm{ch}"] \arrow[d, "\dim"'] & \mathcal{W}^*(-) \arrow[dd, "\bmod p"] \\
H^*(-;\mathbb{Z}_{(p)}) \arrow[d, "\bmod p"'] & \\
H^*(-;\mathbb{F}_p) \arrow[r, "\mathcal{P}^{-1}"'] & \mathcal{V}^*(-)
\end{tikzcd}$$

(See [1] and, for example, [5].)

We choose a complex orientation for ℓ^* lifting the standard orientation on integral cohomology. Let $e_\ell(\lambda) \in \ell^2(Z)$ be the associated Euler class of a complex line bundle λ over a finite complex Z and $e(\lambda) \in H^2(Z;\mathbb{Z}_{(p)})$ be the standard Euler class.

By looking at the Hopf line bundle over $\mathbb{C}P^\infty$ we see that

$$\mathrm{ch}(e_\ell(\lambda)) = \sigma^{-1}(e(\lambda)),$$

for some formal power series $\sigma^{-1}(X) = \sum t_n w^n X^{(p-1)n+1}$, in $\mathbb{Z}_{(p)}[w][[X]]$, with $t_0 = 1$. The inverse formal power series $\sigma(X)$ has a similar form $\sum s_n w^n X^{(p-1)n+1}$

with $s_0 = 1$. We shall also need $\sigma_T(X) = \sum s_n(wT)^n X^{(p-1)n+1}$. From the commutative diagram above one deduces that σ and σ_T^{-1} lift the power series π and π_T^{-1} in $\mathbb{F}_p[w][[X]]$ corresponding to the total Steenrod power and its inverse, (2.11).

The Adams operations ψ^k are defined on $\ell^*(-)$ for k prime to p. We have

$$\mathrm{ch} \circ \psi^k = \psi^k_{\mathcal{W}} \circ \mathrm{ch},$$

where the corresponding operator $\psi^k_{\mathcal{W}}$ on $\mathcal{W}^*(-)$ is the identity on $H^*(-;\mathbb{Z}_{(p)})$ and $\psi^k_{\mathcal{W}}(w) = k^{p-1}w$.

In ℓ^*-theory we can write $\psi^k(e_\ell(\lambda)) = \Psi^k(e_\ell(\lambda))$, where

$$\Psi^k(X) = \sum c_n v^n X^{(p-1)n+1} \in \mathbb{Z}_{(p)}[v][[X]], \quad c_0 = 1.$$

The relations in Theorem 2.1 arise from the equality

$$\mathrm{ch}(\psi^k(e_\ell(\lambda)) = \psi^k_{\mathcal{W}}(\mathrm{ch}(e_\ell(\lambda))).$$

This gives

$$\Psi^k_{\mathcal{W}}(\sigma^{-1}(e)) = \sigma^{-1}_{k^{p-1}}(e),$$

where $\Psi^k_{\mathcal{W}}$ is obtained from Ψ^k by substituting pw for v. But by considering $\mathbb{C}P^\infty$, this is a general equality in $\mathbb{Z}_{(p)}[w][[X]]$ when e is replaced by X, and so

$$\Psi^k_{\mathcal{W}}(X) = \sigma^{-1}_{k^{p-1}}(\sigma(X)).$$

We consider the coefficient of $w^n X^{n(p-1)+1}$ in $\sigma_T^{-1}(\sigma(X))$, which is a polynomial $f_n(T) \in \mathbb{Z}_{(p)}[T]$. Now $f_n(k^{p-1})$, the coefficient of $w^n X^{n(p-1)+1}$ in $\Psi^k_{\mathcal{W}}(X)$, is divisible by p^n. The reduction of $f_n(T)$ mod p is the coefficient of $w^n X^{n(p-1)+1}$ in $\pi_T^{-1}(\pi(X))$. So Lemma 3.1 applies to give a non-computational proof of the essential step in the proof of Theorem 2.1 that $F_n(T)e$ is divisible by $(T-1)^{\phi(n)}$.

When $p = 2$, $\mathcal{P}^i$ in this section must be replaced by Sq^{2i} and the cohomology theories $\mathcal{V}^*(-)$ and $\mathcal{W}^*(-)$ re-defined with the class w of degree -2. Then the results above are valid. The connection with Theorem 2.1 can be made, as in [6], by noting that a homogeneous polynomial in the Sq^{2i} vanishes on the cohomology of a product of complex projective spaces if and only if the same polynomial in the Sq^i vanishes on the corresponding product of real projective spaces.

Remarks on formal groups. The formal power series occurring in Theorem 2.10 as the automorphisms of the cohomology theory $\mathcal{V}^*(-)$ are the strict automorphisms of the additive formal group law F over $\mathbb{F}_p[w]$: $F(X,Y) = X+Y$. (The degree of X is 1 if $p = 2$ and 2 if p is odd.) The ring of endomorphisms of F is the ring $\mathbb{F}_p[[\mathcal{F}]]$ of formal power series in the Frobenius $\mathcal{F}$, $\mathcal{F}(X) = X^p$. The automorphism group is the group of units in this ring with constant term 1 and is free abelian over (the p-adic integers) $\mathbb{Z}_p$ of rank $p-1$. More generally, if we work over a commutative $\mathbb{F}_p$-algebra R, the endomorphism ring is the twisted formal power series ring $R[[\mathcal{F}]]$, with $\mathcal{F}r = r^p\mathcal{F}$ for $r \in R$. The calculations in the proof of Theorem 2.1 are carried out in this ring, with $R = \mathbb{F}_p[T]$.

In Section 3, the power series σ and σ^{-1} are isomorphisms between the additive formal group law F over $\mathbb{Z}_{(p)}[w]$ and the group law F_ℓ coming from the chosen complex orientation of ℓ^*:

$$F(\sigma(X), \sigma(Y)) = \sigma(X) + \sigma(Y) = \sigma(F_\ell(X,Y)).$$

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