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COHOMOLOGY OF GROUPS WITH METACYCLIC SYLOW p-SUBGROUPS

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ABSTRACT. We determine the cohomology algebras $H^*(G;\mathbf{F}_p)$ for all groups G with a metacyclic Sylow p-subgroup. The complete p-local stable decomposition of the classifying space BG is also determined.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let P be a non-abelian metacyclic p-group of odd order and G a finite group with P as a Sylow p-subgroup. In this note we classify all possible mod-p cohomology algebras $H^*(G)$ and determine complete p-local stable splittings for the classifying spaces BG. Much of the topological part of this work was done by the first author in [D]; recent results on Swan groups [MP] have enabled us to show that in all cases $H^*(G)$ is given by a ring of invariants. Similar but less complete information for metacyclic 2-groups was obtained in [D1, MP2, M].

A metacyclic p-group is a p-group P which is an extension of a cyclic group by a cyclic group. Following [D] we say that P is *split* if P can be expressed by some split extension. We recall that up to isomorphism any non-abelian metacyclic p-group can be expressed as

$$P = P(p^m, p^n, p^l+1, p^q) = \langle x, y \mid x^{p^m} = 1, y^{p^n} = x^{p^q}, yxy^{-1} = x^{p^l+1}\rangle$$

for positive integers m, n, l, q satisfying $l, q \leq m$, $(p^l+1)^{p^n} \equiv 1 \bmod p^m$, $(p^l+1)p^q \equiv p^q \bmod p^m$, $n+l \geq m$ and $q+l \geq m$. In these terms P splits unless $m \neq q$ and $l < q < n$ [D, Thm. 3.1].

Let $W_G(P) = N_G(P)/P \cdot C_G(P)$; then $W_G(P) \leq Out(P)$. If P is split, then $Out(P) \cong O_pOut(P) \rtimes \mathbf{Z}/(p-1)$ where $O_pOut(P)$ is a Sylow p-subgroup [D, Prop. 3.2]. Therefore $W_G(P) = Z/d$ where d is a divisor of $p-1$. If P is non-split, $Out(P)$ is a p-group and so $W_G(P) = 1$. We denote by $\mathbf{F}_p[\cdot]$ and $E[\cdot]$ the polynomial and exterior algebras over $\mathbf{F}_p$.

Theorem 1.1. *As an algebra, $H^*(G)$ has one of the following forms:*
(1) *If P is split and $l \neq m-n$, then*

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[u_d, v] \otimes E[a_d, b]$$

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where $|u_d| = 2d$, $|v| = 2$, $|a_d| = 2d-1$, $|b| = 1$.

(2) *If P is split and $l = m - n$, then*

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[v, z] \otimes E[b, \alpha_{2i-1}, i = 1, \ldots, p]/R$$

where the relations R are given by

$$\alpha_{2i-1}\alpha_{2j-1} = 0, \quad 1 \le i, j \le p,$$

$$\alpha_{2i-1}v = 0, \quad 1 \le i \le p-1,$$

and $|b| = 1$, $|v| = 2$, $|z| = 2p$, $|\alpha_{2i-1}| = 2i - 1 + 2pd(i)$, *where* $0 \le d(i) < d$ *is the residue of* $-i$ *mod* d.

(3) *If P is non-split, then $H^*(G) = H^*(P)$ is isomorphic to the algebra of* (1) *with $d = 1$ if $m = l + q$ and to that of* (2) *with $d = 1$ if $m < l + q$.*

Generators for these cohomology groups are specified explicitly in the proof.

Remark 1.2. Groups exemplifying the cases above are easily given by $G = P \rtimes \mathbf{Z}/d$.

R. Lyons has suggested other, more natural examples, which occur as automorphism groups of Chevalley groups. For example, let $\mathbf{F}_q$ be a finite field of characteristic different from p such that the Sylow p-subgroup of $PSL_2(\mathbf{F}_q)$ has order p, i.e., $q^2 - 1$ is divisible by p but not by p^2. Then the Sylow p-subgroup of $H = PSL_2(\mathbf{F}_{q^p})$ is cyclic of order p^2. Let ϕ be the Frobenius automorphism of $\mathbf{F}_{q^p}$ of order p. Then it is easy to see that ϕ fixes a cyclic subgroup $C \le H$ of order $p^q + 1$ which contains one such Sylow p-subgroup. Thus $G = Aut(H) = PSL_2(\mathbf{F}_q) \rtimes \mathbf{Z}/p\langle\phi\rangle$ has $P = M_3(p)$ as a Sylow p-subgroup. Furthermore $N_H(C)$ is a dihedral group [Hu, II, 8.4 Satz] containing the permutation matrix of order two. Since this matrix is fixed by ϕ, $W_G(P) = \mathbf{Z}/2$ and $H^*(G)$ is of type (2) in Theorem 1.1 with $d = 2$.

The group cohomology of a group G is the cohomology of the classifying space BG of G. The space BG is stably homotopy equivalent to a wedge product of indecomposable spectra,

$$BG \simeq X_1 \vee X_2 \vee \cdots \vee X_n.$$

A complete stable decomposition of BG is a splitting into indecomposable spectra. The decomposition is unique up to stable homotopy type and ordering. If G is a p-group, then all of these spectra are p-local. Otherwise, if P is a Sylow p-subgroup of G, then a simple transfer argument shows the p-localization of BG is a stable summand of BP,

$$BP \simeq BG_p \vee Y,$$

where BG_p is the p-localization of BG. Hence BG_p consists of some, but possibly not all, of the summands of BP. Note $H^*(BG_p; \mathbf{F}_p) = H^*(BG; \mathbf{F}_p)$.

Each indecomposable spectrum X of BP corresponds up to conjugacy to a primitive idempotent e in the ring of stable self-maps $\{BP, BP\}$. The spectrum X is the infinite mapping telescope or homotopy colimit of e,

$$X \simeq eBP = Tel(BP \xrightarrow{e} BP) = hocolim(BP \xrightarrow{e} BP \xrightarrow{e} \cdots).$$

For more information see either [BF] or [MP1].

For the remainder of the paper all spectra are localized at the prime p. If P is a Swan group, then $BG \simeq BN_G(P) \simeq B(P \rtimes W_G(P))$. Thus determining the stable homotopy type of BG involves determining which summands have their cohomology left invariant by the action of the Weyl group of G.

$Z_pOut(P) \subseteq \{BP, BP\}$ is a subring, in fact a retract. Therefore, certain indecomposable summands of BP correspond to simple modules of the outer automorphism group $Out(P)$. A summand corresponding to a simple $Out(P)$-module is said to *originate* in BP. A summand originating in BP does not occur as the summand of the classifying space of any proper subgroup of P.

In this paragraph we introduce some notation for Theorem 1.3 below. $L(2,k)$ originates in $B(\mathbf{Z}/p\times\mathbf{Z}/p)$ and corresponds to $St\otimes(det)^k$ where St is the Steinberg module for $\mathbf{F}_pGL_2(\mathbf{F}_p)$ and det is the determinant module. It is well known that the group ring $\mathbf{F}_p[\mathbf{Z}/(p-1)]$ has a complete set of orthogonal primitive idempotents $e_0, ..., e_{p-2}$ [D]. Lifting these idempotents to $Z_p[\mathbf{Z}/(p-1)]$ determines a complete stable splitting of

$$B\mathbf{Z}/p^n \simeq \bigvee_{i=0}^{p-2} L(1,n,i),$$

where $L(1,n,i)$ originates in $B\mathbf{Z}/p^n$. For more information on these summands see [HK] and [D].

If P is a split metacyclic p-group, then since $W_G(P)$ is a p'-group, we have $W_G(P) \leq \mathbf{Z}/(p-1)$. Thus the primitive idempotents $e_0, ..., e_{p-2}$ above determine a stable splitting of BG. If P is non-split, then BP is stably indecomposable [D, Thm. 1.3].

Among the split metacyclic groups there is one which plays a special role, the extra-special modular group $M_3(p) = P(p^2, p, p+1, 1)$. It is characterized by its order and exponent which are p^3 and p^2 respectively.

Theorem 1.3. (1) *If P is split and $P \neq M_3(p)$, then*

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n), \quad e_iBP = X_i, \quad l = m-n, \ 1 \leq i \leq p-2.$$

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n) \vee L(1,n,0), \ \ e_iBP = \ X_i \vee L(1,n,i), \quad l \neq m-n, \ 1 \leq i \leq p-2.$$

(2) *If $P = M_3(p)$, then*

$$e_0BP = X_0 \vee \bigvee_{i=0}^{p-2} L(2,i) \vee L(1,1,i), \quad e_iBP = X_i, \qquad 1 \leq i \leq p-2,$$

where X_i originates in BP.

(3) *In both cases this yields a complete stable decomposition of BP and*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} e_{id}BP.$$

Corollary 1.4. *Localized at p, the complete stable decomposition of BG is given by one of the following:*

(1) *If P is non-split, then $BG \simeq BP$.*

(2) *If P is split and $P \neq M_3(p)$, then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{j=0}^{p-2} L(1,n,j), \qquad l = m-n.$$

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee L(1,n,id) \vee \bigvee_{j=0}^{p-2} L(1,n,j), \qquad l \neq m-n.$$

(3) *If $P = M_3(p)$, then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{i=0}^{p-2} L(2,i) \vee L(1,1,i).$$

If P is split and $P \neq M_3(p)$, then Theorem 1.3 and Corollary 1.4 were proved by the first author [D, Thm. 1.3].

Throughout we assume P is a non-abelian metacyclic p-group and p is an odd prime. All cohomology is taken with simple coefficients in $\mathbf{F}_p$ and all spaces are considered stably, localized at p.

2. Proofs

The classifying space BP is indecomposable if and only if P is non-split [D, Thm. 1.1]. Thus, since BG is a summand in BP, we have $BG \simeq BP$ and $H^*(G) \cong H^*(P)$. In this case the cohomology algebras are given by [Hb, Thm. B if $m < l+q$ and Thm. E if $m = l+q$]. This completes the proof of both theorems for the non-split case.

Before turning to the proof of Theorem 1.1 we recall the notion of a Swan group [MP]. A p-group P is called a *Swan group* if the cohomology of any group G with P as a Sylow p-subgroup is given by its invariants, i.e.,

$$res : H^*(G;\mathbf{F}_p) \xrightarrow{\cong} H^*(P;\mathbf{F}_p)^{W_G(P)}.$$

The following result of Dietz and Glauberman [MP] is fundamental to our classification.

Theorem 2.1. *If P is a metacyclic group of odd order, then P is a Swan group.*

Proof of Theorem 1.1 *for P split.* Let $\Phi(P) = \langle x^p, y^p \rangle$ be the Frattini subgroup. Then $P/\Phi(P) \cong \mathbf{Z}/p \times \mathbf{Z}/p = \langle \overline{x}, \overline{y} \rangle$. Thus quotienting by $\Phi(P)$ induces a homomorphism $Out(P) \xrightarrow{\pi} Aut(P/\Phi(P)) = GL_2(\mathbf{F}_p)$. By [D, Prop. 3.2] if P is split, then $Out(P) \cong O_p Out(P) \rtimes \mathbf{Z}/(p-1)$; moreover,

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \in Im(\pi). \tag{1}$$

Now consider the extension

$$1 \to N \to P \to K \to 1$$

where $N \cong \mathbf{Z}/p^m = \langle x \rangle$ and $K \cong \mathbf{Z}/p^n = \langle \overline{y} \rangle$. The Lyndon-Hochschild-Serre spectral sequence has $E_2 = \mathbf{F}_p[u,v] \otimes E[a,b]$ where $H^*(N) = \mathbf{F}_p[u] \otimes E[a]$, $H^*(K) = \mathbf{F}_p[v] \otimes E[b]$, $|a| = |b| = 1$, $|u| = |v| = 2$. Explicitly, a, b are given as canonical homomorphisms dual to x, y respectively, and $u = \beta_n(a)$, $v = \beta_m(b)$ are their respective Bocksteins.

If $l \neq m-n$, then this spectral sequence collapses at E_2 [Dh, Thm.1]. Since P is a Swan group, we need only compute invariants. Let ζ be a generator of $\mathbf{Z}/(p-1) \leq Out(P)$ so that $\gamma = \zeta^{p-1/d}$ generates $W_G(P) = \mathbf{Z}/d$. By (1) $\gamma^*(a) = c \cdot a$ where $c^d = 1$ is a primitive d-th root of unity; $\gamma^*(u) = c \cdot u$ by application of the Bockstein. Similarly γ^* is trivial on v, b. Computing we find $\gamma^*(u^k v^l a^\epsilon b^\delta) = u^k v^l a^\epsilon b^\delta$ iff $k + \epsilon \equiv 0 \bmod d$. Theorem 1.1 (1) follows with $u_d = u^d$, $a_d = u^{d-1}a$.

If $l = m-n$ the spectral sequence collapses at E_3 [Dh, Thm.2] and we have $E_3 = \mathbf{F}_p[z,v] \otimes E[b, \xi_{2i-1}, i = 1, ..., p]/R$ where $z = u^p$, $\xi_{2i-1} = au^{i-1}$. Relations are given by

$$\xi_{2i-1}\xi_{2j-1} = 0, \quad 1 \leq i,j \leq p,$$
$$\xi_{2i-1}v = 0, \quad 1 \leq i \leq p-1.$$

In this case, $\gamma^*(z) = c \cdot z$, $\gamma^*(\xi_{2i-1}) = c^i \cdot \xi_{2i-1}$. For $d > 1$ we have invariants z^d and $\alpha_{2i-1} = \xi_{2i-1} z^{d(i)}$, where $0 \leq d(i) < d$ is the residue of $-i \bmod d$. The result follows. For $d = 1$, $H^*(G) = H^*(P)$ and the result holds setting $\alpha_{2i-1} = \xi_{2i-1}$ since $d(i) = 0$ in this case. $\square$

Proof of Theorem 1.3 *for* $P = M_3(p)$. In this case we have [D, Thm. 1.1 (3)]

$$BP \simeq \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{k=0}^{p-2} L(2,k) \vee \bigvee_{k=0}^{p-2} L(1,1,k).$$

Since P is a Swan group, we may assume $G = N_G(P)$ and $C_G(P) < P$, i.e., $P \lhd G$ and $G = P \rtimes C$, where $C \leq \mathbf{Z}/(p-1)$. From (1) in the proof of Theorem 1.1 it is clear that the subgroup $\langle x \rangle \rtimes C$ is normal in $G = P \rtimes C$. Therefore, $\mathbf{Z}/p\langle y \rangle$ is a retract of G; hence, $B\mathbf{Z}/p$ is a summand of BG for every G with $M_3(p)$ as a Sylow p-subgroup. Thus

$$B\mathbf{Z}/p = \bigvee_{k=0}^{p-2} L(1,1,k)$$

is a summand of $e_0 BP = B(P \rtimes \mathbf{Z}/(p-1))$.

We are reduced to showing $L(2,k)$ is a summand of $e_0 BP$. Let $Q = \langle x^p, y \rangle \cong \mathbf{Z}/p \times \mathbf{Z}/p$. Since $L(2,k)$ corresponds to the simple $\mathbf{F}_p Out(Q)$-module $M_k = St \otimes (det)^k$, we can use the criterion developed in [MP1]. That is, since Q is not a retract of P and $C_P(Q) = Q$, we must show

$$\overline{N_G(Q)/Q} \cdot M_k \neq 0$$

where $\overline{H} = \sum h$ summed over $h \in H \leq \mathbf{F}_p(H)$. Since $C_G(Q)/Q$ is a p'-group, this is equivalent to

$$\overline{N_G(Q)/C_G(Q)} \cdot M_k \neq 0$$

where $N_G(Q)/C_G(Q) \leq GL_2(\mathbf{F}_p)$. An explicit description of the Steinberg module St may be given as follows: $St = \langle u^{p-1}, u^{p-2}v, \ldots, uv^{p-2}, v^{p-1}\rangle$ is the $\mathbf{F}_p$-module of polynomials in indeterminates u, v of homogeneous degree $p-1$ with $GL_2(\mathbf{F}_p)$ acting on $\langle u, v\rangle$ in the standard way [G]. Furthermore

$$\overline{N_G(Q)/C_G(Q)} = \overline{N_G(Q)/PC_G(Q)} \cdot \overline{P/Q}.$$

According to [D, Prop. 4.6 and Proof]

$$\overline{P/Q} \cdot M_k = \langle v^{p-1}\rangle.$$

Since $N_G(Q)/C_G(Q)$ is a p'-group normalizing P/Q, we may assume it is isomorphic to a subgroup of the Borel subgroup of upper triangular matrices, i.e., the matrices of the form

$$w = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Thus $w(v^{p-1}) = b^{p-1}v^{p-1} = v^{p-1}$, and so $\overline{N_G(Q)/PC_G(Q)} \cdot v^{p-1} \neq 0$. □

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