

COHOMOLOGY OF GROUPS WITH METACYCLIC SYLOW p -SUBGROUPS

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ABSTRACT. We determine the cohomology algebras $H^*(G; \mathbf{F}_p)$ for all groups G with a metacyclic Sylow p -subgroup. The complete p -local stable decomposition of the classifying space BG is also determined.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let P be a non-abelian metacyclic p -group of odd order and G a finite group with P as a Sylow p -subgroup. In this note we classify all possible mod- p cohomology algebras $H^*(G)$ and determine complete p -local stable splittings for the classifying spaces BG . Much of the topological part of this work was done by the first author in [D]; recent results on Swan groups [MP] have enabled us to show that in all cases $H^*(G)$ is given by a ring of invariants. Similar but less complete information for metacyclic 2-groups was obtained in [D1, MP2, M].

A metacyclic p -group is a p -group P which is an extension of a cyclic group by a cyclic group. Following [D] we say that P is *split* if P can be expressed by some split extension. We recall that up to isomorphism any non-abelian metacyclic p -group can be expressed as

$$P = P(p^m, p^n, p^l + 1, p^q) = \langle x, y \mid x^{p^m} = 1, y^{p^n} = x^{p^q}, yxy^{-1} = x^{p^l+1} \rangle$$

for positive integers m, n, l, q satisfying $l, q \leq m$, $(p^l+1)^{p^n} \equiv 1 \pmod{p^m}$, $(p^l+1)p^q \equiv p^q \pmod{p^m}$, $n+l \geq m$ and $q+l \geq m$. In these terms P splits unless $m \neq q$ and $l < q < n$ [D, Thm. 3.1].

Let $W_G(P) = N_G(P)/P \cdot C_G(P)$; then $W_G(P) \leq \text{Out}(P)$. If P is split, then $\text{Out}(P) \cong O_p \text{Out}(P) \rtimes \mathbf{Z}/(p-1)$ where $O_p \text{Out}(P)$ is a Sylow p -subgroup [D, Prop. 3.2]. Therefore $W_G(P) = \mathbf{Z}/d$ where d is a divisor of $p-1$. If P is non-split, $\text{Out}(P)$ is a p -group and so $W_G(P) = 1$. We denote by $\mathbf{F}_p[\cdot]$ and $E[\cdot]$ the polynomial and exterior algebras over \mathbf{F}_p .

Theorem 1.1. *As an algebra, $H^*(G)$ has one of the following forms:*

- (1) *If P is split and $l \neq m - n$, then*

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[u_d, v] \otimes E[a_d, b]$$

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where $|u_d| = 2d$, $|v| = 2$, $|a_d| = 2d - 1$, $|b| = 1$.

(2) If P is split and $l = m - n$, then

$$H^*(G) \cong H^*(P)^{W_G(P)} = \mathbf{F}_p[v, z] \otimes E[b, \alpha_{2i-1}, i = 1, \dots, p]/R$$

where the relations R are given by

$$\alpha_{2i-1}\alpha_{2j-1} = 0, \quad 1 \leq i, j \leq p,$$

$$\alpha_{2i-1}v = 0, \quad 1 \leq i \leq p - 1,$$

and $|b| = 1$, $|v| = 2$, $|z| = 2p$, $|\alpha_{2i-1}| = 2i - 1 + 2pd(i)$, where $0 \leq d(i) < d$ is the residue of $-i \pmod{d}$.

(3) If P is non-split, then $H^*(G) = H^*(P)$ is isomorphic to the algebra of (1) with $d = 1$ if $m = l + q$ and to that of (2) with $d = 1$ if $m < l + q$.

Generators for these cohomology groups are specified explicitly in the proof.

Remark 1.2. Groups exemplifying the cases above are easily given by $G = P \rtimes \mathbf{Z}/d$.

R. Lyons has suggested other, more natural examples, which occur as automorphism groups of Chevalley groups. For example, let \mathbf{F}_q be a finite field of characteristic different from p such that the Sylow p -subgroup of $PSL_2(\mathbf{F}_q)$ has order p , i.e., $q^2 - 1$ is divisible by p but not by p^2 . Then the Sylow p -subgroup of $H = PSL_2(\mathbf{F}_{q^p})$ is cyclic of order p^2 . Let ϕ be the Frobenius automorphism of \mathbf{F}_{q^p} of order p . Then it is easy to see that ϕ fixes a cyclic subgroup $C \leq H$ of order $p^a + 1$ which contains one such Sylow p -subgroup. Thus $G = \text{Aut}(H) = PSL_2(\mathbf{F}_q) \rtimes \mathbf{Z}/p\langle\phi\rangle$ has $P = M_3(p)$ as a Sylow p -subgroup. Furthermore $N_H(C)$ is a dihedral group [Hu, II, 8.4 Satz] containing the permutation matrix of order two. Since this matrix is fixed by ϕ , $W_G(P) = \mathbf{Z}/2$ and $H^*(G)$ is of type (2) in Theorem 1.1 with $d = 2$.

The group cohomology of a group G is the cohomology of the classifying space BG of G . The space BG is stably homotopy equivalent to a wedge product of indecomposable spectra,

$$BG \simeq X_1 \vee X_2 \vee \cdots \vee X_n.$$

A complete stable decomposition of BG is a splitting into indecomposable spectra. The decomposition is unique up to stable homotopy type and ordering. If G is a p -group, then all of these spectra are p -local. Otherwise, if P is a Sylow p -subgroup of G , then a simple transfer argument shows the p -localization of BG is a stable summand of BP ,

$$BP \simeq BG_p \vee Y,$$

where BG_p is the p -localization of BG . Hence BG_p consists of some, but possibly not all, of the summands of BP . Note $H^*(BG_p; \mathbf{F}_p) = H^*(BG; \mathbf{F}_p)$.

Each indecomposable spectrum X of BP corresponds up to conjugacy to a primitive idempotent e in the ring of stable self-maps $\{BP, BP\}$. The spectrum X is the infinite mapping telescope or homotopy colimit of e ,

$$X \simeq eBP = \text{Tel}(BP \xrightarrow{e} BP) = \text{hocolim}(BP \xrightarrow{e} BP \xrightarrow{e} \cdots).$$

For more information see either [BF] or [MP1].

For the remainder of the paper all spectra are localized at the prime p . If P is a Swan group, then $BG \simeq BN_G(P) \simeq B(P \rtimes W_G(P))$. Thus determining the stable homotopy type of BG involves determining which summands have their cohomology left invariant by the action of the Weyl group of G .

$Z_p \text{Out}(P) \subseteq \{BP, BP\}$ is a subring, in fact a retract. Therefore, certain indecomposable summands of BP correspond to simple modules of the outer automorphism group $\text{Out}(P)$. A summand corresponding to a simple $\text{Out}(P)$ -module is said to *originate* in BP . A summand originating in BP does not occur as the summand of the classifying space of any proper subgroup of P .

In this paragraph we introduce some notation for Theorem 1.3 below. $L(2, k)$ originates in $B(\mathbf{Z}/p \times \mathbf{Z}/p)$ and corresponds to $St \otimes (\det)^k$ where St is the Steinberg module for $\mathbf{F}_p GL_2(\mathbf{F}_p)$ and \det is the determinant module. It is well known that the group ring $\mathbf{F}_p[\mathbf{Z}/(p-1)]$ has a complete set of orthogonal primitive idempotents e_0, \dots, e_{p-2} [D]. Lifting these idempotents to $Z_p[\mathbf{Z}/(p-1)]$ determines a complete stable splitting of

$$B\mathbf{Z}/p^n \simeq \bigvee_{i=0}^{p-2} L(1, n, i),$$

where $L(1, n, i)$ originates in $B\mathbf{Z}/p^n$. For more information on these summands see [HK] and [D].

If P is a split metacyclic p -group, then since $W_G(P)$ is a p' -group, we have $W_G(P) \leq \mathbf{Z}/(p-1)$. Thus the primitive idempotents e_0, \dots, e_{p-2} above determine a stable splitting of BG . If P is non-split, then BP is stably indecomposable [D, Thm. 1.3].

Among the split metacyclic groups there is one which plays a special role, the extra-special modular group $M_3(p) = P(p^2, p, p+1, 1)$. It is characterized by its order and exponent which are p^3 and p^2 respectively.

Theorem 1.3. (1) *If P is split and $P \neq M_3(p)$, then*

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n), \quad e_iBP = X_i, \quad l = m - n, \quad 1 \leq i \leq p - 2.$$

$$e_0BP = X_0 \vee B(\mathbf{Z}/p^n) \vee L(1, n, 0), \quad e_iBP = X_i \vee L(1, n, i), \quad l \neq m - n, \quad 1 \leq i \leq p - 2.$$

(2) *If $P = M_3(p)$, then*

$$e_0BP = X_0 \vee \bigvee_{i=0}^{p-2} L(2, i) \vee L(1, 1, i), \quad e_iBP = X_i, \quad 1 \leq i \leq p - 2,$$

where X_i originates in BP .

(3) *In both cases this yields a complete stable decomposition of BP and*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} e_{id}BP.$$

Corollary 1.4. *Localized at p , the complete stable decomposition of BG is given by one of the following:*

- (1) *If P is non-split, then $BG \simeq BP$.*
- (2) *If P is split and $P \neq M_3(p)$, then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l = m - n.$$

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee L(1, n, id) \vee \bigvee_{j=0}^{p-2} L(1, n, j), \quad l \neq m - n.$$

- (3) *If $P = M_3(p)$, then*

$$BG \simeq \bigvee_{i=0}^{(p-1/d)-1} X_{id} \vee \bigvee_{i=0}^{p-2} L(2, i) \vee L(1, 1, i).$$

If P is split and $P \neq M_3(p)$, then Theorem 1.3 and Corollary 1.4 were proved by the first author [D, Thm. 1.3].

Throughout we assume P is a non-abelian metacyclic p -group and p is an odd prime. All cohomology is taken with simple coefficients in \mathbf{F}_p and all spaces are considered stably, localized at p .

2. PROOFS

The classifying space BP is indecomposable if and only if P is non-split [D, Thm. 1.1]. Thus, since BG is a summand in BP , we have $BG \simeq BP$ and $H^*(G) \cong H^*(P)$. In this case the cohomology algebras are given by [Hb, Thm. B if $m < l + q$ and Thm. E if $m = l + q$]. This completes the proof of both theorems for the non-split case.

Before turning to the proof of Theorem 1.1 we recall the notion of a Swan group [MP]. A p -group P is called a *Swan group* if the cohomology of any group G with P as a Sylow p -subgroup is given by its invariants, i.e.,

$$\text{res} : H^*(G; \mathbf{F}_p) \xrightarrow{\cong} H^*(P; \mathbf{F}_p)^{W_G(P)}.$$

The following result of Dietz and Glauberman [MP] is fundamental to our classification.

Theorem 2.1. *If P is a metacyclic group of odd order, then P is a Swan group.*

Proof of Theorem 1.1 for P split. Let $\Phi(P) = \langle x^p, y^p \rangle$ be the Frattini subgroup. Then $P/\Phi(P) \cong \mathbf{Z}/p \times \mathbf{Z}/p = \langle \bar{x}, \bar{y} \rangle$. Thus quotienting by $\Phi(P)$ induces a homomorphism $\text{Out}(P) \xrightarrow{\pi} \text{Aut}(P/\Phi(P)) = GL_2(\mathbf{F}_p)$. By [D, Prop. 3.2] if P is split, then $\text{Out}(P) \cong O_p \text{Out}(P) \rtimes \mathbf{Z}/(p-1)$; moreover,

$$(1) \quad \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \in \text{Im}(\pi).$$

Now consider the extension

$$1 \rightarrow N \rightarrow P \rightarrow K \rightarrow 1$$

where $N \cong \mathbf{Z}/p^m = \langle x \rangle$ and $K \cong \mathbf{Z}/p^n = \langle \bar{y} \rangle$. The Lyndon-Hochschild-Serre spectral sequence has $E_2 = \mathbf{F}_p[u, v] \otimes E[a, b]$ where $H^*(N) = \mathbf{F}_p[u] \otimes E[a]$, $H^*(K) = \mathbf{F}_p[v] \otimes E[b]$, $|a| = |b| = 1$, $|u| = |v| = 2$. Explicitly, a, b are given as canonical homomorphisms dual to x, y respectively, and $u = \beta_n(a)$, $v = \beta_m(b)$ are their respective Bocksteins.

If $l \neq m - n$, then this spectral sequence collapses at E_2 [Dh, Thm.1]. Since P is a Swan group, we need only compute invariants. Let ζ be a generator of $\mathbf{Z}/(p-1) \leq \text{Out}(P)$ so that $\gamma = \zeta^{p-1/d}$ generates $W_G(P) = \mathbf{Z}/d$. By (1) $\gamma^*(a) = c \cdot a$ where $c^d = 1$ is a primitive d -th root of unity; $\gamma^*(u) = c \cdot u$ by application of the Bockstein. Similarly γ^* is trivial on v, b . Computing we find $\gamma^*(u^k v^l a^\epsilon b^\delta) = u^k v^l a^\epsilon b^\delta$ iff $k + \epsilon \equiv 0 \pmod d$. Theorem 1.1 (1) follows with $u_d = u^d$, $a_d = u^{d-1}a$.

If $l = m - n$ the spectral sequence collapses at E_3 [Dh, Thm.2] and we have $E_3 = \mathbf{F}_p[z, v] \otimes E[b, \xi_{2i-1}, i = 1, \dots, p]/R$ where $z = u^p$, $\xi_{2i-1} = au^{i-1}$. Relations are given by

$$\begin{aligned} \xi_{2i-1}\xi_{2j-1} &= 0, & 1 \leq i, j \leq p, \\ \xi_{2i-1}v &= 0, & 1 \leq i \leq p-1. \end{aligned}$$

In this case, $\gamma^*(z) = c \cdot z$, $\gamma^*(\xi_{2i-1}) = c^i \cdot \xi_{2i-1}$. For $d > 1$ we have invariants z^d and $\alpha_{2i-1} = \xi_{2i-1}z^{d(i)}$, where $0 \leq d(i) < d$ is the residue of $-i \pmod d$. The result follows. For $d = 1$, $H^*(G) = H^*(P)$ and the result holds setting $\alpha_{2i-1} = \xi_{2i-1}$ since $d(i) = 0$ in this case. □

Proof of Theorem 1.3 for $P = M_3(p)$. In this case we have [D, Thm. 1.1 (3)]

$$BP \simeq \bigvee_{i=0}^{p-2} X_i \vee \bigvee_{k=0}^{p-2} L(2, k) \vee \bigvee_{k=0}^{p-2} L(1, 1, k).$$

Since P is a Swan group, we may assume $G = N_G(P)$ and $C_G(P) < P$, i.e., $P \triangleleft G$ and $G = P \rtimes C$, where $C \leq \mathbf{Z}/(p-1)$. From (1) in the proof of Theorem 1.1 it is clear that the subgroup $\langle x \rangle \rtimes C$ is normal in $G = P \rtimes C$. Therefore, $\mathbf{Z}/p\langle y \rangle$ is a retract of G ; hence, $B\mathbf{Z}/p$ is a summand of BG for every G with $M_3(p)$ as a Sylow p -subgroup. Thus

$$B\mathbf{Z}/p = \bigvee_{k=0}^{p-2} L(1, 1, k)$$

is a summand of $e_0BP = B(P \rtimes \mathbf{Z}/(p-1))$.

We are reduced to showing $L(2, k)$ is a summand of e_0BP . Let $Q = \langle x^p, y \rangle \cong \mathbf{Z}/p \times \mathbf{Z}/p$. Since $L(2, k)$ corresponds to the simple $\mathbf{F}_p \text{Out}(Q)$ -module $M_k = St \otimes (\det)^k$, we can use the criterion developed in [MP1]. That is, since Q is not a retract of P and $C_P(Q) = Q$, we must show

$$\overline{N_G(Q)/Q} \cdot M_k \neq 0$$

where $\overline{H} = \sum h$ summed over $h \in H \leq \mathbf{F}_p(H)$. Since $C_G(Q)/Q$ is a p' -group, this is equivalent to

$$\overline{N_G(Q)/C_G(Q)} \cdot M_k \neq 0$$

where $N_G(Q)/C_G(Q) \leq GL_2(\mathbf{F}_p)$. An explicit description of the Steinberg module St may be given as follows: $St = \langle u^{p-1}, u^{p-2}v, \dots, uv^{p-2}, v^{p-1} \rangle$ is the \mathbf{F}_p -module of polynomials in indeterminates u, v of homogeneous degree $p-1$ with $GL_2(\mathbf{F}_p)$ acting on $\langle u, v \rangle$ in the standard way [G]. Furthermore

$$\overline{N_G(Q)/C_G(Q)} = \overline{N_G(Q)/PC_G(Q)} \cdot \overline{P/Q}.$$

According to [D, Prop. 4.6 and Proof]

$$\overline{P/Q} \cdot M_k = \langle v^{p-1} \rangle.$$

Since $N_G(Q)/C_G(Q)$ is a p' -group normalizing P/Q , we may assume it is isomorphic to a subgroup of the Borel subgroup of upper triangular matrices, i.e., the matrices of the form

$$w = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Thus $w(v^{p-1}) = b^{p-1}v^{p-1} = v^{p-1}$, and so $\overline{N_G(Q)/PC_G(Q)} \cdot v^{p-1} \neq 0$. \square

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