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MULTIPLE PATH-VALUED CONDITIONAL YEH-WIENER INTEGRALS

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ABSTRACT. In this paper we establish various results involving parallel line-valued conditional Yeh-Wiener integrals of the type $E(F(x)|x(s_j,\cdot)=\eta_j(\cdot), j=1,\dots,n)$ where $0<s_1<\dots<s_n$. We then develop a formula for converting these multiple path-valued conditional Yeh-Wiener integrals into ordinary Yeh-Wiener integrals. Next, conditional Yeh-Wiener integrals for functionals F of the form

$$F(x)=\exp\left\{\int_0^S\int_0^T \phi(s,t,x(s,t))\,dt\,ds\right\}$$

are evaluated by solving an appropriate Wiener integral equation. Finally, a Cameron-Martin translation theorem is obtained for these multiple path-valued conditional Yeh-Wiener integrals.

1. INTRODUCTION

For $Q=[0,S]\times[0,T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s,t)$ on Q such that $x(0,t)=x(s,0)=0$ for every (s,t) in Q. Yeh [11] defined a Gaussian measure m_y on $C(Q)$ (later modified in [14]) such that as a stochastic process $\{x(s,t),(s,t)\in Q\}$ has mean $E[x(s,t)]=\int_{C(Q)}x(s,t)m_y(dx)=0$ and covariance $E[x(s,t)x(u,v)]=\min\{s,u\}\min\{t,v\}$. Let $C_W\equiv C[0,T]$ denote the standard Wiener space on $[0,T]$ with Wiener measure m_w. Yeh [13] introduced the concept of the conditional Wiener integral of F given X, $E(F|X)$, and for the case $X(x)=x(T)$ obtained some very useful results including a Kac-Feynman integral equation.

A very important class of functions in quantum mechanics consists of functions on $C[0,T]$ of the type

$$G(x)=\exp\left\{\int_0^T\theta(s,x(s))\,ds\right\}$$

where $\theta\colon[0,T]\times\mathbb{R}\to\mathbb{C}$.

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Yeh [13] shows that under suitable regularity conditions on θ, the conditional Wiener integral

$$H(t,\xi) = (2\pi t)^{-1/2} \exp\left\{-\frac{\xi^2}{2t}\right\} E\left(\exp\left\{\int_0^t \theta(s,x(s))\,ds\right\} |x(t)=\xi\right) \tag{1.1}$$

satisfied the Kac-Feynman integral equation

$$\begin{aligned} H(t,\xi) =&(2\pi t)^{-1/2} \exp\left\{-\frac{\xi^2}{2t}\right\} \\ &+ \int_0^t [2\pi(t-s)]^{-1/2} \int_{\mathbb{R}} \theta(s,\eta) H(s,\eta) \exp\left\{-\frac{(\eta-\xi)^2}{2(t-s)}\right\} d\eta\,ds \end{aligned} \tag{1.2}$$

whose solution can be expressed as an infinite series of terms involving Lebesgue integrals. Then using (1.1), one can use the series solution of (1.2) to evaluate the conditional Wiener integral

$$E\left(\exp\left\{\int_0^t \theta(s,x(s))\,ds\right\} |x(t)=\xi\right).$$

The corresponding problem in Yeh-Wiener space, namely to evaluate

$$E\left(\exp\left\{\int_0^t \int_0^s \phi(u,v,x(u,v))\,du\,dv\right\} |x(x,t)=\xi\right), \tag{1.3}$$

turned out to be substantially different from the corresponding one-parameter problem. After many attempts to solve this problem by several mathematicians, the first really successful solution was given by Park and Skoug [9] by introducing a sample path-valued conditional Yeh-Wiener integral of the type

$$E\left(\exp\left\{\int_0^t \int_0^s \phi(u,v,x(u,v))\,du\,dv\right\} |x(s,\cdot)=\eta(\cdot)\right), \tag{1.4}$$

which satisfies a Wiener integral equation similar to that of Cameron and Storvick [2]. The Wiener integral equation is then solved to evaluate (1.4), and finally (1.3) is obtained by integrating (1.4) appropriately.

In this paper we consider parallel line-valued conditional Yeh-Wiener integrals of the type

$$E(F(x)|x(s_j,\cdot)=\eta_j(\cdot),\ j=1,\dots,n), \tag{1.5}$$

where $F \in L_1(C(Q),m_y)$ and $0=s_0<s_1<\cdots<s_n=S$. In section 2 we develop a formula for converting these conditional Yeh-Wiener integrals into ordinary Yeh-Wiener integrals and in section 3 we use this formula to evaluate (1.5) for various functionals F. In section 4, we evaluate (1.5) for functionals F of the form $F(x) = \exp\{\int_0^S \int_0^T \phi(u,t,x(u,t))\,dt\,du\}$ by solving an appropriate Wiener integral equation. Finally, in section 5, a Cameron-Martin type translation theorem is obtained for conditional Yeh-Wiener integrals of the type (1.5).

2. Parallel line-valued conditional Yeh-Wiener integrals

Let $\sigma\colon 0=s_0<s_1<\cdots<s_n=S$ be any partition of $[0,S]$. For $x \in C(Q)$, define $X_\sigma(x)$ by

$$X_\sigma(x) = (x(s_1,\cdot),\dots,x(s_n,\cdot)). \tag{2.1}$$

We also define an s-sectional function x_σ of $x \in C(Q)$ by

$$(2.2) \quad x_\sigma(s,\cdot) = x(s_{j-1},\cdot) + \frac{s - s_{j-1}}{s_j - s_{j-1}}[x(s_j,\cdot) - x(s_{j-1},\cdot)], \qquad s_{j-1} \le s \le s_j,\ j = 1,\ldots,n.$$

Similarly for $\overline{\eta}(\cdot) = (\eta_1(\cdot),\ldots,\eta_n(\cdot))$, $\eta_j(\cdot) \in C[0,T]$, we define

$$(2.3) \quad \overline{\eta}_\sigma(s,\cdot) = \eta_{j-1}(\cdot) + \frac{s - s_{j-1}}{s_j - s_{j-1}}[\eta_j(\cdot) - \eta_{j-1}(\cdot)], \qquad s_{j-1} \le s \le s_j,\ j = 1,\ldots,n,$$

where $\eta_0(\cdot) \equiv 0$.

Our first result, which plays a key role throughout this paper, involves the stochastic independence between $x - x_\sigma$ and $X_\sigma(x)$ and between $x - x_\sigma$ and x.

Theorem 1. *If $\{x(s,t),(s,t) \in Q\}$ is the standard Yeh-Wiener process, then $x - x_\sigma$ and $X_\sigma(x)$ are stochastically independent on Q. In addition $x - x_\sigma$ and x are stochastically independent on distinct rectangles $[s_{i-1},s_i] \times [0,T]$ and $[s_{j-1},s_j] \times [0,T]$ of Q, where the s_k are partition points in σ.*

Proof. For $s_{j-1} \le s \le s_j$ we note that

$$(2.4) \quad x(s,t) - x_\sigma(s,t) = x(s,t) - x(s_{j-1},t) - \frac{s - s_{j-1}}{s_j - s_{j-1}}[x(s_j,t) - x(s_{j-1},t)].$$

Thus, to show that $x - x_\sigma$ and $X_\sigma(x)$ are independent, it is sufficient to show that (2.4) is independent of $x(s_k,t')$ for $k = 1,\ldots,n$. Using the formula

$$E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$$

it is easy to establish that

$$E\{x(s_k,t')[x(s,t) - x_\sigma(s,t)]\} = 0.$$

Since uncorrelated Gaussian processes are independent, it follows that $x - x_\sigma$ and $X_\sigma(x)$ are independent. The independence of $x - x_\sigma$ and x on distinct rectangles of Q follows similarly. $\square$

The following corollary is an immediate consequence of Theorem 1.

Corollary. *The two processes $x - x_\sigma$ and x_σ are independent Gaussian processes on Q.*

Our next theorem, in which we express a conditional Yeh-Wiener integral in terms of a Yeh-Wiener integral, also plays a key role throughout this paper.

Theorem 2. *Let $F \in L_1(C(Q))$. Then*

$$E(F(x)|X_\sigma(x) = \overline{\eta}(\cdot)) = E[F(x - x_\sigma + \overline{\eta}_\sigma].$$

Proof. Under the conditioning $X_\sigma(x) = \overline{\eta}(\cdot)$, we have that $x_\sigma = \overline{\eta}_\sigma$, and so

$$E(F(x)|X_\sigma(x) = \overline{\eta}(\cdot)) = E(f(x - x_\sigma + \overline{\eta}_\sigma)|X_\sigma(x) = \overline{\eta}(\cdot)).$$

The result now follows because $x - x_\sigma$ and $X_\sigma(x)$ are independent by Theorem 1. $\square$

The following theorem shows how to recover $E[F(x)]$ from the conditional Yeh-Wiener integral.

Theorem 3. *Let $F \in L_1(C(Q))$ and let $w_1, \dots, w_n$ be n independent Wiener processes on $[0,T]$ which are independent of the Yeh-Wiener process x. Let $\overline{\eta}(\cdot) = (\eta_1(\cdot), \dots, \eta_n(\cdot))$ where, for $j = 1, \dots, n$, $\eta_j(\cdot) = \sum_{k=1}^{j}(s_k - s_{k-1})^{1/2} w_k(\cdot)$. Then*

$$E_{w_1,\dots,w_n}\{E(F(x)|X_\sigma(x) = \overline{\eta}(\cdot))\} = E[F(x)].$$

Proof. By Theorem 2, we have that $E(F(x)|X_\sigma(x) = \overline{\eta}(\cdot)) = E[F(x - x_\sigma + \overline{\eta}_\sigma)]$. Let $y(s,t) = x(s,t) - x_\sigma(s,t) + \overline{\eta}_\sigma(s,t)$. Then y, as a process depending on x and the w_j's, has mean $E[y(s,t)] = 0$ and covariance $E[y(s,t)y(u,v)] = \min\{s,u\}\min\{t,v\}$. Thus y is a standard Yeh-Wiener process on Q, and so

$$E_{w_1,\dots,w_n}\{E_x[F(x - x_\sigma + \overline{\eta}_\sigma)]\} = \int_{C(Q)} F(y) m_y(dy) = E[F(x)],$$

which completes the proof. □

In the following sections, we shall consider stochastic integrals of $h \in L_2(Q)$ with respect to x_σ, defined by (2.2). Such stochastic integrals have convenient representations with respect to $\hat{h}$, the s-sectional average of h.

Definition. Let σ be as before. For each function $h \in L_2(Q)$, define the s-sectional average of h by

$$\hat{h}(s,t) = \begin{cases} \frac{1}{s_j - s_{j-1}} \int_{s_{j-1}}^{s_j} h(u,t)\,du, & s_{j-1} < s \le s_j,\ j = 1,2,\dots,n, \\ 0, & st = 0. \end{cases} \tag{2.5}$$

The following theorem gives some useful and interesting formulas involving $\hat{h}$, h and x_σ. The proof is rather straightforward and hence omitted. A similar observation was made in [8, p. 456]. In particular, the observation in (2.9) that the stochastic integrals $\int_q h\,dx_\sigma$ and $\int_q \hat{h}\,dx_\sigma$ both equal the stochastic integral $\int_q \hat{h}\,dx$ is very useful in evaluating various expectations.

Theorem 4. *Let $h \in L_2(Q)$. Then*

$$\int_Q h(s,t)\hat{h}(s,t)\,ds\,dt = \int_Q [\hat{h}(s,t)]^2\,ds\,dt, \tag{2.6}$$

$$\|h\|_1 \ge \|\hat{h}\|_1 \text{ and } \|h\|_1 = \|\hat{h}\|_1 \text{ if either } h \ge 0 \text{ or } h \le 0 \text{ a.e. on } Q, \tag{2.7}$$

$$\|h - \hat{h}\|_2^2 = \|h\|_2^2 - \|\hat{h}\|_2^2 \ge 0, \tag{2.8}$$

and

$$\int_Q h\,dx_\sigma = \int_Q \hat{h}\,dx = \int_Q \hat{h}\,dx_\sigma \quad \text{for every } x \in C(Q). \tag{2.9}$$

3. Examples

In this section we illustrate that Theorem 2 makes the evaluation of the multiple path-valued conditional Yeh-Wiener integral of certain functionals rather simple.

Example 1. For $x \in C(Q)$, let $F(x) = \int_Q x(s,t)\,ds\,dt$. Then, by Theorem 2 and the Fubini Theorem, we obtain

$$I \equiv E\left(\int_Q x(s,t)\,ds\,dt | X_\sigma(x) = \overline{\eta}(\cdot)\right) = \sum_{j=1}^{n} \frac{s_j - s_{j-1}}{2} \int_0^T [\eta_j(t) - \eta_{j-1}(t)]\,dt.$$

In particular, if $n=1$, then $I = \frac{S}{2}\int_0^T \eta_1(t)\,dt$, which agrees with the computation in [9].

Example 2. Let $F(x) = \int_Q x^2(s,t)\,ds\,dt$. Then, proceeding as in Example 1 above, we obtain

$$\begin{aligned} I &\equiv E\left(\int_Q x^2(s,t)\,ds\,dt | X_\sigma(x) = \overline{\eta}(\cdot)\right) \\ &= \frac{T^2}{12}\sum_{j=1}^{n}(s_j - s_{j-1})^2 + \int_Q \overline{\eta}_\sigma^2(s,t)\,ds\,dt. \end{aligned}$$

If $n=1$, then $I = \frac{S^2T^2}{12} + \frac{S}{3}\int_0^T \eta_1^2(t)\,dt$, which agrees with the calculation in [9].

Example 3. Let $M(L_2(Q))$ be the class of all countably additive complex-valued Borel measures on $L_2(Q)$. The Banach algebra $\mathcal{S}(2)$ consists of the functionals on $C(Q)$ of the form

$$F(x) = \int_{L_2(Q)} \exp\left\{i\int_Q v(s,t)\,dx(s,t)\right\}\,d\delta(v)$$

with $\delta \in M(L_2(Q))$. Using Theorem 2, the Fubini Theorem, and letting $\hat{v}$ denote the s-sectional average function of v given by (2.5), we see that for each $F \in \mathcal{S}(2)$,

$$\begin{aligned} J &\equiv E(F(x)|X_\sigma(x) = \overline{\eta}(\cdot)) \\ &= \int_{L_2(Q)} \exp\left\{i\int_Q v\,d\overline{\eta}_\sigma\right\} E\left[\exp\left\{i\int_Q (v-\hat{v})\,dx\right\}\right]\,d\delta(v). \end{aligned}$$

Hence, an application of a familiar Yeh-Wiener integration formula yields

$$J = \int_{L_2(Q)} \exp\left\{-\frac{1}{2}(\|v\|^2 - \|\hat{v}\|^2)\right\}\exp\left\{i\int_Q v\,d\overline{\eta}_\sigma\right\}\,d\delta(v). \tag{3.1}$$

Example 4. In this example we show that Theorem 3 holds for all $F \in \mathcal{S}(2)$ in Example 3 above. For let

$$\eta_j(\cdot) = \sum_{k=1}^{j}(s_k - s_{k-1})^{1/2} w_k(\cdot), \qquad j = 1, \ldots, n.$$

Then, using (2.3) we have,

$$\int_Q v(s,t)d\overline{\eta}_\sigma(s,t) = \sum_{j=1}^{n}\int_0^T\left[\int_{s_{j-1}}^{s_j} v(s,t)(s_j - s_{j-1})^{-1/2}\,ds\right]\,dw_j(t).$$

Substituting this into (3.1), and then integrating with respect to $w_1, \ldots, w_n$, yields

$$\begin{aligned}
E_{w_1,\ldots,w_n}(J) &= \int_{L_2(Q)} \exp\left\{-\frac{1}{2}[\|v\|_2^2 - \|\hat{v}\|_2^2]\right\} \\
&\quad \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\int_0^T \left[\int_{s_{j-1}}^{s_j} v(s,t)(s_j - s_{j-1})^{-1/2}\, ds\right]^2 dt\right\} d\delta(v) \\
&= \int_{L_2(Q)} \exp\left\{-\frac{1}{2}\left[\|v\|_2^2 - \|\hat{v}\|_2^2\right]\right\} \exp\left\{-\frac{1}{2}\int_Q \hat{v}^2(s,t)\, ds\, dt\right\} d\delta(v) \\
&= \int_{L_2(Q)} \exp\left\{-\frac{1}{2}\|v\|_2^2\right\} d\delta(v) \\
&= E\left[\int_{L_2(Q)} \exp\left\{i\int_Q v(s,t)\, dx(s,t)\right\} d\delta(v)\right] = E[F(x)].
\end{aligned}$$

4. Evaluation of $E(\exp\{\int_Q \phi(u,t,x(u,t))\, du\, dt\} | X_\sigma(x) = \overline{\eta}(\cdot))$

Let $\phi(s,t,v)$ be a bounded continuous function on $Q \times \mathbb{R}$, and let

$$\theta(u, x(u,\cdot)) = \int_0^T \phi(u,t,x(u,t))\, dt. \tag{4.1}$$

Park and Skoug [9] have shown that the function G defined on $[0,S] \times C[0,T]$ by

$$G(s,\eta(\cdot)) \equiv E\left(\exp\left\{\int_0^s \theta(u, x(u,\cdot))\, du\right\} | x(s,\cdot) = \eta(\cdot)\right) \tag{4.2}$$

satisfies the Wiener integral equation

$$\begin{aligned}
G(s,\eta(\cdot)) = 1 + \int_0^s E_w &\left[\theta\left(u, \sqrt{u\left(1-\frac{u}{s}\right)}w(\cdot) + \frac{u}{s}\eta(\cdot)\right)\right. \\
&\left.\cdot\, G\left(u, \sqrt{u\left(1-\frac{u}{s}\right)}w(\cdot) + \frac{u}{s}\eta(\cdot)\right)\right] du,
\end{aligned} \tag{4.3}$$

whose solution is given by

$$G(s,\eta(\cdot)) = \sum_{k=1}^{\infty} H_k(s,\eta(\cdot)), \tag{4.4}$$

where the sequence $\{H_k\}$ is given inductively by $H_0(s,\eta(\cdot)) \equiv 1$, and

$$H_{k+1}(s,\eta(\cdot)) = \int_0^s E_w\left[(\theta \cdot H_k)\left(u, \sqrt{u\left(1-\frac{u}{s}\right)}w(\cdot) + \frac{u}{s}\eta(\cdot)\right)\right] du,$$

where $(\theta \cdot H_k)(u,v) = \theta(u,v)H_k(u,v)$. Furthermore, if $|\theta(s,v)| \leq M$ on $[0,S] \times \mathbb{R}$, then

$$|H_k(s,\eta(\cdot))| \leq \frac{(Ms)^k}{k!} \leq \frac{(MS)^k}{k!} \quad \text{for } k = 0,1,2,\dots . \tag{4.5}$$

Under the same assumptions on ϕ and θ, we now proceed to find the multi-conditional expectation

$$E\left(\exp\left\{\int_0^S \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x) = \overline{\eta}(\cdot)\right).$$

We start with the following lemmas.

Lemma 1. *Let $\sigma\colon 0 = s_0 < s_1 < \cdots < s_n = S$ be any partition on $[0,S]$, and let θ be given by* (4.1)*. Then for $j = 1,2,\dots,n$,*

$$\begin{aligned} &E\left(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x) = \overline{\eta}(\cdot)\right) \\ &\quad = E\left(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|x(s_{j-1},\cdot) = \eta_{j-1}(\cdot),\ x(s_j,\cdot) = \eta_j(\cdot)\right). \end{aligned}$$

Proof. By Theorem 2, (2.2), and (2.3),

$$\begin{aligned} &E\bigg(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x) = \overline{\eta}(\cdot)\bigg) \\ &\quad = E\bigg[\exp\bigg\{\int_{s_{j-1}}^{s_j} \theta\bigg(u, x(u,\cdot) - x(s_{j-1},\cdot) - \frac{u - s_{j-1}}{s_j - s_{j-1}}[x(s_j,\cdot) - x(s_{j-1},\cdot)] \\ &\qquad\qquad + \eta_{j-1}(\cdot) + \frac{u - s_{j-1}}{s_j - s_{j-1}}[\eta_j(\cdot) - \eta_{j-1}(\cdot)]\bigg)\,du\bigg\}\bigg] \\ &\quad = E\bigg(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|x(s_{j-1},\cdot) = \eta_{j-1}(\cdot), x(s_j,\cdot) = \eta_j(\cdot)\bigg). \end{aligned}$$

□

Lemma 2. *Under the assumptions of Lemma 1,*

$$\begin{aligned} &E\left(\exp\left\{\int_0^S \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x) = \overline{\eta}(\cdot)\right) \\ &\quad = \prod_{j=1}^n E\left(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|x(s_{j-1},\cdot) = \eta_{j-1}(\cdot), x(s_j,\cdot) = \eta_j(\cdot)\right). \end{aligned}$$

Proof. One can easily establish that the processes $\{x(s,\cdot)-x_\sigma(s,\cdot), s_{j-1}\le s\le s_j\}$, $j=1,\dots,n$, are independent. Thus

$$\begin{aligned}
&E\left(\exp\left\{\int_0^S \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x)=\overline{\eta}(\cdot)\right)\\
&\quad= E\left[\prod_{j=1}^n \exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot)-x_\sigma(u,\cdot)+\overline{\eta}_\sigma(u,\cdot))\,du\right\}\right]\\
&\quad= \prod_{j=1}^n E\left(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u,x(u,\cdot))\,du\right\}|X_\sigma(x)=\overline{\eta}(\cdot)\right).
\end{aligned}$$

The result now follows by Lemma 1. □

Next, consider the conditional expectation for $s_1<s\le s_2$,

$$I\equiv E\left(\exp\left\{\int_{s_1}^s \theta(u,x(u,\cdot))\,du\right\}|x(s_1,\cdot)=\eta_1(\cdot), x(s,\cdot)=\eta(\cdot)\right). \tag{4.6}$$

Since the Yeh-Wiener process has stationary increments in each components, it follows that

(4.7)

$$\begin{aligned}
I&=E\left(\exp\left\{\int_0^{s-s_1} \theta(u+s_1,x(u,\cdot)+\eta_1(\cdot))\,du\right\}|x(s-s_1,\cdot)=\eta(\cdot)-\eta_1(\cdot)\right)\\
&\equiv G^*(s-s_1,\eta(\cdot)-\eta_1(\cdot);s_1,\eta_1(\cdot)).
\end{aligned}$$

Comparing (4.7) and (4.2), one can easily deduce that the function G^* satisfies the Wiener integral equation

(4.8)

$$\begin{aligned}
&G^*(s-s_1,\eta(\cdot)-\eta_1(\cdot);s_1,\eta_1(\cdot))\\
&= 1+\int_0^{s-s_1} E_w\left[\theta\left(u+s_1,\sqrt{u\left(1-\frac{u}{s-s_1}\right)}w(\cdot)+\frac{u}{s-s_1}[\eta(\cdot)-\eta_1(\cdot)]+\eta_1(\cdot)\right)\right.\\
&\qquad\left.\cdot G^*\left(u,\sqrt{u\left(1-\frac{u}{s-s_1}\right)}w(\cdot)+\frac{u}{s-s_1}[\eta(\cdot)-\eta_1(\cdot)];s_1,\eta_1(\cdot)\right)\right]du
\end{aligned}$$

whose solution is given by

$$G^*(s-s_1,\eta(\cdot)-\eta_1(\cdot);s_1,\eta_1(\cdot))=\sum_{k=0}^\infty H_k^*(s-s_1,\eta(\cdot)-\eta_1(\cdot);s_1,\eta_1(\cdot)), \tag{4.9}$$

where the sequence $\{H_k^*\}$ is given inductively by $H_0^*\equiv 1$, and

$$\begin{aligned}
&H_{k+1}^*(s-s_1,\eta(\cdot)-\eta_1(\cdot);s_1,\eta_1(\cdot))\\
&= \int_0^{s-s_1} E_w\left[\theta\left(u+s_1,\sqrt{u\left(1-\frac{u}{s-s_1}\right)}w(\cdot)+\frac{u}{s-s_1}[\eta(\cdot)-\eta_1(\cdot)]+\eta_1(\cdot)\right)\right.\\
&\qquad\left.\cdot H_k^*\left(u,\sqrt{u\left(1-\frac{u}{s-s_1}\right)}w(\cdot)+\frac{u}{s-s_1}[\eta(\cdot)-\eta_1(\cdot)];s_1,\eta_1(\cdot)\right)\right]du.
\end{aligned}$$

Obviously each H_k^* is bounded by the bound of H_k given in formula (4.5). Thus, under the same assumptions on θ, the series in formula (4.9) converges absolutely and uniformly for $s_1 \le s \le s_2$. In particular, the series corresponding to $G^*(s_2 - s_1, \eta_2(\cdot) - \eta_1(\cdot); s_1, \eta_1(\cdot))$ converges absolutely.

Exactly the same argument gives rise to an absolutely convergent series expansion for

$$\begin{aligned} &G^*(s_j - s_{j-1}, \eta_j(\cdot) - \eta_{j-1}(\cdot); s_{j-1}, \eta_{j-1}(\cdot)) \\ &\qquad = E\left(\exp\left\{\int_{s_{j-1}}^{s_j} \theta(u, x(u,\cdot))\,du\right\} | x(s_{j-1},\cdot) = \eta_{j-1}(\cdot), x(s_j,\cdot) = \eta_j(\cdot)\right) \end{aligned}$$

for $j = 1, 2, \ldots, n$. Hence by Lemma 2, we finally obtain that

$$\begin{aligned} E&\left(\exp\left\{\int_0^S \int_0^T \phi(u,t,x(u,t))\,dt\,du\right\} | x_\sigma(x) = \overline{\eta}(\cdot)\right) \\ &= E\left(\exp\left\{\int_0^S \theta(u, x(u,\cdot))\,du\right\} | X_\sigma(x) = \overline{\eta}(\cdot)\right) \\ &= \prod_{j=1}^n G^*(s_j - s_{j-1}, \eta_j(\cdot) - \eta_{j-1}(\cdot); s_{j-1}, \eta_{j-1}(\cdot)). \end{aligned}$$

5. Translation of parallel line-valued conditional Yeh-Wiener integrals

The Cameron-Martin Translation Theorem for Yeh-Wiener integrals [12] states that if $x_0(s,t) = \int_0^t \int_0^s h(u,v)\,du\,dv$ on Q for $h \in L^2(Q)$, and if T_1 is the transformation of $C(Q)$ into itself defined by

$$T_1(x) = x + x_0 \quad \text{for } x \in C(Q),$$

then, for any Yeh-Wiener integrable function F on $C(Q)$ and any Yeh-Wiener measurable set Γ,

$$\int_\Gamma F(z) m_y(dz) = \int_{T_1^{-1}(\Gamma)} F(x + x_0) J(x_0, x) m_y(dx), \tag{5.1}$$

where

$$J(x_0, x) = \exp\left\{-\frac{1}{2}\int_Q h^2(u,v)\,du\,dv\right\} \exp\left\{-\int_Q h(u,v)\,dx(u,v)\right\}.$$

In particular, if $\Gamma = C(Q)$, then (5.1) becomes

$$E[F(z)] = E[F(x + x_0) J(x_0, x)]. \tag{5.2}$$

The following theorem is the conditional version of (5.2).

Theorem 5. *Let $h \in L_2(Q)$ be given and let $x_0(s,t) = \int_0^t \int_0^s h(u,v)\,du\,dv$ on Q. Let $F \in L_1(C(Q), m_y)$. Then for each partition $\sigma\colon 0 = s_0 < s_1 < \cdots < s_n = S$ of $[0,S]$,*

$$\begin{aligned} &E(F(z) | X_\sigma(z) = \overline{\eta}(\cdot)) \\ &\qquad = \exp\left\{-\frac{1}{2}\|\hat{h}\|_2^2 + \int_Q h\,d\overline{\eta}_\sigma\right\} E(F(x + x_0) J(x_0, x) | X_\sigma(x + x_0) = \overline{\eta}(\cdot)). \end{aligned}$$

Proof. First, using Theorem 3, we see that

$$E(F(z)|X_\sigma(z)=\overline{\eta}(\cdot))=E[F(z-z_\sigma+\overline{\eta}_\sigma)].$$

Since $(x+x_0)_\sigma = x_\sigma + (x_0)_\sigma$, we may apply (5.2) to get that

$$E[F(z-z_\sigma+\overline{\eta}_\sigma)] = E[F(x+x_0-x_\sigma-(x_0)_\sigma+\overline{\eta}_\sigma)J(x_0,x)]. \tag{5.3}$$

Next, we rewrite $J(x_0,x)$ in the form,

$$\begin{aligned} J(x_0,x) = &\exp\left\{-\frac{1}{2}\|h\|_2^2\right\}\exp\left\{-\int_Q h\,d(x-x_\sigma+\overline{\eta}_\sigma-(x_0)_\sigma)\right\} \\ &\cdot\exp\left\{-\int_Q h\,dx_\sigma\right\}\exp\left\{\int_Q h\,d\overline{\eta}_\sigma\right\}\exp\left\{-\int_Q h\,d(x_0)_\sigma\right\}. \end{aligned}\tag{5.4}$$

But $x-x_\sigma$ and x_σ are independent processes on Q by the Corollary to Theorem 1, and so it follows from (5.3) and (5.4) that

(5.5)

$$\begin{aligned} &E[F(z-z_\sigma+\overline{\eta}_\sigma)] \\ &\quad= \exp\left\{-\frac{1}{2}\|h\|_2^2+\int_Q h\,d\overline{\eta}_\sigma-\int_Q h\,d(x_0))_\sigma\right\}E\left[\exp\left\{-\int_Q h\,dx_\sigma\right\}\right] \\ &\quad\cdot E\left[F(x+x_0-x_\sigma-(x_0)_\sigma+\overline{\eta}_\sigma)\exp\left\{-\int_Q h\,d(x-x_\sigma+\overline{\eta}_\sigma-(x_0)_\sigma)\right\}\right]. \end{aligned}$$

Since $\int_Q h\,dx_\sigma = \int_Q \hat{h}\,dx$ by (2.9), $\int_Q h\,dx_\sigma$ is a normal random variable with variance $\|\hat{h}\|_2^2$, and so

$$E\left[\exp\left\{-\int_Q h\,dx_\sigma\right\}\right] = \exp\left\{\frac{1}{2}\|\hat{h}\|_2^2\right\}. \tag{5.6}$$

Next, using (2.9), (2.6), and the definition of x_0, we see that

$$\int_Q h\,d(x_0)_\sigma = \int_Q \hat{h}\,d(x_0) = \int_Q \hat{h}h = \|\hat{h}\|_2^2. \tag{5.7}$$

Now, using Theorem 2, we obtain that

$$\begin{aligned} &E(F(x+x_0)J(x_0,x)|X_\sigma(x+x_0)=\overline{\eta}(\cdot)) \\ &\qquad= \exp\left\{-\frac{1}{2}\|h\|_2^2\right\}E\left[F(x+x_0-x_\sigma-(x_0)_\sigma+\overline{\eta}_\sigma)\right. \\ &\qquad\qquad\left.\cdot\exp\left\{-\int_Q h\,d(x-x_\sigma+\overline{\eta}_\sigma-(x_0)_\sigma)\right\}\right]. \end{aligned}\tag{5.8}$$

Finally, substitution of (5.6) through (5.8) into (5.5) yields

$$\begin{aligned} E[F(z-z_\sigma+\overline{\eta}_\sigma)] = &\exp\left\{-\frac{1}{2}\|\hat{h}\|_2^2+\int_Q h\,d\overline{\eta}_\sigma\right\} \\ &\cdot E(F(x+x_0)J(x_0,x)|X_\sigma(x+x_0)=\overline{\eta}(\cdot)), \end{aligned}$$

which completes the proof. □

References

1. R. H. Cameron and D. A. Storvick, *Two related integrals over spaces of continuous functions*, Pacific J. Math. **55** (1974), 19–37. MR **51:**5881
2. ______, *An operator-valued Yeh-Feynman integral and a Feynman integral equation*, Japan J. Math. **6** (1980), 283–342. MR **84m:**81065
3. K. S. Chang, J. M. Ahn, and J. S. Chang, *An evaluation of the conditional Yeh-Wiener integral*, Pacific J. Math. **124** (1986), 107–117. MR **88e:**60094
4. D. M. Chung and J. M. Ahn, *Conditional Yeh-Wiener integrals*, J. Korean Math. Soc. **20** (1983), 209–221. MR **85j:**28013
5. J. L. Doob, *Stochastic processes*, Wiley, New York, 1965 (originally published in 1953). MR **15:**445b
6. T. J. Huffman, C. Park and D. L. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673. MR **95d:**28017
7. C. Park, *On Fredholm transformations in Yeh-Wiener space*, Pacific J. Math. **40** (1972), 173–195. MR **46:**3739
8. C. Park and D. L. Skoug, *Conditional Yeh-Wiemer integrals with vector valued conditioning functions*, Proc. Amer. Math. Soc. **105** (1989), 450–461. MR **89m:**60198
9. ______, *Sample path-valued conditional Yeh-Wiener integrals and a Wiener integral equation*, Proc. Amer. Math. Soc. **115** (1992), 479–487. MR **92i:**28016
10. ______, *An operator-valued Yeh-Wiener integral, and a Kac-Feynman Wiener-Integral equation*, Proc. Amer. Math. Soc. **120** (1994), 929–942. MR **94h:**28009
11. J. Yeh, *Wiener measure in a space of functions of two variables*, Trans. Amer. Math. Soc. **95** (1960), 433–450. MR **23:**A2735
12. ______, *Cameron-Martin translation theorems in the Wiener space of functions of two variables*, Trans. Amer. Math. Soc. **107** (1963), 409–420. MR **32:**6565
13. ______, *Inversion of conditional Wiener integrals*, Pacific J. Math. **59** (1975), 623–638. MR **52:**10988
14. ______, *Stochastic processes and the Wiener integral*, Marcel Dekker, New York, 1983.

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