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DUALITY AND PERFECT PROBABILITY SPACES

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ABSTRACT. Given probability spaces $(X_i, \mathcal{A}_i, P_i), i = 1, 2$, let $\mathcal{M}(P_1, P_2)$ denote the set of all probabilities on the product space with marginals P_1 and P_2 and let h be a measurable function on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$. Continuous versions of linear programming stemming from the works of Monge (1781) and Kantorovich-Rubinštein (1958) for the case of compact metric spaces are concerned with the validity of the duality

$$\sup\{\int h\ dP : P \in \mathcal{M}(P_1, P_2)\}$$
$$= \inf\{\sum_{i=1}^{2} \int h_i\ dP_i : h_i \in \mathcal{L}^1(P_i)\ \ and\ \ h \leq \oplus_i\ h_i\}$$

(where $\mathcal{M}(P_1, P_2)$ is the collection of all probability measures on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with P_1 and P_2 as the marginals). A recently established general duality theorem asserts the validity of the above duality whenever at least one of the marginals is a perfect probability space. We pursue the converse direction to examine the interplay between the notions of duality and perfectness and obtain a new characterization of perfect probability spaces.

1. Introduction

Let $(X, \mathcal{A}, P)$ be a probability space. P is called *perfect* (equivalently, the space $(X, \mathcal{A}, P)$ is called *perfect*) if, for every $\mathcal{A}$-measurable, real-valued function f on X we can find a Borel subset B_f of the real line such that $B_f \subset f(X)$ with $P(f^{-1}(B_f)) = 1$. Introduced by Gnedenko and Kolmogorov [2] "to achieve complete harmony between the abstract theory of measure and the theory of measures in metric spaces", perfect probability spaces form the technically most pleasing class of probability spaces (see Ramachandran [10]).

A succinct history of the notion of duality from its origins in the works of Monge [5] and Kantorovich-Rubinštein [3], along with a variety of applications in probability theory can be found in Kellerer [4] (see also Dudley [1] and Rachev [9]). Ramachandran and Rüschendorf [11] have recently established that a general duality theorem holds whenever one of the underlying spaces is perfect.

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In this note we investigate the converse direction and obtain a new characterization of perfect probability spaces using the notion of duality which brings to light the interplay between the notions of duality and perfectness.

2. Notation and preliminaries

We use customary measure-theoretic terminology and notation (as, for instance, in Neveu [6]). All measures that we consider are probabilities. For properties of perfect measures we refer the reader to Ramachandran [10]. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of $[0,1]$ and by λ the Lebesgue measure on $([0,1],\mathcal{B})$. Δ denotes the diagonal in $[0,1]\times[0,1]$. If P is a probability on $(X,\mathcal{A})$, then P_* and P^* denote respectively the inner and the outer measures induced by P. A σ-algebra $\mathcal{A}_0$ is said to be countably generated (or c.g. for short) if $\mathcal{A}_0 = \sigma(\{A_n, n \geq 1\})$, in which case $\varphi : (X,\mathcal{A}_0) \to ([0,1],\mathcal{B})$ defined by $\varphi(x) = \sum_{n=1}^{\infty}(2/3^n)1_{A_n}(x)$ is called the Marczewski function; φ is measurable with $\varphi(x_1) \neq \varphi(x_2)$ if x_1 and x_2 belong to different atoms of $\mathcal{A}_0$, and so we can identify $(X,\mathcal{A}_0)$ with $(\varphi(X),\mathcal{B}\cap\varphi(X))$. We say that $(X,\mathcal{A},P)$ is a thick subspace of $(X_1,\mathcal{A}_1,P_1)$ and write $(X,\mathcal{A},P) \subset (X_1,\mathcal{A}_1,P_1)$ whenever $X \subset X_1$, $\mathcal{A} = \mathcal{A}_1 \cap X =$ the trace of $\mathcal{A}_1$ on X, $P_1^*(X) = 1$ and $P = P_1^*|\mathcal{A}$.

Let $(X_i,\mathcal{A}_i,P_i)$, $i = 1,2$, be two probability spaces. A probability μ on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is said to have marginals P_1 and P_2 if

$$\mu(A_1 \times X_2) = P_1(A_1) \text{ for all } A_1 \in \mathcal{A}_1, \text{ and}$$
$$\mu(X_1 \times A_2) = P_2(A_2) \text{ for all } A_2 \in \mathcal{A}_2.$$

Let $\mathcal{M}(P_1,P_2) = \{\ \mu \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \mu \text{ has marginals } P_1 \text{ and } P_2\ \}$. $\pi_i : X_1 \times X_2 \to X_i$ denote the canonical projections for $i = 1,2$. The abbreviation $\oplus g_i$ is used for $\sum_{i=1}^{2} g_i \circ \pi_i$.

For a bounded, $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable function h on $X_1 \times X_2$, the <u>*marginal problem*</u> is concerned with

$$S(h) = \sup\{\int_{X_1\times X_2} h d\mu : \mu \in \mathcal{M}(P_1,P_2)\}$$

while the <u>*dual problem*</u> deals with

$$I(h) = \inf\{\sum_{i=1}^{2}\int_{X_i} h_i dP_i : h_i \in \mathcal{L}^1(P_i) \ \textit{and} \ h \leq \oplus_i h_i\}.$$

The measure-theoretic version of the transportation problem dating back to Monge [5] seeks the validity of the duality

$$S(h) = I(h). \tag{D}$$

The main duality theorem of Kellerer [4] deals essentially with second countable or metrizable spaces $X_i, i = 1,2$, with tight (or Radon) probabilities defined on the Borel sets in which case (D) is shown to hold for a suitably large class containing all the bounded, measurable functions. The following result of Ramachandran and Rüschendorf [11] is the most general duality theorem of this type.

Theorem 1. *If at least one of the underlying probability spaces is perfect, then* (D) *holds for all bounded, measurable functions.*

We pursue the converse direction in the next section.

3. Main results

We first construct an example to show that the direct converse of Theorem 1 does not hold; that is, (D) can hold for all bounded measurable functions while both the underlying probability spaces are nonperfect.

Example 1. Pachl [8] has constructed a subset X of $[0,1]$ such that
(i) $\lambda^*(X) = 1 = \lambda^*([0,1] - X)$ and
(ii) if $\overline{\mu}$ is a probability on $([0,1]\times[0,1], \mathcal{B}\otimes\mathcal{B})$ with $\overline{\mu} \in \mathcal{M}(\lambda,\lambda)$, then $\overline{\mu}^*(X\times X) = 1$.

Let $X_i = X$, $\mathcal{A}_i = \mathcal{B}\cap X$ and $P_i = \lambda^*|\mathcal{A}_i$ for $i = 1, 2$. Then P_i is clearly nonperfect for both $i = 1, 2$. For $\mu \in \mathcal{M}(P_1, P_2)$ let $\overline{\mu}$ be defined by $\overline{\mu}(C) = \mu(C \cap (X \times X)), C \in \mathcal{B}\otimes\mathcal{B}$. Then it is easy to check that $\overline{\mu} \in \mathcal{M}(\lambda,\lambda)$ and that $\mu \to \overline{\mu}$ is a 1-1 correspondence between $\mathcal{M}(P_1, P_2)$ and $\mathcal{M}(\lambda,\lambda)$. Further, since $\mathcal{A}_1\otimes\mathcal{A}_2 = \mathcal{B}\otimes\mathcal{B}\cap(X\times X)$, it can be checked (starting with $1_C, C \in \mathcal{A}_1\otimes\mathcal{A}_2$ and using standard measure-theoretic arguments) that for every h on $X_1\times X_2$ which is $\mathcal{A}_1\otimes\mathcal{A}_2$ - measurable there is $\overline{h}$ on $[0,1]\times[0,1]$ which is $\mathcal{B}\otimes\mathcal{B}$ - measurable such that $\overline{h}|_{X_1\times X_2} = h$ and $\int h d\mu = \int \overline{h} d\overline{\mu}$. Hence it follows that for every bounded, $\mathcal{A}_1\otimes\mathcal{A}_2$-measurable h,

$$\begin{aligned} S(h) &= \sup\{\textstyle\int h d\mu : \mu \in \mathcal{M}(P_1,P_2)\} \\ &= \sup\{\textstyle\int \overline{h} d\overline{\mu} : \overline{\mu} \in \mathcal{M}(\lambda,\lambda)\} \\ &= S(\overline{h}) \\ &= I(\overline{h}) \quad \text{by Theorem 1} \\ &\geq I(h) \quad (\text{by the definition of } I(h)) \\ &\geq S(h) \quad (\text{by the definitions of } I(h) \text{ and } S(h)) \end{aligned}$$

whereby (D) holds.

In order to bring out the interplay between the notions of perfectness and duality we introduce

Definition 1. A probability space $(X_1, \mathcal{A}_1, P_1)$ is said to be a duality space if for every $(X_2, \mathcal{A}_2, P_2)$ the duality (D) holds for all bounded, measurable functions.

Theorem 1 can now be recast as

Theorem 2. *Every perfect probability space is a duality space.*

The next major step is to establish

Proposition 1. *If* $(X, \mathcal{A}, P)$ *is nonperfect, then there exists* $(X_1, \mathcal{A}_1, P_1)$ *such that*
(i) $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$ *and*
(ii) $(X_1, \mathcal{A}_1, P_1)$ *is not a duality space.*

Proof. Since $(X, \mathcal{A}, P)$ is nonperfect, there exists a c.g. sub-σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that $P_0 = P|\mathcal{A}_0$ is nonperfect (see $P3$ of Ramachandran [10, p. 26]). Let φ be the Marczewski function on $(X, \mathcal{A}_0)$ and let $Q_0 = P_0\varphi^{-1}$. Then φ provides the Marczewski identification of the space $(X, \mathcal{A}_0, P_0)$ with $(\varphi(X), \mathcal{B}\cap\varphi(X), Q_0) \subset ([0,1], \mathcal{B}, \overline{Q_0})$ where $\overline{Q_0}$ is defined on $\mathcal{B}$ by $\overline{Q}_0(B) = Q_0(B\cap\varphi(X))$ $(= P_0\varphi^{-1}(B))$, $B \in \mathcal{B}$. Notice that, by construction, $\overline{Q}_0^*(\varphi(X)) = 1$.

Suppose that $\overline{Q}_{0*}(\varphi(X)) = 1$. Then we can find a set $Y \subset \varphi(X)$, $Y \in \mathcal{B}$, with $\overline{Q_0}(Y) = 1$. Since Y is a Borel subset of [0,1], every probability on $(Y, \mathcal{B}\cap Y)$ is perfect (see Theorem 2.3.1 of Ramachandran [10]); hence, $Q_0|_Y$ is perfect which in turn would imply that Q_0 is perfect. However, Q_0 is nonperfect since

P_0 is nonperfect (see $P7$ of Ramachandran [10, p. 27]), and so we conclude that $\overline{Q}_{0*}(\varphi(X)) = \alpha < 1$.

Let $f : x \longmapsto (x, \varphi(x), \varphi(x)), x \in X$.

Then f: $(X, \mathcal{A}, P) \to (X \times \varphi(X) \times [0,1], \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}, Pf^{-1})$ has the following properties:

1. f is 1-1.

2. Since $G_\varphi = \{(x, \varphi(x)), x \in X\} =$ Graph of $\varphi \in \mathcal{A} \otimes \mathcal{B} \cap (X \times \varphi(X)) = \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X))$, it follows that

$$f(X) = (G_\varphi \times [0,1]) \cap (X \times (\Delta \cap (\varphi(X) \times [0,1]))) \in \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}.$$

3. $Pf^{-1}(f(X)) = P(X) = 1$.

4. $f^{-1}(A \times (B \cap \varphi(X)) \times C) = A \cap \varphi^{-1}(B) \cap \varphi^{-1}(C) \in \mathcal{A}$ for all $A \in \mathcal{A}, B \in \mathcal{B}$ and $C \in \mathcal{B}$.

5. For $A \in \mathcal{A}, f(A) = (A \times \varphi(X) \times [0,1]) \cap f(X) \in \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}$.

Hence, we have

$$\begin{array}{rcl} (X, \mathcal{A}, P) & \overset{f}{\longleftrightarrow} & (f(X), (\mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}) \cap f(X), Pf^{-1}) \\ & \subset & (X \times \varphi(X) \times [0,1], \mathcal{A} \otimes (\mathcal{B} \cap \varphi(X)) \otimes \mathcal{B}, Pf^{-1}) \\ & \overset{def}{=} & (X_1, \mathcal{A}_1, P_1). \end{array}$$

P_1 has marginals P, Q_0 and $\overline{Q}_0$ respectively on $\mathcal{A}, \mathcal{B} \cap \varphi(X)$ and $\mathcal{B}$ (by 4 above). Define

$$X_2 = [0,1] \times (\varphi(X))^c, \qquad \mathcal{A}_2 = \mathcal{B} \otimes (\mathcal{B} \cap (\varphi(X))^c).$$

Before defining P_2 observe that $\overline{Q}_0^*((\varphi(X))^c) = 1 - \alpha > 0$ and $\overline{Q}_{0*}((\varphi(X))^c) = 0$. Choose and fix $B_0 \in \mathcal{B}$, $B_0 \subset \varphi(X)$ such that $\overline{Q}_0(B_0) = \overline{Q}_{0*}(\varphi(X)) = \alpha$. Let Q_1 on $\mathcal{B}$ be defined by

$$Q_1(B) = \overline{Q}_0(B | B_0^c) = \frac{\overline{Q}_0(B - B_0)}{1 - \alpha}, \quad B \in \mathcal{B}.$$

Then $Q_1(B_0^c) = 1$ and $B_0^c \supset (\varphi(X))^c$. If $B \in \mathcal{B}$ with $B \supset (\varphi(X))^c$, then $B^c \subset \varphi(X) \Rightarrow \overline{Q}_0(B^c - B_0) = 0 \Rightarrow Q_1(B^c) = 0 \Rightarrow Q_1(B) = 1$ and so $Q_1^*((\varphi(X))^c) = 1$. Similarly, it can be seen that $Q_{1*}((\varphi(X))^c) = 0$. Now define P_2 on $\mathcal{A}_2$ by

$$P_2(A_2) = Q_1^*(\pi_2(A_2 \cap \triangle)), A_2 \in \mathcal{A}_2.$$

Marginals of P_2 are Q_1 and Q_1^* respectively. Having constructed suitable $(X_i, \mathcal{A}_i, P_i)$, $i = 1, 2$, with $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$, we now show that $(X_1, \mathcal{A}_1, P_1)$ fails to be a duality space.

Consider

$$X_1 \times X_2 = (X \times \varphi(X) \times [0,1]) \times ([0,1] \times (\varphi(X))^c)$$

and define

$$H = X \times \varphi(X) \times \triangle \times (\varphi(X))^c \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

Let $h = 1_H$. Since $P_1(X \times (\triangle \cap (\varphi(X) \times [0,1]))) = 1 = P_2(\triangle \cap X_2)$, for every $\mu \in \mathcal{M}(P_1, P_2)$ we get $\mu(X \times (\triangle \cap (\varphi(X) \times [0,1])) \times (\triangle \cap X_2)) = 1$. But $H \cap (X \times (\triangle \cap (\varphi(X) \times [0,1])) \times (\triangle \cap X_2)) = \emptyset$ and so $\mu(H) = 0$ for all $\mu \in \mu(P_1, P_2)$. Hence $S(h) = 0$.

By (3.3) of Kellerer [4] (see also Strassen [12]), we have

$$I(h) = \inf\{P_1(A_1) + P_2(A_2) : H \subset (A_1 \times X_2) \cup (X_1 \times A_2), A_i \in \mathcal{A}_i\ ,\ i = 1, 2\}.$$

Since $H \subset (\emptyset \times X_2) \cup (X_1 \times X_2)$, we get

$$I(h) \leq P_1(\emptyset) + P_2(X_2) = 1.$$

If $H \subset (A_1 \times X_2) \cup (X_1 \times A_2)$ with $A_i \in \mathcal{A}_i\ ,\ i = 1, 2$, then let

$$B_2 = \pi_2(A_2 \cap \triangle) \in \mathcal{B} \cap (\varphi(X))^c.$$

Hence, $B_2 = B \cap (\varphi(X))^c$ for some $B \in \mathcal{B}$ and so

$$P_2(A_2) = Q_1^*(B_2) = Q_1(B) = \frac{\overline{Q}_0(B - B_0)}{1 - \alpha}. \tag{1}$$

Carefully look at

$$D = X \times \varphi(X) \times (B^c \cap (\varphi(X))^c).$$

$$\begin{aligned}(x, y_1, y_2) \in D \quad &\Rightarrow \quad (x, y_1, y_2, y_2, y_2) \in H \\ &\Rightarrow \quad (x, y_1, y_2, y_2, y_2) \in A_1 \times X_2 \\ &\qquad\ ((y_2, y_2) \notin A_2 \text{ since } \ y_2 \notin B_2) \\ &\Rightarrow \quad (x, y_1, y_2) \in A_1.\end{aligned}$$

It follows that $D \subset A_1$. Hence

$$P_1(A_1) \geq P_1^*(D) = \overline{Q}_0^*(B^c \cap (\varphi(X))^c) = \overline{Q}_0(B^c - B_0). \tag{2}$$

(To conclude that $\overline{Q}_0^*(C \cap (\varphi(X))^c) = \overline{Q}_0(C - B_0)$ for all $C \in \mathcal{B}$ first note that $\overline{Q}_0(C - B_0) \geq \overline{Q}_0^*(C \cap (\varphi(X))^c)$ since $B_0 \subset \varphi(X)$; on the other hand, if $C \cap (\varphi(X))^c \subset C_1 \in \mathcal{B}$, then $(C - B_0) - (C_1 - B_0) \subset \varphi(X) - B_0 \Rightarrow \overline{Q}_0((C - B_0) - (C_1 - B_0)) = 0 \Rightarrow \overline{Q}_0(C - B_0) = \overline{Q}_0((C - B_0) \cap (C_1 - B_0)) \leq \overline{Q}_0(C_1 - B_0) \leq \overline{Q}_0(C_1)$.)

Adding (1) and (2),

$$\begin{aligned}P_1(A_1) + P_2(A_2) \quad &\geq \quad \tfrac{\overline{Q}_0(B - B_0)}{1 - \alpha} + \overline{Q}_0(B^c - B_0) \\ &\geq \quad \overline{Q}_0(B - B_0) + \overline{Q}_0(B^c - B_0) \\ &= \quad \overline{Q}_0(B_0^c) = 1 - \alpha.\end{aligned}$$

Thus $I(h) \geq 1 - \alpha > 0$ and as a consequence $(X_1, \mathcal{A}_1, P_1)$ fails to be a duality space. □

Our main result which yields a new characterization of perfectness of measures is

Theorem 3. *Let $(X, \mathcal{A}, P)$ be a probability space. Then the following statements are equivalent:*

(a) *$(X, \mathcal{A}, P)$ is perfect.*
(b) *If $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$, then $(X_1, \mathcal{A}_1, P_1)$ is perfect.*
(c) *If $(X, \mathcal{A}, P) \subset (X_1, \mathcal{A}_1, P_1)$, then $(X_1, \mathcal{A}_1, P_1)$ is a duality space.*

Proof. (a) $\Rightarrow$ (b). If f_1 is an $\mathcal{A}_1$-measurable, real-valued function on X_1, then letting $f = f_1|_X$, choose a Borel set B_f of the real line such that $B_f \subset f(X) \subset f_1(X_1)$ with $P(f^{-1}B_f) = 1$. Take $B_{f_1} = B_f$.

(b) $\Rightarrow$ (c) follows from Theorem 2.

(c) $\Rightarrow$ (a) is Proposition 1. □

References

1. Dudley, R.M.: Probabilities and Metrics. Lecture Notes Series No. 45. *Aarhus: Matematisk Institut,* 1976. MR **58:**7764
2. Gnedenko, B.V., and Kolomogorov, A.N. (1954). Limit distributions for sums of independent random variables. Addison Wesley, Cambridge. MR **16:**52d
3. Kantorovich, L.V., and Rubinštein, G.S. (1958). On a space of completely additive functions (in Russian). *Vestnik Leningrad Univ* . **13/7**, 52-59. MR **21:**808
4. Kellerer, H.G. (1984). Duality Theorems for marginal problems. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **67**, 399-432. MR **86i:**28010
5. Monge, G. (1781). Mémoire sur la théorie des déblais ét ramblais. *Mem. Math. Phys. Acad. Roy. Sci. Paris* , 666-704.
6. Neveu, J. (1965). Mathematical foundations of the calculus of probability. Holden Day, London. MR **33:**6660
7. Pachl, J. (1979). Two classes of measures. *Colloq. Math.* **42**, 331-340. MR **82b:**28012
8. Pachl, J. (1981). Correction to the paper " Two classes of measures. " *Colloq. Math.* **45**, 331-333. MR **84c:**28009
9. Rachev, S.T. : Probability metrics and the stability of stochastic models. New York: Wiley 1991 MR **93b:**60012
10. Ramachandran, D. (1979). Perfect Measures, I and II. ISI Lecture Notes Series, **5** and **7**, New Delhi, Macmillan. MR **81h:**6005b; MR **81h:**6005a
11. Ramachandran, D. and Rüschendorf, L. (1995). A general duality theorem for marginal problems. *Probab. Theory Relat. Fields*, **101**, 311-319.
12. Strassen, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Statist.* **36**, 423-439. MR **31:**1693

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