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KILLING AND SUBORDINATION

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ABSTRACT. Given the one-to-one correspondence between nearly Borel right processes and non-symmetric Dirichlet forms, we prove in the present paper that the killing transform of Markov processes is equivalent to strong subordination of the respective Dirichlet forms and give a characterization of so-called bivariate smooth measures.

1. INTRODUCTION

In my previous paper [Yi] I formulated a so-called generalized Feynman-Kac formula for Dirichlet forms as follows. Let X be a Borel right process which is associated with a quasi-regular (non-symmetric) Dirichlet form $(a, \mathcal{D}_a)$ on $L^2 := L^2(E; m)$. Let M be a decreasing multiplicative functional of X and ν_M the bivariate Revuz measure of M relative to m. If (X, M) denotes the subprocess of X killed by M, then the Dirichlet form $(b, \mathcal{D}_b)$ associated with (X, M) is given by

$$\begin{aligned} \mathcal{D}_b &= \mathcal{D}_a \cap L^2(\rho_M) \cap L^2(\lambda_M); \\ b(u, v) &= a(u, v) + \nu_M(u \otimes v), \ u, v \in \mathcal{D}_b, \end{aligned} \tag{1.1}$$

where ρ_M and λ_M are the right and left marginal measures of ν_M, respectively, and $u \otimes v(x, y) := u(x)v(y)$.

In this paper we will study the inverse problem: given two quasi-regular Dirichlet forms $(a, \mathcal{D}_a)$ and $(b, \mathcal{D}_b)$, what conditions make the process associated with $(b, \mathcal{D}_b)$ the subprocess of that with $(a, \mathcal{D}_a)$ killed by a decreasing multiplicative functional?

After collecting some standard definitions, notations and results in §2, we show in §3 that the subordination of Dirichlet forms is equivalent to the killing transform for Markov processes. In §4 we give a necessary and sufficient condition that a bivariate measure is a bivariate Revuz measure of a decreasing multiplicative functional.

2. PRELIMINARIES

Throughout this paper we assume E to be a separable and metrizable topological space, $\mathcal{E}$ its Borel σ-algebra, m a σ-finite positive measure on $(E, \mathcal{E})$. Write $L^2 := L^2(E; m)$ and denote by $C_c(E)$ the set of all continuous functions on E with compact

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support. Let $(a, \mathcal{D}_a)$ be a Dirichlet form on L^2 defined as in [MR]. Particularly $\mathcal{D}_a$ is a dense linear subspace of L^2 and the norm $||\cdot||_a$ on $\mathcal{D}_a$, defined by $||u||_a := [a_1(u,u)]^{\frac{1}{2}}$, satisfies the Markovian property: for any normal contraction ϕ it holds that $\phi(u) \in \mathcal{D}_a$ and $||\phi(u)||_a \leq ||u||_a$. Set $\mathcal{D}_a^+ := \{u \in \mathcal{D}_a;\ u \geq 0\}$. Define for $F \subset E$, F closed,

$$\mathcal{D}_a|_F := \{u \in \mathcal{D}_a :\ u = 0 \text{ a.e. } m \text{ on } F^c\}.$$

An increasing sequence $\{F_n\}$ of closed subsets of E is called an a-nest if $\bigcup_n \mathcal{D}_a|_{F_n}$ is dense in $\mathcal{D}_a$ with respect to $||\cdot||_a$. A subset $N \subset E$ is called a-exceptional if $N \subset \bigcap_n F_n^c$ for some a-nest $\{F_n\}$. We say that a property of points in E holds a-quasi-everywhere (a-q.e., in abbreviation) if the property holds off some a-exceptional set. Given an a-nest $\{F_n\}$ we define

$$C(\{F_n\}) := \{f :\ A \longrightarrow \mathbb{R},\ \bigcup_n F_n \subset A \subset E, f|_{F_n} \text{ is continuous for any } n \in \mathbb{N}\}.$$

An a-q.e.defined function f on E is called a-quasi-continuous if there exists an a-nest $\{F_n\}$ such that $f \in C(\{F_n\})$.

Definition 2.1. A Dirichlet form $(a, \mathcal{D}_a)$ is called quasi-regular if (i) there exists an a-nest $\{E_n\}$ consisting of compact sets; (ii) there exists an $||\cdot||_a$-dense subset of $\mathcal{D}_a$ whose elements have a-quasi-continuous m-versions; (iii) there exist $u_n \in \mathcal{D}_a$, $n \in \mathbb{N}$, having a-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an a-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E - N$.

An important result in [MR] is that a Dirichlet form $(a, \mathcal{D}_a)$ is quasi-regular if and only if it is associated with a unique Borel right Markov process

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$$

on $(E, \mathcal{E})$, with sub-Markovian semigroup (P_t) and resolvent (U^q), as follows: for any $f \in L^2$, $u \in \mathcal{D}_a$ and $q > 0$, $U^q f$ is an a-quasi-continuous element in $\mathcal{D}_a$ and $a_q(u, U^q f) = (u, f)$. A set $B \in \mathcal{E}$ is called m-polar if $P^m(T_B < \zeta) = 0$, where T_B is the hitting time. It is easy to check that B is m-polar if and only if it is a-exceptional. Moreover it is known that any exceptional set N is contained in a properly exceptional N_1 in the sense that $E \setminus N_1$ is absorbing. (See [Fu].) Having $(a, \mathcal{D}_a)$ at hand, we define a notion of multiplicative functionals which is a little weaker than what is usually used in theory of Markov processes.

Definition 2.2. A real valued function $M = (M_t(\omega) : t \geq 0, \omega \in \Omega)$ is called a decreasing multiplicative functional of $(a, \mathcal{D}_a)$ (or X) (MF in abbreviation) if (i) $M_t(\cdot)$ is $(\mathcal{F}_t)$-adapted; (ii) there exist a set $\Lambda \in \mathcal{F}_\infty$ and a properly a-exceptional set $N \subset E$ such that $P^x(\Lambda) = 1$ for all $x \in E - N$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$ and moreover for each $\omega \in \Lambda$, $M_\cdot(\omega)$ is right continuous on $[0, \infty]$, $0 \leq M_t(\omega) \leq 1$ for any $t < \zeta(\omega)$, $M_0(\omega) = 1$ and $M_{t+s}(\omega) = M_s(\omega) M_t(\theta_s \omega)$ for any $t, s \geq 0$.

Clearly M is a perfect MF in the ordinary sense but with respect to the restricted Borel right process $X|_{E-N'}$ for a properly m-polar Borel set N'. This makes no difference in the view of P^m. Hence we can use most of results on bivariate Revuz measures and related analysis developed in [Yi] freely. Given $M \in \mathrm{MF}$, the bivariate Revuz measure of M relative to m is defined as

$$\nu_M(F) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m[\int_0^t F(X_{s-}, X_s) d(-M_s)], \ F \in (\mathcal{E} \times \mathcal{E})^+, \tag{2.1}$$

and is actually that of $(1 - M_t)$, which is an M-additive functional or an additive functional of M in some books. (Refer to [FG] for existence of Revuz measures in this case.) Denote by ρ_M and λ_M the right and left marginal measures of ν_M, respectively. We also get a right process with transition semigroup (Q_t) given by $Q_t f := P^{\cdot}[f(X_t)M_t]$ for any $f \in \mathcal{E}$, and the corresponding resolvent is denoted by (V^q). This process is usually called the subprocess of X killed by M, and is denoted by (X, M). The condition that $M_0 \equiv 1$ amounts to the assumption that the state space of the killed process is the same as that of the original process. Ruled out here is killing at the first hitting time of a non-m-polar Borel set. We know from the work of Silverstein [Si] that in the symmetric case, if $(b, \mathcal{D}_b)$ is obtained by killing the process associated with $(a, \mathcal{D}_a)$ at the first hitting time of a Borel set, then $\mathcal{D}_b \subset \mathcal{D}_a$, $b = a$ on $\mathcal{D}_b \times \mathcal{D}_b$, and $\mathcal{D}_b \cap L^\infty$ is an ideal in the algebra $\mathcal{D}_a \cap L^\infty$. Moreover the converse assertion is also true. This result of Silverstein can certainly be extended to the near-symmetry context of this paper; therefore the main results following can be extended to accommodate the most general killing transformation without great difficulties. But to keep the article to a reasonable length, I won't give details here.

If X is associated with a quasi-regular Dirichlet form $(a, \mathcal{D}_a)$, it has automatically a weak dual $\hat{X}$ relative to m which is associated with the dual form $(\hat{a}, \mathcal{D}_{\hat{a}})$ of $(a, \mathcal{D}_a)$ defined by $\mathcal{D}_{\hat{a}} := \mathcal{D}_a$ and $\hat{a}(u, v) := a(v, u), \ u, v \in \mathcal{D}_{\hat{a}}$. As a convention, notations with 'hat' refer to $\hat{X}$ and assume the same meanings as to X. It is known from [Yi] that ν_M is dual to $\hat{\nu}_{\hat{M}}$; i.e.,

$$\hat{\nu}_{\hat{M}}(dx, dy) = \nu_M(dy, dx), \tag{2.2}$$

and the generalized Revuz formula holds:

$$\nu_M(\hat{V}^1 f \otimes g) = (f, U^1_M g), \ f, g \in \mathcal{E}^+, \tag{2.3}$$

where $(\hat{V}^q)$ is the resolvent of $(\hat{X}, \hat{M})$ and U^q_M the potential operator of $(1 - M_t)$.

3. Subordination

We begin this section with a definition.

Definition 3.1. Given two Dirichlet forms $(a, \mathcal{D}_a)$ and $(b, \mathcal{D}_b)$ on L^2, we say that $(b, \mathcal{D}_b)$ is subordinate to $(a, \mathcal{D}_a)$ if $\mathcal{D}_b \subset \mathcal{D}_a$ and $b(u, v) \geq a(u, v)$ for any $u, v \in \mathcal{D}_b^+$, and $(b, \mathcal{D}_b)$ is strongly subordinate to $(a, \mathcal{D}_a)$ if, in addition, $\mathcal{D}_b$ is dense in $\mathcal{D}_a$ w.r.t. $||\cdot||_a$.

The following lemma is essential in later discussions.

Lemma 3.2. *If $(b, \mathcal{D}_b)$ is subordinate to $(a, \mathcal{D}_a)$, then there exists an $C > 0$ such that $||u||_a \leq C \cdot ||u||_b$ for any $u \in \mathcal{D}_b$.*

Proof. Let I be the inclusion operator from $(\mathcal{D}_b, ||\cdot||_b)$ to $(\mathcal{D}_a, ||\cdot||_a)$; i.e., $I(u) = u$ for any $u \in \mathcal{D}_b$. It suffices to show that I is continuous. Let $\{u_n\} \subset \mathcal{D}_b$ and

$u_n \longrightarrow 0$ in $||\cdot||_b$-norm. By the Markovian property of $||\cdot||_b$we find that $u_n^+ \longrightarrow 0$ and $u_n^- \longrightarrow 0$, both in $||\cdot||_b$-norm. Since $(b, \mathcal{D}_b)$ is subordinate to $(a, \mathcal{D}_a)$, we have $||u_n^+||_a \leq ||u_n^+||_b$ and $||u_n^-||_a \leq ||u_n^-||_b$. Thus $u_n^+ \longrightarrow 0$ and $u_n^- \longrightarrow 0$, both in $||\cdot||_a$-norm, and then we have $u_n \longrightarrow 0$ in $||\cdot||_a$-norm. □

Corollary 3.3. *If $(b, \mathcal{D}_b)$ is strongly subordinate to $(a, \mathcal{D}_a)$, then any b-nest is an a-nest. Therefore any b-quasi-continuous function is a-quasi-continuous, and if $(b, \mathcal{D}_b)$ is quasi-regular then $(a, \mathcal{D}_a)$ is too.*

Proof. Let $\{F_n\}$ be a b-nest. Then $\mathcal{D} := \bigcup_n \mathcal{D}_b|_{F_n}$ is dense in $(\mathcal{D}_b, ||\cdot||_b)$. By Lemma 3.2 we know that $\mathcal{D}$ is dense in $(\mathcal{D}_b, ||\cdot||_a)$. Also $\mathcal{D}_b$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$. Hence $\mathcal{D}$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$ and so is $\bigcup_n \mathcal{D}_a|_{F_n}$, since it contains $\mathcal{D}$; i.e., $\{F_n\}$ is an a-nest. □

We shall first prove that killing transform implies subordination.

Theorem 3.4. *Let X be a Borel right process associated with a quasi-regular Dirichlet form $(a, \mathcal{D}_a)$ on L^2. If $M \in MF$, then (i) the bilinear form $(b, \mathcal{D}_b)$ defined in (1.1) is a quasi-regular Dirichlet form on L^2 with which the subprocess (X, M) is associated; (ii) $\mathcal{D}_b$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$. Therefore $(b, \mathcal{D}_b)$ is strongly subordinate to $(a, \mathcal{D}_a)$.*

Proof. It was shown in [Yi] that $(b, \mathcal{D}_b)$ is a Dirichlet form on L^2 with which (X, M) is associated. Hence it suffices to show that (i) $(b, \mathcal{D}_b)$ is quasi-regular and (ii) $\mathcal{D}_b$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$. But the proof of (i) is a step by step exercise if we follow that of IV.4.5 and IV 4.6 of [MR] and replace $e^{-A_t^\mu}$ there by M_t. We will not write them here.

(ii) Fix $f \in \mathcal{E}$ with $0 < f \leq 1$ and $m(f) < \infty$. By (2.2) and (2.3) we have

$$\lambda_M(\hat{V}^1 f) = \nu_M(V^1 f \otimes 1) = (f, U_M^1 1) \leq m(f) < \infty;$$
$$\rho_M(V^1 f) = \nu_M(1 \otimes V^1 f) = \hat{\nu}_{\hat{M}}(V^1 f \otimes 1) = (f, \hat{U}_{\hat{M}} 1) \leq m(f) \leq \infty,$$

where $(\hat{V}^q)$ and (V^q) are resolvents of $(\hat{X}, \hat{M})$ and (X, M), respectively. Set $F_n := \{x : \ V^1 f(x) \geq \frac{1}{n},\ \hat{V}^1 f(x) \geq \frac{1}{n}\}$. Then F_n is finely closed, $E - \bigcup_n F_n$ is an a-exceptional set, and both $\rho_M(F_n)$ and $\lambda_M(F_n)$ are finite for each $n \in \mathbb{N}$.

Now let $g \in L^2$ be nonnegative, bounded and $\mathcal{E}$-measurable, and $u := U^1 g$, $u_n := u - P^1_{F_n^c} u$, where $P^1_{F_n^c}$ is the balayage operator. It is clear that

$$u_n = P^{\cdot} \int_0^{T_n} e^{-t} g(X_t) dt,$$

where $T_n := T_{F_n^c}$, and hence $u_n = 0$ a-q.e. on F_n^c. Obviously we have $u_n \in \mathcal{D}_a \cap L^2(\rho_M) \cap L^2(\lambda_M)$ or $u_n \in \mathcal{D}_b$ for each n. Since $T_n \uparrow \zeta$ a.e. P^m, $u_n \longrightarrow u$ m-a.e. On the other hand $a_1(u_n, u_n) = a_1(u - P^1_{F_n^c}, u) \leq a_1(u, u)$. Hence by I.2.12 of [MR] there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}$ such that its Cesaro mean $\frac{1}{n}\sum_{k=1}^n u_{n_k} \longrightarrow u$ in $||\cdot||_a$. This gives the conclusion that $\mathcal{D}_b$ is dense in $U^1(L^2)$ with respect to $||\cdot||_a$, but the latter is dense in $(\mathcal{D}_a, ||\cdot||_a)$. Therefore $\mathcal{D}_b$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$. □

Remark. The idea of Theorem 3.1(ii) comes from an unpublished note of Fitzsimmons, who proved this denseness in the case that $(a, \mathcal{D}_a)$ is symmetric and M is continuous and never vanishes. But it is this denseness that plays a very important role in exploring the inverse problem.

The converse to Theorem 3.4 is also true and can be stated as follows.

Theorem 3.5. *Let X and Y be two Borel right processes which are associated with two quasi-regular Dirichlet forms $(a,\mathcal{D}_a)$ and $(b,\mathcal{D}_b)$, respectively. If $(b,\mathcal{D}_b)$ is strongly subordinate to $(a,\mathcal{D}_a)$, then there exists an $M\in MF$ such that Y is the subprocess of X killed by M.*

Proof. Let $f,g\in L^2$ be nonnegative Borel measurable functions, and let $(\hat{U}^q)$, (U^q) and (V^q) be the resolvents of $\hat{X}$, X and Y, respectively. Since $\hat{U}^q f\in\mathcal{D}_a$ for $q>0$, we can choose $v_n\in\mathcal{D}_b$, $n\in\mathbb{N}$, with $v_n\longrightarrow\hat{U}^q f$ in $||\cdot||_a$-norm. Then $v_n^+\longrightarrow\hat{U}^q f$ in $||\cdot||_a$-norm and, clearly, in L^2 norm. Now $b_q(v_n^+,V^q g)\geq a_q(v_n^+,V^q g)$ for any $n\in\mathbb{N}$. Hence $(v_n^+,g)\geq a_q(v_n^+,V^q g)$, and then $(\hat{U}^q f,g)\geq a_q(\hat{U}^q f,V^q g)=(f,V^q g)$ with n tending to infinity. By duality we have $(f,U^q g)\geq(f,V^q g)$ for any $q>0$; in particular, $U^q g\geq V^q g$ m-a.e. for $q>0$ and $g\in C_c^+(E)$, the set of non-negative elements in $C_c(E)$. Let us take $\{E_k\}$ as a b-nest consisting of compact sets and $F:=\bigcup_k E_k$. Then $\{E_k\}$ is an a-nest. By IV.3.2(iii) and the proof of IV.3.5 of [MR] we can see that (i) there exists a metric ρ on Y, compatible with the original topology, such that each E_k is compact with ρ, and hence F and $C_c(F)$ are separable; (ii) both X and Y actually live on F. Let $\{q_i\}$ be a dense set in $(0,\infty)$ and $\{g_j\}$ a dense set in $C_c^+(F)$. Since each $V^{q_i}g_j$ is b-quasi-continuous, it is a-quasi-continuous by Corollary 3.3. Then it follows that there exists an a-nest $\{F_n\}$ such that $\{U^{q_i}g_j,V^{q_i}g_j:\ i,j\in\mathbb{N}\}\subset C(\{F_n\})$. Hence $U^{q_i}g_j(x)\geq V^{q_i}g_j(x)$ for any $x\in\bigcup F_n$ and $i,j\in\mathbb{N}$. Let $N:=\cap F_n^c$, which is clearly an a-exceptional set. By the continuity of $U^{\cdot}f(x)$ and $V^{\cdot}f(x)$ for $f\in C_c(F)$ and $x\in E$, we find that $U^q g_j(x)\geq V^q g_j(x)$ for any $q>0$, $j\in\mathbb{N}$ and $x\in F-N$. The similar reasoning gives

$$U^q g(x)\geq V^q g(x),\quad q>0,\ g\in C_c(Y),\ x\in F-N. \tag{3.1}$$

Thus (V^q) is exactly subordinate to (U^q) (in terminology of [BG] and [Sh]) and by [Sh] there exists $M\in\mathrm{MF}$ such that

$$V^q f(x)=P^x\int_0^\infty e^{-qt}M_t f(X_t)dt,\ x\in F-N,\ f\in b\mathcal{E}^+,q\geq 0. \tag{3.2}$$

It follows that Y is the subprocess of X killed by M. □

A Dirichlet form $(b,\mathcal{D}_b)$ on L^2 is called reflectly subordinate to $(a,\mathcal{D}_a)$ if $\mathcal{D}_b\subset\mathcal{D}_a$ and $a=b$ on $\mathcal{D}_b$. A simple example of reflect subordination is a Brownian motion on a the bounded domain $D\subset\mathbb{R}^n$ with its so-called reflect Brownian motion on the boundary. The following corollary is a result on subordination. It also says that a general subordination can be decomposed into a combination of a strong subordination and a reflect subordination.

Corollary 3.6. *Let $(a,\mathcal{D}_a)$ and $(b,\mathcal{D}_b)$ be two Dirichlet forms on L^2. If $(b,\mathcal{D}_b)$ is quasi-regular and subordinate to $(a,\mathcal{D}_a)$, then there exists a measure σ on $E\times E$ such that*

$$b(u,v)=a(u,v)+\sigma(u\otimes v),\ u,v\in\mathcal{D}_b.$$

Proof. Let $\mathcal{D}_{b'}$ be the closure of $\mathcal{D}_b$ in $(\mathcal{D}_a,||\cdot||_a)$. Define $b'(u,v):=a(u,v)$ for any $u,v\in\mathcal{D}_{b'}$. Clearly $(b',\mathcal{D}_{b'})$ is a Dirichlet form on L^2 and by Corollary 3.3 it is quasi-regular. It follows from Theorem 3.5 that there exists an $M\in\mathrm{MF}$ such that

$b(u,v) = b'(u,v) + \nu_M(u \otimes v) = a(u,v) + \sigma(u \otimes v)$ for $u, v \in \mathcal{D}_b$, where σ is taken to be ν_M. □

Remark. Let $(A, D(A))$ and $(B, D(B))$ be the respective generators of $(a, \mathcal{D}_a)$ and $(b, \mathcal{D}_b)$. Benyaiche [Be] proved that in the case of classical symmetry if $(b, \mathcal{D}_b)$ is subordinate to $(a, \mathcal{D}_a)$ and $D(A) \cap D(B)$ is dense in $C_c(E)$, then the conclusion of Corollary 3.6 holds.

4. Bivariate smooth measures

From the transfer method developed in Chapter VI of [MR] we know that a quasi-regular Dirichlet form $(a, \mathcal{D}_a)$ admits a Beurling-Deny type decomposition along the diagonal of $\mathcal{D}_a \times \mathcal{D}_a$

$$(4.1)\quad a(u,u) = a^c(u,u) + \frac{1}{2}\int_{E\times E}[u(x)-u(y)]^2 J(dx,dy) + \int_E (u(x))^2 k(dx),\ u \in \mathcal{D}_a,$$

where a^c is the diffusion part, J the jumping measure and k the killing measure of $(a, \mathcal{D}_a)$. If X is the Borel right process associated with $(a, \mathcal{D}_a)$, then J coincides with the canonical measure ν of X relative to m, which is defined by

$$(4.2)\qquad \nu(F) := \lim_{t\to 0}\frac{1}{t}P^m \sum_{s\le t} F(X_{s-}, X_s)1_{\{X_{s-}\neq X_s\}},\ F \in (\mathcal{E}\times\mathcal{E})^+.$$

A family $A = (A_t)_{t\ge 0}$ of positive functions on Ω is said to be a positive continuous additive functional (PCAF in abbreviation) of X if it satisfies IV(4.6) in [MR], and a positive measure μ on $(E, \mathcal{E})$ is said to be smooth if μ does not charge a-exceptional sets and there exists an a-nest $\{F_n\}$ of compact sets such that $\mu(F_n) < \infty$ for each n. It is known that μ is smooth if and only if it is the Revuz measure of a PCAF (see Chapter IV of [MR]).

Recently there have been many works on characterization of smooth measures. As an application of Theorem 3.5 we will give a characterization which was proved in [Ke] and [St] by different, but rather analytic approaches. Let $\tilde{u}$ stand for an a-quasi-continuous m-version of $u \in \mathcal{D}_a$. Given a positive measure μ not charging m-polar sets, we define

$$(4.3)\qquad \begin{aligned} \mathcal{D}_b &:= \mathcal{D}_a \cap L^2(\mu),\\ b(u,v) &:= a(\tilde{u},\tilde{v}) + \mu(\tilde{u}\cdot\tilde{v}),\ u,v \in \mathcal{D}_b. \end{aligned}$$

Then $(b, \mathcal{D}_b)$ is a well-defined bilinear form on L^2. It is actually a Dirichlet form in the wide sense, namely, $\mathcal{D}_b$ may not be dense in L^2.

Theorem 4.1. *Let $(a, \mathcal{D}_a)$ be a quasi-regular Dirichlet form and μ a positive measure not charging m-polar sets. Then μ is smooth if and only if $(b, \mathcal{D}_b)$ defined in (4.3) is a quasi-regular Dirichlet form on L^2 and $\mathcal{D}_b$ is dense in $(\mathcal{D}_a, ||\cdot||_a)$.*

Proof. The 'only if' is a direct consequence of Theorem 3.4. Conversely by Theorem 3.5 there exists $M \in$ MF such that $\nu_M(dx,dy) = \delta_x(dy)\mu(dx)$ where δ_x is the singleton at x. By the representation theorem of ν_M in [Yi] $\nu_S = 0$, where $S(\omega) := \inf\{t > 0 : M_t > 0\}$. Also by the equivalence theorem in [Yi] we have $S = \zeta$ a.s. P^m; i.e. M never vanishes (before ζ). Similar arguments show that the pure jump factor of M is m-equivalent to 1. Hence M is continuous and never vanishes. Let $A_t := \log M_t$. Then $A = (A_t)$ is a PCAF and μ is nothing but the Revuz measure of A. $\square$

The following lemma is easy to check.

Lemma 4.2. *Let $(a, \mathcal{D}_a)$ and $(b, \mathcal{D}_b)$ be two Dirichlet forms on L^2. If $\mathcal{D}_a = \mathcal{D}_b$ and $||\cdot||_a$ is equivalent to $||\cdot||_b$, then $(a, \mathcal{D}_a)$ is quasi-regular if and only if $(b, \mathcal{D}_b)$ is.*

Given a positive measure σ on $E \times E$, let σ_r be its right marginal measure, σ_l its left marginal measure and $\bar{\sigma} := \frac{1}{2}(\sigma_r + \sigma_l)$. We call σ a bivariate smooth measure if (i) $\bar{\sigma}$ is smooth; (ii) $\sigma|_{D^c} \le J$, where D is the diagonal of $E \times E$.

Theorem 4.3. *Let X be a Borel right process associated with a quasi-regular Dirichlet form $(a, \mathcal{D}_a)$. A positive measure σ on $E \times E$ is a bivariate smooth measure if and only if σ is the bivariate Revuz measure of an $M \in MF$.*

Proof. We will first show the 'if' part. Assume that $\sigma = \nu_M$ for an $M \in$ MF. It is easy to see that σ_r and σ_l do not charge m-polar sets and neither does $\bar{\sigma}$. The assertion that $\sigma|_{D^c} \le J$ is clear from the definition. Let $(b, \mathcal{D}_b)$ be the Dirichlet form associated with (X, M). Then by Theorem 3.4 $(b, \mathcal{D}_b)$ is quasi-regular and strongly subordinate to $(a, \mathcal{D}_a)$. Define $\mathcal{D}_{\bar{b}} := \mathcal{D}_b$ and $\bar{b}(u, v) := a(u, v) + \bar{\sigma}(u \otimes v)$. Clearly $||u||_b \le ||u||_{\bar{b}}$ for any $u \in \mathcal{D}_b$. On the other hand let $\{u_n\} \subset \mathcal{D}_b$ be a sequence which converges to 0 in $||\cdot||_b$. Since $b(u, u) = a^c(u, u) + \int [u(x) - u(y)]^2 (J - \sigma)(dx, dy) + (\bar{\sigma} + k)(u^2)$ for $u \in \mathcal{D}_b$, $\lim_n \bar{\sigma}(u_n^2) = 0$ and $\lim_n \int [u_n(x) - u_n(y)]^2 \sigma(dx, dy) \le \lim_n 4\bar{\sigma}(u_n^2) = 0$. Hence $\{u_n\}$ converges to 0 in $||\cdot||_a$ and in $||\cdot||_{\bar{b}}$; i.e., there exists a constant $M > 0$ such that $||u||_{\bar{b}} \le M \cdot ||u||_b$ for any $u \in \mathcal{D}_b$. Then $||\cdot||_b$ is equivalent to $||\cdot||_{\bar{b}}$. Thus $(\bar{b}, \mathcal{D}_{\bar{b}})$ is quasi-regular and strongly subordinate to $(a, \mathcal{D}_a)$. By Lemma 4.1 $\bar{\sigma}$ is smooth.

Conversely if σ is a bivariate smooth measure, we define $\mathcal{D}_b := \mathcal{D}_a \cap L^2(\bar{\sigma})$, $b(u, v) := a(u, v) + \sigma(u \otimes v)$ and $(\bar{b}, \mathcal{D}_{\bar{b}})$ as above. Then $(\bar{b}, \mathcal{D}_{\bar{b}})$ is quasi-regular and strongly subordinate to $(a, \mathcal{D}_a)$. Hence by the arguments above we can show that $(b, \mathcal{D}_b)$ is also quasi-regular and strongly subordinate to $(a, \mathcal{D}_a)$. Finally Theorem 3.5 tells that σ is a bivariate Revuz measure of some $M \in$ MF. $\square$

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References

[Be] Benyaiche, A., *Subordination des formes de Dirichlet*, technical report, Tunis.

[BG] Blumenthal,R., Getoor, R.K., *Markov processes and potential theory*, Academic Press, New York - London, 1968. MR **41:**9348

[FG] Fitzsimmons, P.J., Getoor, R.K., *Revuz measures and time changes*, Math. Zeit. **199** (1988), 233-256. MR **89h:**60124

[Fu] Fukushima, M., *Dirichlet forms and Markov processes*, North-Holland, Amsterdam - Oxford - New York, 1980. MR **81f:**60105

[Ku] Kuwae, K., *Permanent sets of measures charging no exceptional sets and the Feynman-Kac formula*, Forum Math. **7** (1995), 575–605. CMP 95:17

[MR] Ma, Z., Röckner, M., *Introduction to the theory of* (*non-symmetric*) *Dirichlet forms*, Springer-Verlag, Berlin - Heidelberg - New York, 1992. MR **94d:**60119

[Sh] Sharpe, M.J., *General theory of Markov processes*, Academic Press, New York, 1988. MR **89m:**60169

[Si] Silverstein, M.L., *Symmetric Markov processes*, Lecture Notes in Math. 426, Springer-Verlag, Berlin - Heidelberg - New York, 1974. MR **52:**6891

[St] Stollmann, P., *Smooth perturbation of regular Dirichlet forms*, Proc. Amer. Math. Soc. **116** (1992), 747-752. MR **93a:**34014

[Yi] Ying, J., *Bivariate Revuz measures and the Feynman-Kac formula*, to appear in Ann. Inst. Henri Poincare: Probab. Stat.

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