PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 7, July 1996

MULTINOMIAL EXPANSIONS AND THE PYTHAGOREAN THEOREM

ALLAN FRYANT

(Communicated by J. Marshall Ash)

Dedicated to the memory of Professor Morris Marden

ABSTRACT. An addition formula for homogeneous polynomials is used to obtain a generalization of the Pythagorean theorem and a new view of the multinomial expansion.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, and let P denote the set of all polynomials

$$p(x) = \sum_{j=1}^{m} a_j x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}, \qquad a_j \in \mathbf{R}.$$

Letting $\partial/\partial x = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$, we define on P the inner product

$$(1)\qquad \begin{aligned} (p,q) &= p\left(\frac{\partial}{\partial x}\right) q(x)\bigg|_{x=0} \\ &= \left(\sum_{j=1}^{m} a_j \frac{\partial^{\alpha_j^1}}{\partial x_1^{\alpha_j^1}} \frac{\partial^{\alpha_j^2}}{\partial x_2^{\alpha_j^2}} \dots \frac{\partial^{\alpha_j^n}}{\partial x_n^{\alpha_j^n}}\right) q(x)\Bigg|_{x=0}. \end{aligned}$$

Let $H_k \subset P$ denote the vector space of homogeneous polynomials of degree k. Note that the dimension of H_k is

$$\dim H_k = d_k = \frac{(n+k-1)!}{(n-1)!\,k!}.$$

We first obtain a simple analog of the Funk-Hecke theorem [1, p. 247] for homogeneous polynomials.

Theorem 1. *If $p(x)$ is a homogeneous polynomial of degree k, then*

$$((x_1y_1 + x_2y_2 + \dots + x_ny_n)^k, p(y)) = k!\,p(x).$$

Proof. Expanding the power,

$$\begin{aligned} (x_1y_1 + \dots + x_ny_n)^k &= \sum_\alpha \frac{k!}{\alpha_1!\,\alpha_2!\dots\alpha_n!}(x_1y_1)^{\alpha_1}(x_2y_2)^{\alpha_2}\dots(x_ny_n)^{\alpha_n} \\ &= k!\sum_\alpha \frac{x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\dots\alpha_n!} y_1^{\alpha_1}y_2^{\alpha_2}\dots y_n^{\alpha_n}. \end{aligned}$$

Received by the editors October 7, 1994.
1991 *Mathematics Subject Classification.* Primary 33E99; Secondary 26C99.
Key words and phrases. Homogeneous polynomials, addition formula.

Thus if $p(y) = \sum_\alpha a_\alpha y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n}$, we have

$$\begin{aligned}
&((x_1y_1 + \cdots + x_ny_n)^k, p(y)) \\
&\quad = \left(k! \sum_\alpha \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \dots \alpha_n!} \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} \sum_\beta a_\beta y_1^{\beta_1} \dots y_n^{\beta_n} \right) \\
&\quad = k! \sum_\alpha a_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n} = k!\, p(x).
\end{aligned}$$

Our next result is an addition formula for orthonormal homogeneous polynomials.

Theorem 2 (Addition Formula)**.** *Let $\{p_j(x)\}_{j=1}^{d_k}$ be an orthonormal basis for H_k. Then*

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^k = k! \sum_{j=1}^{d_k} p_j(x)p_j(y). \tag{2}$$

Proof. Let $x = (x_1, x_2, \dots, x_n)$ and $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^n)$. For convenience we write $x^{\alpha_j} = x_1^{\alpha_j^1} x_2^{\alpha_j^2} \dots x_n^{\alpha_j^n}$. Then expanding the power, we have

$$(x_1y_1 + \cdots + x_ny_n)^h = \sum c_j x^{\alpha_j} y^{\alpha_j},$$

where x^{α_j} and y^{α_j} are homogeneous monomials of degree k. Since $\{p_j(x)\}_{j=1}^{d_k}$ is a basis for H_k, there exist constants a_{jm} such that

$$y^{\alpha_j} = \sum_{m=1}^{d_k} a_{jm} p_m(y).$$

Thus

$$\begin{aligned}
(x_1y_1 + \cdots + x_ny_n)^k &= \sum_j c_j x^{\alpha_j} \left(\sum_m a_{jm} p_m(y) \right) \\
&= \sum_j g_j(x) p_j(y),
\end{aligned}$$

where the last expression is a rearrangement of the previous sum. The $g_j(x)$ are linear combinations of the monomials x^{α_j}, and thus are homogeneous polynomials of degree k in x. Further, since the $p_j(x)$ are orthonormal,

$$((x_1y_1 + \cdots + x_ny_n)^k, p_l(y)) = \left(\sum_j g_j(x)p_j(y), p_l(y) \right) = g_l(x).$$

But by the identity given in Theorem 1,

$$((x_1y_1 + \cdots + x_ny_n)^k, p_l(y)) = k!\, p_l(x).$$

Thus,

$$g_j(x) = k! p_j(x), \qquad j = 1, 2, \dots, d_k,$$

which completes the proof.

If $\alpha_j = (\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^n)$, for convenience we write $\alpha_j! = \alpha_j^1!\,\alpha_j^2! \ldots \alpha_j^n!$. The result of Theorem 2 shows that the multinomial expansion

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^k = \sum \frac{k!}{\alpha_j!} x^{\alpha_j} y^{\alpha_j} = k! \sum \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}} \frac{y^{\alpha_j}}{\sqrt{\alpha_j!}}$$

is no more than an addition formula for the orthonormal basis $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{d_k}$ of H_k. The significance of the result (2) lies with the fact that it holds for any orthonormal basis $\{p_j(x)\}_1^{d_k}$, not merely the monomials $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{d_k}$.

The addition formula of Theorem 2 yields an identity which contains the Pythagorean theorem $\sin^2\theta + \cos^2\theta = 1$ as a special case:

Theorem 3 (Pythagorean Identity). *Suppose $\{p_j(x)\}_{j=1}^{d_k}$ is an orthonormal basis for H_k. Then on the unit sphere $|s| = \sqrt{s_1^2 + s_2^2 + \cdots + s_n^2} = 1$,*

$$\sum_{j=1}^{d_k} [p_j(s)]^2 = \frac{1}{k!}.$$

Proof. Letting $x = y$ in the addition formula yields

$$(x_1^2 + x_2^2 + \cdots + x_n^2)^k = k! \sum_{j=1}^{d_k} [p_j(x)]^2,$$

from which the result is immediate.

Most inner products are integrals. Although the inner product we have used here is defined by a derivative, it is far from artificial. Indeed, this inner product allows us to view Taylor series as Fourier series. That is, suppose we have the Taylor series

$$f(x) = \sum_{j=0}^{\infty} a_j x^{\alpha_j}.$$

Rewriting the series as

$$f(x) = \sum_{j=1}^{\infty} b_j \frac{x^{\alpha_j}}{\sqrt{\alpha_j!}},$$

and noting that the monomials $\{x^{\alpha_j}/\sqrt{\alpha_j!}\}_{j=1}^{\infty}$ are orthonormal with respect to the inner product (1), we have

$$(x^{\alpha_j}/\sqrt{\alpha_j!}, f(x)) = b_j.$$

These simple results appear to have considerable analytic significance. To see this, one need only consider the kernel

$$\sum_{k=0}^{\infty} \sum_{j=1}^{d_k} p_k^j(x) p_k^j(y) = \sum_{k=0}^{\infty} \frac{(x \cdot y)^k}{k!} = e^{x \cdot y},$$

and the reproducing formula

$$(f(y), e^{x \cdot y}) = f(x).$$

Here $\{p_k^j(x)\}_{j=1}^{d_k}$, $k = 0, 1, 2, \ldots$, are arbitrary orthonormal bases for the homogeneous polynomials H_k of degree k.

References

1. A. Erdelyi, *Higher transcendental functions*, Vol. 2, McGraw-Hill, New York, 1953. MR **15:**419i

Department of Mathematics, Greensboro College, Greensboro, North Carolina 27401