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ON THE UNIQUENESS PROBLEM OF HARMONIC QUASICONFORMAL MAPPINGS

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ABSTRACT. In this paper, we give an affirmative answer to Sheretov's problem on the uniqueness of harmonic mappings and improve the unique minimal mapping theorem of Reich and Strebel. Meanwhile, we also solve a problem posed by Reich and obtain the uniqueness theorem on related weight functions.

1. INTRODUCTION

For a quasiconformal (q.c.) mapping $w = f(z)$ of the unit disk $U = \{|z| < 1\}$ onto $U = \{|z| < 1\}$ we use the standard notation

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z}, k[f] = \operatorname*{esssup}_{z\in U} |\mu_f(z)|, K[f] = \frac{1+k[f]}{1-k[f]}.$$

If H is a homeomorphism of ∂U onto ∂U we denote by $Q(H)$ the class of q.c. mappings of U onto itself with boundary values H. In order to avoid triviality we assume that $Q(H)$ is non-empty, and that H is not the boundary restriction of a conformal mapping. The homeomorphism H then determines the extremal maximal dilatation $K^* > 1$, defined as

$$K^* = \inf_{f\in Q(H)} K[f].$$

We denote that $Q(H,K) = \{f | f \in Q(H) \text{ and } K[f] \le K\}$ for $K \ge K^*$.

Given a normalized weight function $\rho(w)$, that is, $\rho(w)$ is a continuous (excluding isolated zeros) positive function defined in U and $\iint_U \rho(w)dudv = 1$, we define the Douglas-Dirichlet functional on $Q(H)$ as

$$D_\rho[f] = \iint_U \rho(f(z))(|f_z|^2 + |f_{\bar{z}}|^2)dxdy, f \in Q(H).$$

A q.c. mapping $w = f(z)$ of the unit disk U onto itself is called harmonic with respect to $\rho(w)$ if $\phi(z) = \rho(f(z))f_z\overline{f_{\bar{z}}}$ is a holomorphic function in U.

If $f_0 \in Q(H)$, we call f_0 minimal for D_ρ if $D_\rho[f_0] = \inf_{f\in Q(H)} D_\rho[f]$.

V.G. Sheretov [1] considered a problem related to Shibata'problem [6] and posed that the uniqueness problem of harmonic (with respect to $\rho(w)$) mappings in the class $Q(H)$ under the condition that the class $Q(H, K^*)$ also contains such mappings

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remains open. In this paper, we give an affirmative answer to Sheretov's problem and prove that the harmonic mapping (with respect to $\rho(w)$) in $Q(H)$ is unique. Then we improve the unique minimal mapping theorem of Reich and Strebel. In addition, we also solve a problem posed by Reich [4] and obtain the uniqueness theorem of the related normalized weight functions.

2. Main results and proof

Lemma 1 ([2]). *If $w = F(z)$ is a quasiconformal mapping of the unit disk U onto itself and $F(e^{i\theta}) = e^{i\theta}$ for $\theta \in [0, 2\pi)$, then for any functions $\phi(z)$ holomorphic in U with $\|\phi\| = \iint_U |\phi(z)|dxdy < \infty$, we have*

$$Re \iint_U \frac{\mu_F}{1-|\mu_F|^2}\phi(z)dxdy \le \iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}|\phi(z)|dxdy. \tag{1}$$

Theorem 1. *If $f_0 \in Q(H)$, $f_1 \in Q(H)$, and f_0 is harmonic with respect to the metric ρ, then either $D_\rho[f_1] > D_\rho[f_0]$ or $f_1 = f_0$.*

Proof. If $f_0(z) \in Q(H)$, $f_1(z) \in Q(H)$, and f_0 is harmonic with respect to the metric ρ, then by [3] we know $\phi_0 = \rho(f_0(z))f_{0z}\overline{f_{0\bar z}}$ is holomorphic in U, and

$$\begin{aligned}\|\phi_0\| &= \iint_U |\phi_0(z)|dxdy = \iint_U \rho(f_0)|f_{0z}||f_{0\bar z}|dxdy\\ &= \iint_U \rho(w)\frac{|f_{0z}f_{0\bar z}|}{|f_{0z}|^2-|f_{0\bar z}|^2}dudv = \iint_U \rho(w)\frac{|\mu_{f_0(Z)}|}{1-|\mu_{f_0(z)}|^2}dudv\\ &\le \frac{k[f_0]}{1-k[f_0]^2}.\end{aligned} \tag{2}$$

By assumption of $Q(H)$, we know that f_0 is not conformal in U and $\phi_0(z) \neq 0$.

Otherwise, if $\phi_0(z) = 0$, i.e., $\rho(f_0)f_{0z}\overline{f_{0\bar z}} = 0$, then by the definition of $\rho(w)$, we obtain that $f_0(z)$ is conformal in U. This is a contradiction. Hence $\phi_0(z)$ has at most countable zeros in U, and $\phi_0 \neq 0$ a.e. in U.

Let $F = f_1^{-1} \circ f_0$; then $F(e^{i\theta}) = e^{i\theta}$. By [3] and Lemma 1, we know

$$\begin{aligned}&D_\rho[f_1] - D_\rho[f_0] = D_\rho[f_0 \circ F^{-1}] - D_\rho[f_0]\\ &= 2\iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}\rho(f_0(z))(|f_{0z}|^2+|f_{0\bar z}|^2)dxdy - 4Re\iint_U \frac{\mu_F}{1-|\mu_F|^2}\phi_0(z)dxdy\\ &= 2\iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}|\phi_0(z)|\frac{|f_{0z}|^2+|f_{0\bar z}|^2}{|f_{0z}||f_{0\bar z}|}dxdy - 4Re\iint_U \frac{\mu_F}{1-|\mu_F|^2}\phi_0(z)dxdy\\ &\ge 4\iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}|\phi_0(z)|dxdy - 4Re\iint_U \frac{\mu_F}{1-|\mu_F|^2}\phi_0(z)dxdy\\ &\ge 0.\end{aligned} \tag{3}$$

Then we have $D_\rho[f_1] \ge D_\rho[f_0]$.

If $D_\rho[f_1] > D_\rho[f_0]$, then Theorem 1 holds.

If $D_\rho[f_1] = D_\rho[f_0]$, by (3) we have

$$\iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}(|\mu_{f_0}| + \frac{1}{|\mu_{f_0}|})|\phi_0(z)|dxdy = 2\iint_U \frac{|\mu_F|^2}{1-|\mu_F|^2}|\phi_0(z)|dxdy. \tag{4}$$

Since $\phi_0(z) \neq 0$ a.e. in U and

$$|\mu_{f_0}| + \frac{1}{|\mu_{f_0}|} > 2 \tag{5}$$

a.e. in U, equation (4) implies that $\mu_F = 0$ a.e. in U. So F is a conformal mapping of U onto itself with $F(e^{i\theta}) = e^{i\theta}$. $F(z)$ must be identity, $F(z) = z$. Hence $f_0(z) = f_1(z)$. This completes the proof of Theorem 1. □

From the proof of Theorem 1, we have

Corollary 1. *If $f_0 \in Q(H)$ and f_0 is harmonic with respect to $\rho(w)$, then f_0 is a minimal mapping for D_ρ.*

Corollary 1 indicates that the converse of the minimal mapping theorem of [3] also holds.

Corollary 2. *If there exists $f_0 \in Q(H)$ such that f_0 is minimal for D_ρ, then such f_0 must be unique.*

Proof. If $f_0 \in Q(H)$ and f_0 is minimal for D_ρ, then

$$D_\rho[f_0] = \inf_{f \in Q(H)} D_\rho[f].$$

By [3] we know that f_0 is harmonic with respect to $\rho(w)$. If there exists another $f_1 \in Q(H)$ such that f_1 is minimal for D_ρ, then $D_\rho[f_0] = D_\rho[f_1]$. From the proof of Theorem 1, we have $f_1 = f_0$. □

It should be pointed out that Reich and Strebel [3] proved the unique minimal mapping theorem when there exists a maximal weight function $\rho_0(w)$ by a mean-dilatation inequality.

Theorem 2. *If there exists a harmonic mapping (with respect to $\rho(w)$) in $Q(H)$, then it must be unique.*

Proof. Suppose $f_0 \in Q(H)$ and $f_1 \in Q(H)$ are harmonic with respect to $\rho(w)$. By Corollary 1, we know f_0 and f_1 are minimal for D_ρ. Then by Corollary 2, we have $f_0(z) = f_1(z)$. □

V. G. Sheretov [1] asserted that there exists at least a countable set of mappings $w_{k_n} \in Q(H)$ which are harmonic with respect to the conformal metric $\rho(w)|dw|^2$ under the condition that $Q(H, K^*)$ does not contain such mapping. His proof depends on the existence of harmonic mappings [1, Theorem 6]. But Reich pointed out that Sheretov's proof is not correct, thereby leaving [1, Theorem 6] open to doubt [4]. We give an answer to this problem and point out that Sheretov's assertion is not true.

Theorem 3. *If the class $Q(H, K^*)$ does not contain any harmonic mapping with respect to $\rho(w)$, then there always exists some $K > K^*$ such that, in $Q(H, K)$, there exists no harmonic mapping with respect to $\rho(w)$.*

Proof. Suppose that for every $K > K^*$, there exists a harmonic mapping with respect to $\rho(w)$ in $Q(H, K)$.

We first choose $K_1 > K^*$; then there exists a harmonic mapping (with respect to $\rho(w)$) $f_1 \in Q(H, K_1)$. Since $Q(H, K^*)$ does not contain any harmonic mapping with respect to $\rho(w)$, we must have $f_1 \notin Q(H, K^*)$. So

$$K^* < K[f_1] \leq K_1. \tag{6}$$

Now we set $K_2 = \dfrac{K^* + K[f_1]}{2}$; then

$$K^* < K_2 < K[f_1] \leq K_1. \tag{7}$$

We again have an $f_2 \in Q(H, K_2)$ such that f_2 is a harmonic mapping with respect to $\rho(w)$ and $K[f_2] \leq K_2 < K[f_1]$. We must have $f_1 \neq f_2$, where f_1 and f_2 are harmonic with respect to $\rho(w)$, which is a contradiction to Theorem 2. □

We give an example to indicate that the conditions of Theorem 3 are not empty.

Suppose $\phi_0 = \frac{1}{(1-z)^2}, z \in U$; we have $\|\phi_0\| = \infty$. For $0 < k_0 < 1$, let $f_0(z)$ be a quasiconformal mapping of U onto U with $\mu_{f_0} = k_0 \frac{\overline{\phi_0}}{|\phi_0|}$ and $H = f_0|_{\partial U}$. By Sethares [5], we know f_0 is unique extremal in $Q(H)$ and $k^* = k_0$. That is, $Q(H, K^*) = \{f_0\}$.

It is easy to see that f_0 is not a harmonic mapping with respect to $\rho(w)$. Otherwise, suppose f_0 is harmonic. By [4], there exists a holomorphic function $\phi(z), z \in U$, with $\|\phi\| < \infty$ and $\mu_{f_0} = |\mu_{f_0}| \frac{\overline{\phi}}{|\phi|}$. That is,

$$\frac{\phi}{|\phi|} = \frac{\phi_0}{|\phi_0|}$$

with $\|\phi\| < \infty$ and $\|\phi_0\| = \infty$. This is a contradiction. Hence $Q(H, K^*)$ does not contain a harmonic mapping with respect to $\rho(w)$.

Theorem 4. *If $f_0(z) \in Q(H)$ and there exists a normalized weight function $\rho_0(w)$ such that f_0 is harmonic with respect to $\rho_0(w)$, then the weight function must be unique.*

Proof. Suppose there exists another normalized weight function $\rho_1(w)$ such that f_0 is harmonic with respect to $\rho_1(w)$; then $\phi_0(z) = \rho_0(f_0) f_{0z} \overline{f_{0\bar{z}}}$ and $\phi_1(z) = \rho_1(f_0) f_{0z} \overline{f_{0\bar{z}}}$ are holomorphic in U. By proof of Theorem 1, we know $\phi_0 \neq 0$, $\phi_1 \neq 0$, and $\mu_{f_0} \neq 0$ a.e. in U. From the definition of ρ_0 and ρ_1, there exist a point $z_0 \in U$ and a neighborhood $V \subset U$ of z_0 such that $\phi_0, \phi_1, \rho_0(f_0), \rho_1(f_0)$, and μ_{f_0} don't vanish in V. By [4]

$$\mu_{f_0} = |\mu_{f_0}| \frac{\overline{\phi_0}}{|\phi_0|} = |\mu_{f_0}| \frac{\overline{\phi_1}}{|\phi_1|}; \tag{8}$$

hence

$$\frac{\overline{\phi_0}}{|\phi_0|} = \frac{\overline{\phi_1}}{|\phi_1|} \tag{9}$$

in V. Let $\Phi(z) = \frac{\phi_(z)}{\phi_0(z)}, z \in V$. By (9) we know $\Phi(z)$ must be a constant λ, that is

$$\phi_1(z) = \lambda \phi_0(z), \ z \in V; \tag{10}$$

hence

$$\phi_1(z) = \lambda \phi_0(z), \ z \in U. \tag{11}$$

Hence $\lambda \rho_1(f_0) = \rho_0(f_0)$ a.e. $z \in U$. Because f_0 is a homeomorphism of U onto itself and ρ_0, ρ_1 are continuous and normalized, we obtain $\lambda = 1, \rho_0 = \rho_1$. □

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References

[1] Sheretov,V.G, *On extremal quasiconformal mappings with given boundary correspondence*, Siberian Math.J **19** (1978), 671-678.

[2] Reich,E and Strebel,K, *On quasiconformal mappings which keep the boundary points fixed*, Trans. A.M.S **138** (1969), 211-222. MR **38:**6059

[3] Reich,E and Strebel,K, *On the Gerstenhaber-Rauch principle*, Israel J.Math **57** (1987), 89-100. MR **88g:**30028

[4] Reich,E, *On the variational principle of Gerstenhaber and Rauch*, Ann Acad Sci. Fenn **10** (1985), 469-475. MR **87b:**30036

[5] Sethares,G.C, *The extremal property of certain Teimuller mappings*, Comm.Math.Helv, **4 3** (1968), 98-119. MR **37:**4253

[6] Shibata,K, *On the existence of a harmonic mapping*, Osaka Math. J., **15** (1963), 173-211. MR **28:**1289

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